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STATISTICALLY τ-BOUNDED OPERATORS ON ORDERED TOPOLOGICAL VECTOR SPACES

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Abstract. *In this paper, we introduce statistical bounded sets on topological vector space. Also, we give the three notions of bounded operators from topological vector spaces to ordered topological vector spaces. Moreover, we show some relations between these operators and order bounded operators. Also, we give some algebraic properties of these operators with respect to the uniform convergence topology.*

Keywords: *statistical bounded set; statistical bounded operator; st-bo-operator; st-bb-operator; topological vector space; locally solid Riesz space.*

1. INTRODUCTION AND PRELIMINARIES

The operator theory and the statistical convergence are natural and efficient two tools in the theory of functional analysis. A vector lattice introduced by Riesz in [1] is an ordered vector space, and it has many applications in measure theory, operator theory, and applications in economics [2-4]. On the other hand, the statistical convergence is a generalization of the ordinary convergence of a real sequence [5-8]. One study related to this paper is done by Troitsky [9] where the bb- and nb-bounded operators were defined between topological vector spaces and another one is done by Aydın [10] in which the ob-bounded operator was defined from vector lattices to locally solid vector lattice. Moreover, Hejazian et al. [11] define bounded operators on topological vector spaces. As far as we know, there has been no study for operators from topological vector spaces to topological ordered vector spaces. Our aim in this paper is to fill this gap.

Now, let us give some basic notations and terminologies that will be used in this paper. A neighborhood of an element x in a topological vector space E is a subset of E containing an open set that contains x. In this paper, neighborhoods of zero will often be referred as zero neighborhoods. It is well known that every linear topology τ on a vector space E has a base $\mathcal{N}$ of zero neighborhoods satisfying the following four properties; for each $V \in \mathcal{N}$, we have $\lambda V \subseteq V$ for all scalar $|\lambda| \leq 1$; for any $V_1, V_2 \in \mathcal{N}$ there is another $V \in \mathcal{N}$ such that $V \subseteq V_1 \cap V_2$; for each $V \in \mathcal{N}$ there exists another $U \in \mathcal{N}$ with $U + U \subseteq V$; for any scalar λ and each $V \in \mathcal{N}$, the set λV is also in $\mathcal{N}$. We refer the reader for much more information on linear topology [12]. In this article, unless otherwise, when we mention a zero neighborhood, it means that it always belongs to a base that holds the above properties.

Let E be a real-valued vector space. If there is an order relation "$\leq$" on E, i.e., it is antisymmetric, reflexive and transitive then E is called ordered vector space whenever the following conditions hold: for every $x, y \in E$ such that $x \leq y$, we have $x + z \leq y + z$ and

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$\alpha x \leq \alpha y$ for all $z \in E$ and $0 \leq \alpha \in \mathbb{R}$. An ordered vector space E is called Riesz space or vector lattice if, for any two vectors $x, y \in E$, the infimum $x \wedge y = \inf\{x, y\}$ and the supremum $x \vee y = \sup\{x, y\}$ exist in E. Let E be a vector lattice. Then, for any $x \in E$, the positive part of x is $x^+ := x \vee 0$, the negative part of x is $x^- := (-x) \vee 0$ and absolute value of x is $|x| := x \vee (-x)$. A vector lattice is called order complete if every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). A vector lattice is order complete iff $0 \leq x_\alpha \uparrow \leq x$ implies the existence of $\sup x_\alpha$. For a positive element a in a vector lattice E, the set $[-a, a] = \{x \in E: -a \leq x \leq a\}$ is an order interval. Also, if any subset in E is included in an order interval then it is called order bounded set. An order bounded operator between vector lattices E and F sends order bounded subsets to order bounded subsets. $L_b(E, F)$ denote the set of all order bounded operators between E and F.

Recall that a subset A of a vector lattice E is called solid if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A solid vector subspace of a vector lattice is referred to as an ideal. An order closed ideal is called a band. Let E be vector lattice and τ be a linear topology on it. Then (E, τ) is called a *locally solid vector lattice* (or, *locally solid Riesz space*) if τ has a base which consists of solid sets. A vector lattice E is called *Archimedean* whenever $\frac{1}{n} x \downarrow 0$ holds in E for each $x \in E_+ = \{x \in E: 0 \leq x\}$. In this article, unless otherwise, all vector lattices are assumed to be real and Archimedean.

