***ORIGINAL PAPER***

# NEW RESULTS ON A MIXED CONVECTION BOUNDARY LAYER FLOW OVER A PERMEABLE VERTICAL SURFACE EMBEDDED IN A POROUS MEDIUM

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***Abstract.*** *The objective of this paper is to prove the existence, non-existence and the sign of convex and convex-concave solutions of the third-order non-linear differential equation* $f''' + ff'' + \beta f'(f' - 1) = 0$, *satisfying the boundary conditions* $f(0) = a \in \mathbb{R}$, $f'(0) = b < 0$ *and* $f' \to \lambda$ *as* $t \to +\infty$ *where* $\lambda \in \{0,1\}$ *and* $\beta < 0$. *The problematic arises in the study of the Mixed Convection Boundary Layer flow over a permeable vertical surface embedded in a Porous Medium according to the mixed convection parameter b<0, the permeable parameter* $a \in \mathbb{R}$ *and the temperature parameter* $\beta < 0$**.**

***Keywords:*** *mixed convection; nonlinear differential equation; convex solution; convex-concave solution; shooting technique.*

## 1. INTRODUCTION

Owing to their numerous applications in geothermal energy extraction, oil reservoir modelling, magnetohydrodynamic, casting and welding in manufacturing processes (see [1-3]) or in boundary layer flows (see [4, 5]) etc, the problem of boundary layers related with heated and cooled surfaces embedded in fluid-saturated porous media have attracted considerable attention of researchers during the last few decades. In this paper, our interest focuses on the analysis of the boundary value problems $(P_{\beta;a,b,\lambda})$

$$(P_{\beta;a,b,\lambda}) \begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0, \\ f(0) = a \in \mathbb{R}, \\ f'(0) = b < 0, \\ \lim_{t\to+\infty} f'(t) = \lambda. \end{cases}$$

where $\lambda \in \{0,1\}$ which has been examined in [6-8] with $a = 0$. This problem comes from the study of the mixed convection boundary layer flow along a semi-infinite vertical permeable plate embedded in a saturated porous medium, with a prescribed power law of the distance from the leading edge for the temperature. The parameter $\beta$ is a temperature power-lawprofile and $b$ is the mixed convection parameter, namely $b = \frac{R_a}{P_e} - 1$, with $R_a$ the Rayleigh number and $P_e$ the Péclet number.

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For more details on the physical derivation and the numerical results, the interested reader can consult references [6, 9]. Mathematical results about the problem $(P_{\beta;a,b,\lambda})$ with $\lambda = 1$ can be found in [7, 8, 10-12]. The case where $a \geq 0$, $b \geq 0$, $\beta > 0$ and $\lambda \in \{0,1\}$ was treated by Aïboudi and al. in [10], and the results obtained generalize the ones of [12]. In [11], Brighi and Hoernel established some results about the existence and uniqueness of convex and concave solution of $(P_{\beta;a,b,1})$ where $-2 < \beta < 0$ and $b > 0$. These results can be recovered from [4], where the general equation $f''' + ff'' + g(f') = 0$ is studied. Recently, in [7], the authors prove some theoretical results about the problem $(P_{\beta;0,b,1})$ with

$$-2 < \beta < 0,\ b = 1 + \varepsilon \ \text{ and } \ \varepsilon < -1.$$

In particular, the authors prove that there exist $\varepsilon_* \in (-1{:}807; -1{:}806)$ and $\varepsilon^* \in (-1{:}193; -1{:}192)$, such that:

(i) $(P_{\beta;0,b,1})$ has no convex solution for any $-2 < \beta < 0$ and each $\varepsilon \leq \varepsilon_*$;

(ii) $(P_{\beta;0,b,1})$ has a convex solution for each $-2 < \beta < 0$ and each $\varepsilon \in [\varepsilon *; -1)$.

