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NUMERICAL SOLUTION FOR TWO-DIMENSIONAL NONLINEAR KLEIN-GORDON EQUATION THROUGH MESHLESS SINGULAR BOUNDARY METHOD

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Abstract. *In this study, the singular boundary method (SBM) is employed for the simulation of nonlinear Klein-Gordon equation with initial and Dirichlet-type boundary conditions. The θ-weighted and Houbolt finite difference method is used to discretize the time derivatives. Then the original equations are split into a system of partial differential equations. A splitting scheme is applied to split the solution of the inhomogeneous governing equation into homogeneous solution and particular solution. To solve this system, the method of particular solution in combination with the singular boundary method is used for particular solution and homogeneous solution, respectively. Finally, several numerical examples are provided and compared with the exact analytical solutions to show the accuracy and efficiency of method in comparison with other existing methods.*

Keywords: *nonlinear Klein-Gordon equation; singular boundary method (SBM); method of particular solution; fundamental solution.*

1. INTRODUCTION

Many phenomena in applied sciences and engineering are made models by nonlinear partial differential equations (PDEs). In general, there are no practical methods for solving nonlinear PDEs and finding analytical solutions. Thus, numerical methods are important for solving nonlinear PDEs. Various numerical methods have been used to finding the approximation solution of engineering problems. Compared with the finite element method (FEM), finite volume method (FVM) and boundary element method (BEM) [1], meshfree methods are applied to set up system of algebraic equations for entire problem domain with no need to meshing of the domain discretization in order to use a set of points scattered inside the domain of the problem such as sets of points on the boundaries of the domain to show the domain of the problem and its boundaries. Moreover, some meshless methods are on the basis of collocation approaches(strong forms) like the meshless collocation method based on radial basis func-tions (RBFs) [2-6] and some other kinds of meshfree methods regarding weak forms and hybrid of collocation approach and weak forms, like element free Galerkin (EFG) [7] and meshless local Petrov-Galerkin (MLPG) [8]. Another approach has been utilized for approximated solution of differential equations is spectral collocation method like the spectral meshless radial point interpolation (SMRPI) method [9] the point interpolation method by means of the RBFs is employed to form shape functions. Against to the domain-type meshless approaches as methods based on collocation approaches, there are some boundary-type

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methods like method of fundamental solutions (MFS), boundary collocation method (BCM), regularized meshless method (RMM) and boundary knot method (BKM) [10-11]. This family of boundary-type meshfree methods are very attractive which no need meshing domain and boundary.

The Klein-Gordon equation is a relativistic version of the Schrdinger equation describing free particles, which was proposed by Oskar Klein and Walter Gordon. It has many applications in Physics and Engineering such as quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics and nonlinear optics. The Klein-Gordon equation is an important class of partial differential equations and arises in relativistic quantum mechanics and field theory, which is of great importance for the high-energy physicists, and is used to model many different phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. Finding accurate and efficient methods for solving such Klein-Gordon equation has been an active research undertaking.

In this work, we employ the singular boundary method (SBM) for solving nonlinear Klein-Gordon equation as:

$$u_{tt} + \beta u_t + G(u) = \alpha \Delta u + f(\mathrm{x}, t),\ \mathrm{x} \in \Omega \subset \mathrm{R}^2,\ t > 0, \tag{1}$$

with initial conditions

$$u(\mathrm{x}, 0) = g_1(\mathrm{x}),\ \mathrm{x} \in \Omega, \tag{2}$$

$$\frac{\partial u(\mathrm{x},t)}{\partial t}|t = 0 = g_2(x),\ \mathrm{x} \in \Omega \tag{3}$$

and Dirichlet-type boundary condition

$$u(\mathrm{x}, t) = h(\mathrm{x}, t),\ \mathrm{x} \in \partial\Omega, \tag{4}$$

where Ω is a domain of R^2, x is the space variable, t is the time variable, u(x, t) is the unknown function, G(u) is a nonlinear function of u with different types of nonlinearities, f(x,t) is a given function, g_1 (x) and g_2 (x), are prescribed initial functions, α and β are known constants and h(x, t) is a given function for boundary $\partial\Omega$.