Consider a subset K of the set $\mathbb{N}$ of all natural numbers. Let's define a new set $K_n = \{k \in K: k \leq n\}$. Then we denote $|K_n|$ for the cardinality of the set K_n. If the limit of $\delta(K) = \lim\limits_{n\to\infty} |K_n|/n$ exists then $\delta(K)$ is called the asymptotic density of the set K. Let X be a topological space and (x_n) be a sequence in X. Then (x_n) is said to be statistically convergent to $x \in X$ whenever, for each neighborhood U of x, we have $\delta(\{n \in \mathbb{N}: x_n \notin U\}) = 0$ [13-18]. Similarly, a sequence (x_n) in a locally solid Riesz space (E, τ) is said to be statistically τ-convergent to $x \in E$ if it is provided that, for every τ-neighborhood U of zero, $\lim\limits_{n\to\infty} \frac{1}{n} |\{k \leq n: (x_k - x) \notin U\}| = 0$ holds. Let (x_n) be a sequence in a locally solid Riesz space (E, τ). If there exists some scalar $\lambda > 0$ such that $\delta(\{n \in \mathbb{N}: \lambda x_n \notin U\}) = 0$ holds for every τ-neighborhood U of zero then we say that (x_n) is statistically τ-bounded.

2. BOUNDED OPERATORS

In this section, we introduce the notion of statistical bounded set and st-bo-, st-bb- and st-bs-bounded operators.

Definition 2.1. Let (X, τ) be a topological vector space. A subset $B \subseteq X$ is called statistical bounded (or shortly, st-bounded) set in X if, for every zero neighborhood U, there is a scalar $\lambda > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{b \in B: \lambda b \notin U\}| = 0.$$

It is clear that a subset of an st-bounded set is st-bounded. On the other hand, a subset B in a topological vector space (E, τ) is called topological bounded (or shortly, τ-bounded) if, for every zero neighborhood U in E, there is a positive scalar λ such that $B \subseteq \lambda U$. So, we have the following useful result.

Lemma 2.2. Every τ-bounded set in topological vector spaces is st-bounded.

Let X and Y be topological vector spaces. An operator $T: X \to Y$ is said to be bb-bounded if it maps every bounded set into a bounded set [9]. Motivated by this definition and by the ob-bounded operator [10] and the nb- and bb-bounded operators [11], we give the following notions.

Definition 2.3. Let E be a topological vector space, F be an ordered topological vector space and $T: E \to F$ be an operator. Then T is said to be

a) st-bo-bounded if it maps every statistically bounded set into order bounded set,
b) st-bb-bounded if it maps each statistically bounded set into a topological bounded set,
c) st-bs-bounded if it maps statistically bounded sets into statistically bounded sets.

Consider the following theorems which are two classical results about order bounded subsets in a locally solid vector lattice Theorem 2.19.(i) [12] and Theorem 2.2. [19], respectively.

Theorem 2.4. Let (E, τ) be a locally solid vector lattice. Then every order bounded subset in E is τ-bounded.

Theorem 2.5. Let (E, τ) be an ordered topological vector space that has order bounded τ-neighborhood of zero. Then every τ-bounded subset is order bounded.

We continue with the following several basic results that follow directly from their basic definitions and Lemma 2, Theorem 2.4. and Theorem 2.5., so their proofs are omitted.

Remark 2.6. Let E be a topological vector space and F be an ordered topological vector space. Then we have that

a) every ordered bounded set in a locally solid vector lattice is st-bounded,
b) the st-bb-boundedness implies st-bs-boundedness,
c) the st-bo-boundedness implies st-bb-boundedness if F is a locally solid vector lattice,
d) the st-bo-boundedness implies st-bs-boundedness if F is a locally solid vector lattice,
e) the st-bb-boundedness implies st-bo-boundedness whenever F has order bounded τ-neighborhood of zero,
f) every ordered bounded operator is st-bo-bounded if E is a locally solid vector lattice
g) ordered boundedness implies st-bo-boundedness implies st-bb-boundedness implies st-bs-boundedness if $E = F$ is a locally solid vector lattice, so we have $L_b(E) \subseteq B_{st-bo}(E) \subseteq B_{st-bb}(E) \subseteq B_{st-bs}(E)$,
h) if $E = F$ is a locally solid vector lattice with an order bounded τ-neighborhood of zero then the st-bo-boundedness and the st-bb are coinciding, so we have $L_b(E) \subseteq B_{st-bo}(E) = B_{st-bb}(E) \subseteq B_{st-bs}(E)$.