In [8] one can found interesting new result about the existence of convex solution of $(P_{\beta;0,b,1})$ where $0 < \beta < 1$ under some conditions. In [13] the results obtained by Aïboudi and al generalize the ones of [8]. In [7, 8], the method used by the authors to prove the existence of a convex solution for the case $a = 0$ seems difficult to generalize for $a \neq 0$. The problem $(P_{\beta;a,b,\lambda})$ with $\beta = 0$ is the well known Blasius problem. For a broad view, see [14, 15]. The main goa of this paper is to extend the study of existence and nonexistence of the solutions of $(P_{\beta;a,b,\lambda})$ with $\beta < 0$ and $\lambda \in \{0,1\}$.We will focus our attention on convex and convex-concave solutions of the equation

$$f''' + ff'' + \beta f'(f' - 1) = 0. \tag{1}$$

As usually, to get a convex or convex-concave solution of $(P_{\beta;a,b,\lambda})$, we will use the shooting technique which consists of finding the values of a parameter $c \geq 0$ for which the solution of (1) satisfying the initial conditions $f(0) = a$, $f'(0) = b$ and $f''(0) = c$, exists on $[0; +\infty)$; and is such that $f_c' \to \lambda \in \{0,1\}$ $as\ t \to +\infty$. We denote by $f_c$ the solution of the following initial value problem and by $[0; T_c)$ its right maximal interval of existence:

$$(P_{a,b,c}) \qquad \begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0, \\ f(0) = a, \\ f'(0) = b, \\ f''(0) = c. \end{cases}$$

## 2. ON BLASIUS EQUATION

In this section, we recall some basic properties of the supersolutions of the Blasius equation. Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be a function.

**Definition 1.** We say that $f$ is a supersolution of the Blasius equation

$$f''' + ff'' = 0 \text{ if } f \text{ is of class } C^3$$

and if

$$f''' + ff'' \geq 0 \text{ on } I.$$

**Proposition 1.** Let $t_0 \in \mathbb{R}$. There does not exist nonpositive convex supersolution of the Blasius equation on the interval $[t_0; +\infty)$.

*Proof***:** See [4], Proposition 2.5.

## 3. PRELIMINARY RESULTS

**Proposition 2.** Let f be a solution of the equation (1) on some maximal interval $I = (T-;\ T+)$.

1. If F is any anti-derivative of f on I, then . $(f''e^F)' = -\beta f'(f' - 1)e^F$.
2. Assume that $T_+ = +\infty$ and that $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$: If moreover f is of constant sign at infinity, $f''(t) \to 0$ as $t \to +\infty$.
3. If $T_+ = +\infty$ and if $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$, then $\lambda = 0$ or $\lambda = 1$;
4. If $T_+ < +\infty$, then $f''$ and $f'$ are unbounded near $T_+$ ;
5. If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t) = \mu$, where $\mu = 0$ or 1 then for all $t_0 \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$;

*Proof:* The first item follows immediately from equation (1). For the proof of items 2-5, see [4], Proposition 3:1 with $g(x) = \beta x(x - 1)$.

## 4. THE BOUNDARY VALUE PROBLEM IN THE CASE WHERE $\beta < 0$

In the following we take $a;\ b \in \mathbb{R}$ and $\lambda \in \{0,1\}$ with $b < 0$ and $\beta < 0$. We are interested here in convex and convex-concave solutions of the boundary value problem $\left(P_{\beta;a,b,\lambda}\right)$. As mentioned above in the introduction, we will use the shooting method to find these solutions. To reach our goal, let us use Proposition 3.1*;* items 1 to define the following sets:

$$C_1 = \{c \geq 0;\ f_c' \leq 0 \ and\ f_c'' \geq 0 \ on\ \ [0;\ T_c)\}$$

$$C_2 = \{c \geq 0;\ \exists t_c \in [0;\ T_c), \exists \varepsilon_c > 0 \ s.t\ f_c' < 0 \ \ on\ (0; t_c),\ \ f_c' > 0 \ on\ (t_c; t_c + \varepsilon_c) \ and\ f_c'' > 0 \ on\ \ (0; t_c + \varepsilon_c)\ \}$$

$$C_3 = \{c \geq 0;\ \exists s_c \ \in [0;\ T_c), \exists \varepsilon_c > 0 \ s.t\ f_c'' > 0 \ \ on\ [0; s_c), f_c'' < 0 \ on\ (s_c; s_c + \varepsilon_c) \ and\ \ f_c' \ < 0 \ on\ \ [0; s_c + \varepsilon_c)\ \}.$$

***Remark 1.*** It is easy to prove that $C_2$ and $C_3$ are disjoint nonempty open subsets of $[0, +\infty)$ and that there exist $c_0 > c_* > 0$ such that $C_2 = (c_0, +\infty)$, $C_3 = [0, c_*)$, and $C_1 \cup C_2 \cup C_3 = [0; +\infty)$ (see Appendix A of [4] with $g(x) = \beta x(x - 1)$ and $\beta > 0$).

**Lemma 1.** $f_c$ is a convex solution of the boundary value problem $\left(P_{\beta;a,b,0}\right)$ if and only if $c \in C_1$.