In special case, this equation is called Sine-Gordon equation that for some forms of G(u) such as: sin(u), sin(u) + sin(2u) , sinh(u) + sinh(2u) and exp(u) that characterizes the sine-Gordon, the double sine-Gordon, the double sinh-Gordon and Liouvile equations, respectively. The equation with nonlinear term as $G(u) = \gamma_1 u + \gamma_2 u^m$ is called as the Klein-Gordon equation with quadratic nonlinearity if m = 2 and with cubic nonlinearity if m = 3.

W. Chen and his coworkers proposed the singular boundary method (SBM) as a meshfree boundary collocation method [12-14]. The SBM straightly applies the fundamental solutions as basis functions and it removes the artificial boundary that is used in the MFS. The major idea in the SBM is to present the notion of origin intensity factors (OIFs) to omit the singularities of fundamental solutions on the adaptation of the source points and collocation points upon physical boundary of domain. The SBM is easy in mathematically speaking, simple for programming, accurate and free of integration, more enforceable for problems on complicated shapes with high dimension domains, less time-consuming than the boundary element method (BEM) and at the same time keeps away the fictitious boundary in the MFS and becomes numerically more stable than the MFS because condition number of its interpolation matrix is better [15-17]. As well as the MFS and BEM, the SBM acts when the fundamental solution of the determined problem is attainable [18]. The important notion of the SBM is to utilize the OIF to substitute the singular integration in the BEM to achieve accurate numerical consequences, while keeping high numerical stability also less computation load. Specially, the SBM can achieve to high accurate numerical results employing very few boundary points and small CPU time. So far, the three ways have been extended to calculate the OIFs. The first approach is named the inverse interpolation technique (IIT), in numerical form, calculates OIFs. The second technique is to conclude the analytical formula for evaluating unknown OIFs. And the third one is empirical approach to determining OIFs. In this paper, subtracting and adding-back (SAB) technique is a popular approach to calculate the OIFs. In recent years, some famous problems are solved by SBM [19-23].

The structure of this article is organized as follows: In Section 2, we express briefly mathematical preliminaries and numerical implementation of the method. In Section 3, the time discretization and implementation steps of the method are presented. Some numerical examples are examined for show the accuracy of the method, and results are reported in Section.4. Finally, concluding remarks are given in the last section.

2. A BRIEF REVIEW OF METHODOLOGY AND MATHEMATICAL PRELIMINARIES

In this section we present a numerical process for calculating the particular solution and homogeneous solution. For approximation particular solution and homogeneous solution we used MPS and SBM, respectively.

2.1. METHOD OF PARTICULAR SOLUTIONS (MPS)

An important family of problem dependent radial basis functions are particular solutions [24]. By application of splitting scheme, the solution of the inhomogeneous governing equation split into homogeneous solution and particular solution. The key subject is to make the particular solutions to satisfy the governing equation.
Consider the boundary value problem as follows:

$$\mathrm{L}u(x) = \psi(x),\ x \in \Omega, \tag{5}$$

$$\mathrm{B}u(x) = \omega(x),\ \mathrm{x} \in \Gamma, \tag{6}$$

where L is a differential operator and B is a boundary differential operator. Also, ψ and ω are given functions, Ω is the inner region and $\Gamma = \partial\Omega$ is the boundary of the computational domain.