We denote by $B_{st-bo}(E, F)$ the class of all st-bo-bounded operators from a topological vector space E to an ordered topological vector space F. Also, we denote $B_{st-bb}(E, F)$ and $B_{st-bs}(E, F)$ the class of all st-bb-bounded and the class of all st-bb-bounded operators from E to F, respectively. Now, we can allocate the sets $B_{st-bo}(E, F)$, $B_{st-bb}(E, F)$ and $B_{st-bs}(E, F)$ to the topology of uniform convergence on st-bounded sets. That is, a net (T_α) in $B_{st-bo}(E, F)$ (respectively, in $B_{st-bb}(E, F)$ or $B_{st-bs}(E, F)$) converges to zero on an st-bounded set B if, for each $u \in F_+$(respectively, for every zero neighborhood U or for each st-bounded zero neighborhood V of F), there is an index α_0 such that $T(B) \subseteq [-u, u]$ (respectively, $T(B) \subseteq U$ or $T(B) \subseteq V$) for all $\alpha \geq \alpha_0$. We should note that these classes of operators are not equal, in general. To see that let's consider Example 2.3. [19].

Example 2.7. Let $E = F$ be the ordered topological vector space $\mathbb{R}^2$ with the lexicographic ordering and the usual topology. Then the identity operator I on E is st-bo-bounded, but it fails to be st-bb-bounded. Indeed, consider two elements $x = (-1,0)$ and $y = (1,0)$ in E. Then the order interval $[x, y]$ is clearly an order bounded in E. It can be seen that this interval includes uncountably many infinite vertical rays. Hence, the order interval $[x, y]$ is not topological bounded set in E because E has the usual topology and a topological bounded set cannot contain any infinite vertical ray. Thus, st-bo-boundedness does not imply st-bb-boundedness in general.

Example 2.8. Let $E = F$ be c_0, the space of all null sequences with the usual order and norm topology. Then the identity operator I on E is st-bb-bounded, but it fails to be st-bo-bounded. Indeed, consider the unit ball $B(0,1)$ centered at zero with radius one of c_0. Let's take the sequence (x_n) defined by $x_n = (1,1,1, \dots ,1,0,0,0 \dots)$, where the first n terms are 1 and the rest is zero. Then it can be seen that (x_n) is st-bounded because topological bounded. But it is not order bounded in c_0. Thus, st-bb-boundedness does not imply st-bo-boundedness in general.

3. TOPOLOGICAL ALGEBRAS OF OPERATORS

We give some results about st-bo-, st-bb- and st-bs-bounded operators. In ordered vector spaces, the summation of finite order bounded sets is order bounded, and the summation of finite topological bounded sets is bounded in topological vector spaces. Thus, the summation of st-bo-bounded operators is also st-bo-bounded and the summation of st-bb-bounded operators is also st-bb-bounded. For the st-bs-bounded operators, we have the following proposition.

Proposition 3.1. The summation of st-bs-bounded operators S and T from a topological vector space E to a locally solid Riesz space F is st-bs-bounded.