*Proof:* See Appendix A of [4] with $g(x) = \beta x(x - 1)$.

**Lemma 2.** The set $\mathbf{C_3}$ is empty.

*Proof:* See Lemma A.5 of [4] with $g(x) = \beta x(x-1)$ and $\beta < 0$.
From the previous Lemma, we have $C_1 \cup C_2 = [0;+\infty)$ and $C_1 \cap C_2 = \emptyset$.

*4.1 THE $a \leq 0$ CASE*

**Lemma 3.** The set $C_1$ is empty.
*Proof:* For contradiction, assume that $C_1 \neq \emptyset$ and let $c \in C_1$. From Lemma 1, $f_c$ is a convex solution of the boundary value problem$\left(P_{\beta;a,b,0}\right)$. Hence $f_c$ and $f_c'$ are negative on $[0;+\infty)$. This implies that

$$f''' + ff'' = -\beta f'(f'-1) > 0 \text{ on } [0;+\infty).$$

Hence, $f_c$ is a nonpositive convex supersolution of the Blasius equation on $(0;+\infty)$. This contradicts Proposition 1.

**Remark 2.** From the previous lemma and Lemma 2, $C_2$ =$[0;+\infty)$.

**Remark 3.** From Proposition 3*:*1*;* items 1, 3 and 5, if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $\left(P_{a,b,c}\right)$:

1. $f_c$ is convex on its right maximal interval of existence $[0;\, T_c)$ and $f_c'(t) \to +\infty$ *as* $t \to T_c$ (with $T_c < +\infty$);
2. There exists a point $t_0 \in [0;\, T_c\,)$ such that $f_c''(t_0)$ = *0 and* $0 < f_c'(t_0) < 1$;
3. $f_c$ is a convex solution of $\left(P_{\beta;a,b,1}\right)$.

**Lemma 4.** Let $\beta < 0$, $a \leq 0$ and $b \leq -1$. If $c \geq 0$ and if there exists $t_0 \in [0;\, T_c\,)$ such that $f_c''(t_0) = 0$ and $0 < f_c'(t_0) < 1$; then $f_c(t_0) > 0$.

*Proof:* Let $c \geq 0$ and assume that there exists $t_0 \in [0;\, T_c\,)$ such that

$$f_c''(t_0) = 0 \text{ and } 0 < f_c'(t_0) = \theta < 1.$$

Suppose that $f_c(t_0) \leq 0$. Let us consider the function

$$L_c = 3{f_c''}^2 + 2\,\beta\,{f_c'}^3 - 3\,\beta\,{f_c'}^2.$$

Then, from (1), we have

$$L_c' = -6 f_c {f_c''}^2 > 0 \text{ on } [0;\, t_0)$$

and hence:

$$L_c(0) = 3c^2 + 2\,\beta\, b^3 - 3\,\beta\, b^2 < L_c(t_0) = 2\,\beta\,\theta^3 - 3\,\beta\,\theta^2.$$

It follows that $\theta^2 - b^2 > 0$ which implies that $\theta > 1$. This is a contradiction.

**Lemma 5.** Let $\beta < 0$, $a \leq 0$ and $b \leq \beta$. If $c \geq 0$ and if there exists $t_0 \in [0;\, T_c\,)$ such that $f_c''(t_0) = 0$ and $0 < f_c'(t_0) < 1$; then $f_c(t_0) > 0$.

*Proof:* Let $c \geq 0$ and assume that there exists $t_0 \in [0;\ T_c)$ such that

$$f_c''(t_0) = 0 \text{ and } 0 < f_c'(t_0) = \theta < 1.$$

Suppose that

$$f_c(t_0) \leq 0.$$

Let us consider the function

$$H_c = f_c'' + f_c(f_c' - \beta).$$

Then, from (1), we have

$$H_c' = (1 - \beta) f_c'^2 \geq 0 \text{ on } [0;\ t_0)$$

and hence:

$$0 \leq H_c(0) = c + a(b - \beta) < H_c(t_0) = f_c(t_0)\,(f_c'(t_0) - \beta),$$

thus, $f_c(t_0) > 0$. For the rest of this section, if it is defined, we will set $a_* = -\sqrt{\frac{1-b^2}{\beta-2b}}$.

**Lemma 6.** Let $b \leq -1$ and $c \geq 0$. Let $t_* > 0$ be the first point such that $f_c(t_*) = 0$. If, either $2b \leq \beta < 0$, or $\beta < 2b$ and $a \geq a_*$, then $f_c'(t_*) > 1$.