Suppose $\{\mathrm{x}_i\}_{i=1}^{N}$ be the set of interpolation points containing N_i interior points in Ω and N_b boundary points on Γ, so $N = N_i + N_b$. Let up be a particular solution of Eq.(5), then it satisfies.

$$\mathrm{L}u_p\,(\mathrm{x}) = \psi(\mathrm{x}),\ \mathrm{x} \in \Omega, \tag{7}$$

but does not necessarily need to satisfy the boundary condition. If $u_p$in Eq.(7) can be achieved, then the original equation in Eq.(5) and Eq.(6) can be changed into the following homogeneous equation via the variable substitution u_h= u – u_p, namely

$$\mathrm{L}u_h(\mathrm{x}) = 0, \ \mathrm{x} \in \Omega \tag{8}$$

$$\mathrm{B}u_h(\mathrm{x}) = \omega(\mathrm{x}) - \mathrm{B}u_p(\mathrm{x}), \ \mathrm{x} \in \Gamma, \tag{9}$$

The homogeneous equation Eq.(8) with condition (9) can be solved by employing boundary methods. The mentioned numerical approach for solving PDEs is completely standard equipped when the particular solution and fundamental solution are both attainable. Finally, the solution of Eq.(5) and Eq.(6) can be achieved by summation of particular solution and homogeneous solution as follows:

$$u = u_p + u_h.$$

By MPS, for approximating the variable ψ by a linear superposition of the radial basis functions(RBFs), we assume the solution to Eq.(5) and Eq.(6) can be approximated by a linear superposition of the corresponding particular solutions of the given radial basis function like ϕ, such as

$$\psi(\mathrm{x}) = \sum_{j\,=\,1}^{N} \alpha_j \, \phi\left(\left\|\mathrm{x} - \mathrm{x}_j\right\|\right). \tag{10}$$

where $||.||$ is the Euclidean norm and α_j are unknown coefficients, therefore, an approximated particular solution u_p to Eq.(7) is given by

$$u_p(\mathrm{x}) = \sum_{j\,=\,1}^{N} \alpha_j \, \Phi\left(\left\|\mathrm{x} - \mathrm{x}_j\right\|\right). \tag{11}$$

where

$$\mathfrak{L}\Phi = \phi. \tag{12}$$

2.2. PARTICULAR SOLUTION FOR MODIFIED HELMHOLTZ EQUATION

In this work, we employ Polyharmonic splines of higher order or generalized thin plate spline(GTPS) as radial basis functions, described as:

$$\phi(r) = r^m \ln(r). \qquad m = 2.4.6.\dots in\ R^2. \tag{13}$$

For modified Helmholtz operator as: $\mathfrak{L} = \Delta - \mu^2$ and considering Eq.(12) and Eq.(13) we obtain:

$$(\Delta - \mu^2)\Phi(r) = \phi(r). \tag{14}$$

where Δ denotes the Laplace operator. Finally, we get

$$\Phi(r) = -\frac{1}{\mu^2}\sum_{i=1}^{\frac{m}{2}}\left(\frac{\Delta}{\mu^2}\right)^i . r^m \ln(r) - \frac{(m)!!^2}{\mu^{m+2}} K_0(\mu r). \tag{15}$$

where r is the Euclidean norm between the point x and the origin. Function $K_0(.)$ is the Bessel function of the second kind of order zero.

For TPS $\phi(r) = r^2 \ln(r)$, means $m = 2$ in Eq.(15) the corresponding particular solution is:

$$\Phi(r) = \begin{cases} -\frac{r^2\ln(r)}{\mu^2} - \frac{4}{\mu^4}\left(1 + \ln(r) + K_0(\mu r)\right). & r \neq 0 \\ \frac{4}{\mu^4}\left(-1 + \gamma + \ln\left(\frac{\mu}{2}\right)\right). & r = 0 \end{cases} \tag{16}$$

and for polyharmonic splines of order 2, $\phi(r) = r^4 \ln(r)$. in R^2, the corresponding particular solution is:

$$\Phi(r) = \begin{cases} -\frac{r^4\ln(r)}{\mu^2} - \frac{8r^2}{\mu^4}(2\ln(r) + 1) - \frac{1}{\mu^6}\left(96 + 64\ln(r) + 64K_0(\mu r)\right). & r \neq \\ \frac{1}{\mu^6}\left(-96 + 64\gamma + 64\ln(\frac{\mu}{2})\right). & r = \end{cases} \tag{17}$$