Proof: It is enough to show that the summation of finite statistical bounded sets in locally solid Riesz space is st-bounded. Consider two st-bounded sets B_1 and B_2 in a locally solid Riesz space F. For any zero neighborhood U, there exists another zero neighborhood W such that $W + W \subseteq U$. Hence, we have some positive scalars $\lambda_{w_1}, \lambda_{w_2} > 0$ such that $\delta(K_{w_1}) = \delta(K_{w_2}) = 0$ for sets $K_{w_1} = \{x \in B_1: \lambda_{w_1} x \notin W\}$ and $K_{w_2} = \{y \in B_2: \lambda_{w_2} y \notin W\}$. Let choose a positive scalar $\lambda = \min\{\lambda_{w_1}, \lambda_{w_2}, 1\}$. Then we have $|\lambda x| \leq |\lambda_{w_1} x|$ and $|\lambda y| \leq |\lambda_{w_2} y|$ for $x \in B_1$ and $y \in B_2$, respectively. Since W is solid and absorbing, we have $\lambda x, \lambda y \in W$. Then we have

$$\lambda x + \lambda y \in W + W \subseteq U.$$

Thus we get $\mu(\{x + y: x \in B_1, y \in B_2 \ and \ \lambda(x + y) \notin U\} = 0$. Consequently, $B_1 + B_2$ is a statistically bounded set.

Firstly, we show the continuity of addition and scalar multiplication with the uniform convergence topology.

Theorem 3.2. The operations of addition and scalar multiplication are continuous in $B_{st-bo}(E, F)$, $B_{st-bb}(E, F)$ and $B_{st-bs}(E, F)$.

Proof: The continuity of addition and scalar multiplication in $B_{st-bs}(E,F)$ can be shown in the same way in the proof of Theorem 2.1 [3]. So, it is enough to show continuity for st-bo-operators.

Suppose the nets (T_α) and (S_α) are nets of st-bo-bounded operators that converges to zero uniformly on st-bounded sets. Fix an arbitrary st-bounded set B in E. Then, for any $f \in F_+$, there are some $\lambda_1, \lambda_2 > 0$ such that $T_\alpha(B) \subseteq [-f,f]$ for all $\alpha \geq \alpha_1$ and $S_\alpha(B) \subseteq [-f,f]$ for each $\alpha \geq \alpha_2$. So, there is another index α_0 such that $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_1$ because the index set of nets is directed. Hence, $T_\alpha(B) \subseteq [-f,f]$ and $S_\alpha(B) \subseteq [-f,f]$ for all $\alpha \geq \alpha_0$. Then we have

$$(T_\alpha + S_\alpha)(B) \subseteq [-f,f] + [-f,f] = [-2f,2f]$$

for each $\alpha \geq \alpha_0$.

Now, we show convergence for the scalar multiplication. Consider the st-bounded set B in E. Take a sequence of reals (λ_n) and assume it converges to zero. Since (T_α) is uniform convergent to zero on st-bounded sets, for every positive vector $u \in F_+$, there exists an index α_0 such that $T_\alpha(B) \subseteq [-u,u]$ for all $\alpha \geq \alpha_0$. We know that for enough large n, we have $|\lambda_n| \leq 1$, and so $\lambda_n[-u,u] \subseteq [-u,u]$. Then, for all $\alpha \geq \alpha_0$ and sufficiently large n, we have

$$\lambda_n T(B) = T(\lambda_n B) \subseteq \lambda_n[-u,u] \subseteq [-u,u].$$

Therefore, we get the desired result.

The lattice operations are continuous in the following sense.

Theorem 3.3. Let E be a topological vector space and F be an order complete locally solid vector lattice. Then the lattice operations with the uniform convergence are continuous in $B_{st-bo}(E,F)$, $B_{st-bb}(E,F)$ and $B_{st-bs}(E,F)$ on statistically bounded sets.

Proof: We show the continuity of the lattice operations in $B_{st-bs}(E,F)$. The other cases can be shown similarly. Suppose the nets (S_α) and (T_α) of st-bs-bounded operators converge to the operators S and T uniformly on st-bounded sets, respectively. By applying Theorem 1.8 [12], we have