*Proof:* From Remark 2, Remark 3 and Lemma 4, we know that the point $t_*$ exists.

Let

$$K_c = 2f_c f_c'' - f_c'^2 + f_c^2(2f_c' - \beta).$$

From (1), we obtain

$$K_c' = 2(2-\beta) f_c f_c' < 0 \text{ on } (0,\ t_*).$$

Therefore, $K_c$ is decreasing on $(0,\ t_*)$ and hence $K_c(0) > K_c(t_*)$. It follows that

$$2b \leq \beta < 0$$

then

$$f_c'^2(t_*) > -2ac + b^2 + a^2(\beta - 2b) \geq b^2,$$

which implies that

$$f_c'(t_*) > 1.$$

The same result is obtained where

$$b \leq -1,\ \beta < 2b \text{ and } a \geq a_*.$$

**Theorem 1.** Let $\beta < 0$ and $a; b \in \mathbb{R}$ with $b \leq 0$ and $a \leq 0$.

1) The boundary value problem $(P_{\beta;a,b,0})$ has no convex solution.

2) If $b \leq -1$ and if either $2b \leq \beta < 0$, or $\beta < 2b$ and $a \geq a_*$, then the boundary value problem $(P_{\beta;a,b,1})$ has no convex and no convex-concave solution.

3) If $b \leq -1$ and if, either $2b \leq \beta < 0$, or $\beta < 2b$ and $a \geq a_*$, then, for any $c \geq 0$, the solution $f_c$ of the initial value problem $(P_{a,b,c})$ is convex on its right maximal interval of existence $[0; T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).

*Proof:* The first result follows from Lemma 1 and Lemma 3. The second result follows from Proposition 2, item1, Lemma 4, Lemma 5 and Lemma 6. The third result follows from Remark 2, Remark 3, and Lemma 5.

### *4.2. THE a > 0 CASE*

Let $a; b \in \mathbb{R}$ with $\beta < 2b < 0$ and $a > 0$. We consider the solution $f_c$ of the initial value problem$(P_{a,b,c})$ on the right maximal interval of existence $[0; T_c)$. Let us set $a^* = -\frac{b}{\sqrt{2b-\beta}}$.

**Lemma 7.** Let $a \geq a^*$, $c \geq 0$ and $\beta < 2b < 0$. If $f_c$ is a solution of the initial value problem $(P_{a,b,c})$, then $f_c$ is positive on the right maximal interval of existence $[0; T_c)$.

*Proof:* Assume that there exists $t_* \in (0, T_c)$. Such that $f_c > 0$ on $[0; t_*)$ and $f_c(t_*) = 0$.
Let

$$K_c = 2f_c f_c'' - {f_c'}^2 + {f_c}^2(2f_c' - \beta).$$

From (1), we obtain

$$K_c' = 2(2-\beta) f_c f_c' \text{ on } (0, t_*).$$

Therefore, $K_c$ is increasing on $(0, t_*)$ and hence $K_c(0) < K_c(t_*)$. It follows that

$$0 > -{f_c'}^2(t_*) > a^2(2b-\beta) - b^2.$$

This is a contradiction.

**Remark 4.** From the previous Lemma and Lemma 5.16 of [4], if there exists $t_0 \in [0; T_c)$ such that $f_c''(t_0) = 0$, then $f_c(t_0) > 0$ and $f_c$ is a convex-concave solution of $(P_{\beta;a,b,0})$.

**Lemma 8.** The set $C_2$ is not empty.

*Proof:* Assume $C_2$ is empty, then from Lemma 2, $C_1 = [0; +\infty)$ and $f_c$ is a convex solution of $(P_{\beta;a,b,0})$ for all $c \in [0; +\infty)$.
Let

$$A_c = f_c'' + f_c(f_c' - 1).$$

From (1), we obtain

$$A_c' = (1-\beta) f_c'(f_c' - 1).$$

Since $f_c$ is a convex solution of $(P_{\beta;a,b,0})$, then $f_c' < 0$. Therefore, $A_c$ is increasing on $[0; +\infty)$ and hence $A_c(0) < A_c(t)$ as $t \to +\infty$. It follows that $c < -a.(b-1)$. This is a contradiction.