In (16) and (17) constant γ, is the Euler constant equal to:

$$\gamma = 0.57721566490153286 \ldots$$

Remark 2.1. For calculating Laplace of u_p we can conclude:

$$\Delta u_p(\mathrm{x}) = \sum_{j\,=\,1}^{N} \alpha_j \, \Delta\Phi(r_j).$$

and

$$\Delta\Phi(r_j) = \mu^2\Phi(r_j) + \phi(r_j),$$

finally, by merging two above equations we get:

$$\Delta u_p(\mathrm{x}) = \sum_{j\,=\,1}^{N} \alpha_j \left(\mu^2\Phi(r_j) + \phi(r_j)\right).$$

2.3. SINGULAR BOUNDARY METHOD (SBM)

The SBM belongs to the family of boundary-type meshless method based on the singular fundamental solution that uses as the basis function of its approximation expansion. Compared with the MFS, the source points of the SBM are located upon the physical boundary that are coincided with collocation points while in MFS the source points are located over a fictitious boundary. The major idea in the Singular boundary method (SBM) is to present the notion of origin intensity factors(OIFs) to omit the singularities of fundamental solutions on the adaptation of the collocation and source points on physical boundary of domain.

Consider the homogenous PDE with the following conditions:

$$\begin{cases} \mathfrak{L}u = 0 & \mathrm{x} \in \Omega \subset R^n \\ u = g_0(\mathrm{x}) & \mathrm{x} \in \Gamma_D \\ q(\mathrm{x}) = \dfrac{\partial u(\mathrm{x})}{\partial \mathrm{n}} = g_1(\mathrm{x}). & \mathrm{x} \in \Gamma_N \end{cases} \tag{18}$$

that $\mathfrak{L}$ is a partial differential operator, **n** is the unit outward normal vector, Ω denotes the computational domain that it is a bounded and connected known domain, Γ_D and Γ_N illustrate the Dirichlet boundary(essential) and the Neumann boundary(natural) conditions, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \varnothing$, that $\partial\Omega$ represents the whole physical boundary. Also, functions g_0 and g_1 are given known functions.

If $G(\mathrm{x})$ be the fundamental solution of the operator in Eq.(18), for field points x_i and source points s_j, approximation of u and q are:

$$u(\mathrm{x}_i) = \sum_{j=1}^{N} \alpha_j\, G(\mathrm{x}_i\,.\,\mathrm{s}_j). \qquad \mathrm{x} \in \Omega - \Gamma_D \tag{19}$$

$$u(\mathrm{x}_i) = \sum_{j=1, i\neq j}^{N} \alpha_j\, G(\mathrm{x}_i.\,\mathrm{s}_j) + \alpha_i u_{ii}. \qquad \mathrm{x} \in \Gamma_D \tag{20}$$

$$q(\mathrm{x}_i) = \sum_{j=1, i\neq j}^{N} \alpha_j \frac{\partial G(\mathrm{x}_i.\,\mathrm{s}_j)}{\partial \mathrm{n}} + \alpha_i q_{ii}. \qquad \mathrm{x} \in \Gamma_N \tag{21}$$

where N is the number of source points and α_j is the j-th unknown coefficient. Singularities of the fundamental solution G will occur when $\mathrm{x}_i = \mathrm{s}_j$. To eliminate this problem, the SBM recommends the notion of origin intensity factors (OIFs). The method places all computing points on the same physical boundary. So the source points $\{\mathrm{s}_j\}$ and the collocation points $\{\mathrm{x}_i\}$ are the same set of boundary nodes. When $\mathrm{x}_i = \mathrm{s}_j$, we use origin intensity factors (OIFs) replacing the singular terms in formulation. Where u_{ii} and q_{ii} are defined as the OIFs corresponding to the fundamental solutions and the unit outward normal of fundamental solutions, namely, the diagonal elements of the SBM interpolation matrix. Therefore, to solve all kinds of physical and mechanical problems, the main issue is to specify the OIFs. The origin intensity factor is numerically assigned by a technique, where a sample solution u_s

satisfying the governing equation are imperative, and some sample points y_k are located inside of the physical domain.