$$(S \vee T)(x) = \sup\{Sy + Tz : y + z = x \ \ and \ \ y,z \in E_+\}$$

for every $x \in E_+$. Let's fix an st-bounded set B in E. So, consider another set $A = \{y \in E_+ : \exists z \in E_+, x = y + z \ \ forsome \ \ x \in B_+\}$. Then A is also st-bounded. Indeed, for any zero neighborhood U, there exists a positive scalar λ such that $\mu(K_B) = 0$ for $K_B = \{x \in B : \lambda x \notin U\}$ since B is st-bounded. Consider the set $K_A = \{y \in A : \lambda y \notin U\}$. For a vector $y \in A$, we have $x \in B_+$ and $z \in E_+$ such that $y + z = x$, and so $\lambda y \leq \lambda x$. Then, by using the solidness of U, we get $y \in K_A$ whenever $x \in K_B$. Thus, one can see that the carnality of K_A can not be bigger than two times the carnality of K_B since we have $z \in K_A$ if $y \in K_A$. As a result, we get $\mu(K_A) = 0$ because of $\mu(K_B) = 0$. Therefore, A is statistically bounded set in E. Thus, we have that $S_\alpha \to S$ and $T_\alpha \to T$ converge uniformly on A. Take a fixed $x \in B_+$ and some elements $y,z \in E_+$ such that $x = y + z$. So, by the formula $\sup(M)$-$\sup(N) \leq \sup(M$-$N)$ for any set M,N, we have the following inequality

$$\begin{aligned}(S_\alpha \vee T_\alpha)(x) - (S \vee T)(x) &= \sup\{S_\alpha y + T_\alpha z : y + z = x \ \ and \ \ y,z \in E_+\} \\ &\quad -\sup\{Sy + Tz : y + z = x \ \ and \ \ y,z \in E_+\} \\ &\leq \sup\{(S_\alpha - S)y + (T_\alpha - T)z : y + z = x \ \ and \ \ y,z \in E_+\}.\end{aligned}$$

Now, consider any st-bounded zero neighborhood V in F. Then there is another zero neighborhood W such that $W + W \subseteq V$. One can see that W is also st-bounded. Hence, since $S_\alpha \to S$ and $T_\alpha \to T$ converge uniformly on A and indexed set is directed, we have some index α_0 such that $(S_\alpha - S)(A) \subseteq W$ and $(T_\alpha - T)(A) \subseteq W$ for all $\alpha \geq \alpha_0$. Thus, we have

$$(S_\alpha \wedge T_\alpha)(x) - (S \wedge T)(x) \leq (S_\alpha - S)(x) + (T_\alpha - T)(x) \in W + W \subseteq V$$

for all $\alpha \geq \alpha_0$. Therefore, $S_\alpha \wedge T_\alpha - S \wedge T$ is st-bs-bounded.

It is clear that the product of st-bs-bounded operators is continuous with the topology of uniform convergence on st-bounded sets because every subset of an st-bounded set is st-bounded. For the other cases, we give the next results.

Theorem 3.4. Let E be a topological vector space and F be a locally solid vector lattice. Then the product of st-bo-bounded operators is continuous in $B_{st-bo}(E,F)$ with the topology of uniform convergence on st-bounded sets.

Proof: Suppose the nets of st-bo-bounded operators (T_α) and (S_α) converge to zero uniformly on st-bounded sets. Let's take an st-bounded set B. Then, for fixed positive vector $u \in E_+$, there is an index α_1 such that $T_\alpha(B) \subseteq [-u,u]$ for all $\alpha \geq \alpha_1$. Since every order bounded set in E is topological bounded, we get that $[-u,u]$ is τ-bounded, and so st-bounded; see Theorem 2.1. Thus, $T_\alpha(B)$ is also st-bounded for each $\alpha \geq \alpha_1$. Then $S_\alpha(T_\alpha(B))$ is order bounded for each $\alpha \geq \alpha_1$. So, for a given positive u, we have another index $\alpha_2 \geq \alpha_1$ such that

$$S_\alpha(T_\alpha(B)) \subseteq [-u,u]$$

for all $\alpha \geq \alpha_2$. Therefore, we get the desired result.

Proposition 3.5. Let E be a topological vector space, F be an ordered topological vector space. If a net (T_α) of st-bs-bounded operators order converges uniformly on some st-bounded sets to an operator T then T is st-bs-bounded.