**Remark 5.** From the Remark 4, Proposition 3:1; items 1, 3 and 5, if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $(P_{a,b,c})$:

1. $f_c$ is convex on its right maximal interval of existence [0; Tc) and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$);
2. $f_c$ is convex-concave solution of $(P_{\beta;a,b,0})$;
3. $f_c$ is a convex solution of $(P_{\beta;a,b,1})$.

**Lemma 9.** If $\beta < 2b < 0$ and $a \geq a^*$then there exists $c_0 \in C_2$ such that if $c \geq c_0$ then $f_c$ is a convex solution of (P(a,b,c) ) on its right maximal interval of existence [0; Tc) and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).

*Proof:* From Remark 5 and Lemma 8, we know that, if $c_0 \in C_2$, then $f_c$ is a convex solution of $(P_{\beta;a,b,1})$, a convex-concave solution of $(P_{\beta;a,b,0})$ or $f_c$ is convex on its right maximal interval of existence [0; Tc) and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$). Let $c \in C_2$, be such that $f_c$ is a convex solution of $(P_{\beta;a,b,1})$ or a convex-concave solution of $(P_{\beta;a,b,0})$. Therefore, we have $b < f_c' < 1$ on $[0;+\infty)$ and, from Lemma 6, we have $f_c > 0$.

It follows that

$$( f_c'' + f_c( f_c' - 1))' = (1 - \beta) f_c'( f_c' - 1) \geq -\frac{1}{4}(1 - \beta) \text{ on } [0;+\infty).$$

Integrating between 0 and $t \geq 0$, and using the fact that $f_c > 0$, we obtain

$$f_c'' \geq -\frac{1}{4}(1 - \beta)t + a (b - 1) + c - f_c(t) ( f_c'(t) - 1) \geq -\frac{1}{4}(1 - \beta)t + a (b - 1) + c.$$

Integrating once again we get, for all $t \geq 0$ ,

$$1 > - f_c'(t) > -\frac{1}{8}(1 - \beta)t^2 + (a (b - 1) + c)t + b.$$

Let us set

$$P_c(t) = -\frac{1}{8}(1 - \beta)t^2 + (a (b - 1) + c)t + b - 1.$$

We have $P_c(t) < 0$ for all $t \geq 0$. It means that $P_c$ has no positive roots. Thus c cannot be too large, because, on the contrary, its discriminant

$$\Delta = ((a (b - 1) + c)t + b)^2 + \frac{1}{2}(1 - \beta)(b - 1)$$

and

$$a (b - 1) + c \text{ would be positive,}$$

and hence the polynomial $P_c$ would have two positive roots, a contradiction.

Therefore, there exists $c_0 > 0$ such that for any $c > c_0$, $f_c$ is convex on its right maximal interval of existence [0; Tc) and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$). This completes the proof.

**Theorem 2.** Let $\beta < 2b < 0$, $a \geq a^* > 0$ and $f_c$ be a solution of the initial value problem $(P_{a,b,c})$.

1) For all $c \geq 0$, $f_c$ is positive.

2) There exists $c_0 > 0$ such that for any $c > c_0$, $f_c$ is convex on its right maximal interval of existence [0; Tc) and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).

*Proof:* The first result follows from Lemma 7. The second result follows from the first result, Remark 4, Remark 5, Lemma 8 and Lemma 9.

## 5. CONCLUSIONS

In this work we have presented a set of new and important results for $\beta < 0$ and $b < 0$, we summarize as follows:

1) If $a \leq 0$

(a) The boundary value problem $(P_{\beta;a,b,0})$ has no convex solution on $[0;+\infty)$.

(b) If $b \leq -1$ and if either $2b \leq \beta < 0$ or $\beta < 2b$ and $a \geq a_*$ with $a_* = -\sqrt{\frac{1-b^2}{\beta-2b}}$, then the boundary value problem $(P_{\beta;a,b,1})$ has no convex and no convex-concave solution.

(c) If $b \leq -1$ and either $2b \leq \beta < 0$ or $\beta < 2b$ and $a \geq a_*$, and if $f_c$ is a solution of the initial problem $(P_{a,b,c})$ with $c \geq 0$ then $f_c$ is a convex solution of the boundary value problem $(P_{\beta;a,b,+\infty})$ .

2) For $a > 0$

(a) If $a \geq a^* > 0$ where $a^* = -\frac{b}{\sqrt{2b-\beta}}$ , all solution of the initial value problem $(P_{a,b,c})$ is a positive.

(b) The boundary value problem $(P_{\beta;a,b,+\infty})$ has infinitely many positive convex solutions.

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