By using subtracting and adding-back(SAB) technique, SBM interpolation formula for boundary condition can be regularized accurately. The origin intensity factor is numerically determined , where a sample solution u_s satisfying the governing equation are imperative, and some sample points y_k are located inside the physical domain.

Replacing the sample points y_k with the boundary collocation points x_i, the SBM interpolation matrix of the diffusion problem can be written as

$$u_s(\mathrm{x}_i) = \sum_{j=1,j\neq i}^{N} \beta_j\, G(\mathrm{x}_i.\mathrm{s}_j) + \beta_j u_{ii}. \tag{22}$$

and

$$\frac{\partial u_s(\mathrm{x}_i)}{\partial \mathrm{n}} = \sum_{j=1,j\neq i}^{N} \beta_j \frac{\partial G(\mathrm{x}_i.\mathrm{s}_j)}{\partial \mathrm{n}} + \beta_j q_{ii}. \tag{23}$$

It is noted that only the origin intensity factors u_{ii} and q_{ii} are unknown in the above equation. Thus, the origin intensity factors can be calculated via

$$u_{ii} = \frac{1}{\beta_j}\left[u_s(\mathrm{x}_i) - \sum_{j=1,j\neq i}^{N} \beta_j\, G(\mathrm{x}_i.\mathrm{s}_j)\right]. \tag{24}$$

and

$$q_{ii} = -\frac{1}{L_i}\sum_{j=1,j\neq i}^{N} L_j \frac{\partial G_0(\mathrm{x}_i.\mathrm{s}_j)}{\partial n_s}. \tag{25}$$

where L_i is the half length of the curve between source points s_{i-1} and s_{i+1}. Also G_0 is the fundamental solution of the Laplace equation in 2D.

It is noted that the sample points y_k do not coincide with the source points s_j, and the sample points number should not be fewer than the physical boundary source node number. Finally, the approximated solution is:

$$u(\mathrm{x}) = \sum_{j=1\,.i\neq j}^{N} \alpha_j\, G(\mathrm{x}.\mathrm{s}_j) + \alpha_i u_{ii}. \tag{26}$$

that, Eq.(26) is the solution of equation Eq.(18).

It is emphasized that the source intensity factors only depends on the distribution of the source points, the fundamental solution of the governing equation and the boundary conditions. Theoretically speaking, the source intensity factors remain unchanged with different sample solutions. Therefore, by employing mentioned technique, we circumvent the singularity of the fundamental solution upon the coincidence of the source and collocation points.

Remark 2.2. For the Laplace equation on 2D domain, as

$$\Delta u(\mathrm{x}) = 0, \qquad \mathrm{x} \in \Omega$$

fundamental solution can be written as:

$$G(\mathrm{x}.\mathrm{s}_j) = -\frac{1}{2\pi}\ln(r(\mathrm{x}.\mathrm{s}_j)) = -\frac{1}{2\pi}\ln(\|\mathrm{x}-\mathrm{s}_j\|_2). \qquad \mathrm{x} \in R^2$$

Remark 2.3. For the modified Helmholtz equation on a two-dimensional domain, as

$$(\Delta - \mu^2)u(\mathrm{x}) = 0. \qquad \mathrm{x} \in \Omega$$

fundamental solution can be written as:

$$G(\mathrm{x}\,.\,\mathrm{s}_j) = -\frac{1}{2\pi}K_0(\mu r(\mathrm{x}\,,\mathrm{s}_j)) = -\frac{1}{2\pi}K_0(\mu\|\mathrm{x}-\mathrm{s}_j\|_2). \qquad \mathrm{x} \in R^2$$