Proof: Suppose B is st-bounded set in E. Then, for each st-bounded zero neighborhood U in F, there exists an index α_0 such that $(T - T_\alpha)(B) \subseteq U$ for each $\alpha \geq \alpha_0$. So, we get

$$T(B) \subseteq T_{\alpha_0}(B) + U.$$

Since T_{α_0} is st-bs-bounded, we have $T_{\alpha_0}(B)$ is also st-bounded. Thus, $T_{\alpha_0}(B) + U$ is also st-bounded because sum of st-bounded sets is st-bounded. As a result, $T(B)$ is st-bounded, and so we get the desired result.

Proposition 3.6. Let E be a topological vector space, F be an ordered topological vector space. If a net (T_α) of st-bo-bounded operators order converges uniformly on some st-bounded sets to an operator T then T is st-bo-bounded.

Proof: Let B be st-bounded set in E. Then, for each positive vector u in F, there exists an index α_0 such that $(T - T_\alpha)(B) \subseteq [-u,u]$ for each $\alpha \geq \alpha_0$. So, we get

$$T(B) \subseteq T_{\alpha_0}(B) + [-u,u].$$

Since T_{α_0} is st-bo-bounded, we have $T_{\alpha_0}(B)$ is also order bounded set in F. Thus, $T_{\alpha_0}(B) + [-u, u]$ is also order bounded because the sum of order bounded sets is order bounded. As a result, $T(B)$ is order bounded set in F.

Theorem 3.7. Let E be a topological vector space and F be an ordered topological complete vector space. If every st-bounded set in E is absorbing then $B_{st-bs}(E, F)$ is complete with the topology of uniform convergence on st-bounded sets.

Proof: Let (T_α) be a Cauchy sequence in $B_{st-bs}(E, F)$ with respect to uniform convergence on st-bounded sets. Consider an arbitrary st-bounded set B in E. Hence, for an st-bounded zero neighborhood U in F, there exists α_0 such that $(T_\alpha - T_\beta)(B) \subseteq U$ for each $\alpha, \beta \geq \alpha_0$. Next, take arbitrary vector x in E. Then since B is absorbing, there is some positive scalar λ such that $\lambda x \in B$. So, we get $(T_\alpha(x) - T_\beta)(x)) \subset \frac{1}{\lambda}U$ for every $\alpha, \beta \geq \alpha_0$. So, $(T_\alpha(x))$ is a Cauchy net in F. Then, by choosing an operator T from E to F as $T(x)$ is the limit of $(T_\alpha(x))$ and by applying Proposition 3, T is st-bs-bounded.

Theorem 3.8. Let E be a topological vector space and F be an ordered topological complete vector space. If st-bounded sets in E is also absorbing then $B_{st-bo}(E, F)$ is complete with the topology of uniform convergence on st-bounded sets.

Proof: Suppose (T_α) is a Cauchy sequence in $B_{st-bo}(E, F)$ with respect to uniform convergence on st-bounded sets. Fix an st-bounded set B in E. Thus, for every positive vector $u \in F_+$, there is an index α_0 such that $(T_\alpha - T_\beta)(B) \subseteq [-u, u]$ for each $\alpha, \beta \geq \alpha_0$.
Now, let's take an element $x \in E$. Then we have some positive scalar λ such that $\lambda x \in B$ because B is absorbing. So, we get $(T_\alpha(x) - T_\beta)(x)) \subset [\frac{-u}{\lambda}, \frac{u}{\lambda}]$ for every $\alpha, \beta \geq \alpha_0$. Thus, we have that $(T_\alpha(x))$ is a Cauchy net in F. So, one can choose an operator T from E to F as $T(x)$ is the limit of $(T_\alpha(x))$. Then, by applying Proposition 3.6., we get $T \in B_{st-bo}(E, F)$.

By applying Theorem 2.17. [12] and using the above results, one can observe the following fact.

Corollary 3.9. Suppose E is a locally solid Riesz space and F be an ordered topological complete vector space. If st-bounded sets in E is also absorbing then $B_{st-bo}(E, F)$ and $B_{st-bs}(E, F)$ are complete with the topology of uniform convergence on st-bounded sets.

4. CONCLUSION

In this paper, we obtain some kinds of statistical bounded sets on topological vector space and bounded operators defined from topological vector spaces to ordered topological vector spaces. Also, we get some algebraic properties of operators with respect to the uniform convergence topology.

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