Remark 2.4. Relation between OIFs of Laplace and modified Helmholtz operators is shown that the origin intensity factors OIFs) of the two-dimensional modified Helmholtz equation are relevant with the OIFs of the Laplace equation as following relations:

$$u^{ii} = u_L^{ii} - \frac{1}{2\pi}\ln\left(\frac{\mu}{2}\right) - \frac{\gamma}{2\pi}.$$

$$q^{ii} = q_L^{ii}.$$

where u_L^{ii} are OIFs in Dirichlet boundary conditions and q_L^{ii} are OIFs in Neumann boundary conditions of the Laplace equation. Also, γ is the Euler constant.

3. FINITE DIFFERENCES FOR TIME DISCRETIZATION

In this section, by introducing a uniformly partitioned time mesh, the procedure of time discretization based on the Houbolt finite-difference relation will be used for approximation of the first-order and second-order derivative on time variable of the main equation at two successive time levels k and $k+1$.

Let $\tau = t^{k+1} - t^k$ be the constant length of the time steps and $t^k = k\tau$. For any $t^k \le t \le t^{k+1}$:

$$\frac{\partial u^{k+1}}{\partial t} \cong \frac{11u^{k+1} - 18u^k + 9u^{k-1} - 2u^{k-2}}{6\tau}. \tag{27}$$

$$\frac{\partial^2 u^{k+1}}{\partial t^2} \cong \frac{2u^{k+1} - 5u^k + 4u^{k-1} - u^{k-2}}{\tau^2}. \tag{28}$$

that $u^k = u(\mathrm{x}\,,k\tau)$.

Also, we employ θ-method$(0 \le \theta \le 1)$ for the following approximation as

$$\Delta u(\mathrm{x}.t) \cong \theta\Delta u^{k+1}(\mathrm{x}) + (1-\theta)\Delta u^k(\mathrm{x}).$$

In special case, if $\theta = \frac{1}{2}$ the Crank-Nicolson technique is:

$$\Delta u(\mathrm{x}.t) \cong \frac{\Delta u^{k+1}(\mathrm{x}) + \Delta u^{k}(\mathrm{x})}{2}. \tag{29}$$

where $\Delta u^{k}(\mathrm{x}) = \Delta u(\mathrm{x}, k\,\tau)$ is the Laplacian operator.

By replacing Eq.(15)-(17) in Eq.(1), the main equation can be described as:

$$\frac{2u^{k+1}-5u^{k}+4u^{k-1}-u^{k-2}}{\tau^2} + \beta\frac{11u^{k+1}-18u^{k}+9u^{k-1}-2u^{k-2}}{6\tau} + G(u^{k}) = \alpha\frac{\Delta u^{k+1}+\Delta u^{k}}{2} + \frac{f^{k+1}+f^{k}}{2}.$$

then

$$\Delta u^{k+1} - \mu^2 u^{k+1} = -\Delta u^{k} + C_0 u^{k} + C_1 u^{k-1} + C_2 u^{k-2} + \frac{2}{\alpha}G(u^{k}) + F^{k+1}. \tag{30}$$

where

$$\mu^2 = \frac{4}{\alpha\tau^2} + \frac{11\beta}{3\alpha\tau}.$$

and

$$C_0 = -\frac{10}{\alpha\tau^2} - \frac{6\beta}{\alpha\tau}. \quad C_1 = \frac{8}{\alpha\tau^2} + \frac{3\beta}{\alpha\tau}. \quad C_2 = -\frac{2}{\alpha\tau^2} - \frac{2\beta}{3\alpha\tau}. \quad F^{k+1} = -\frac{f^{k+1}+f^{k}}{\alpha}.$$

If the right hand side of Eq.(18) demonstrated as a function like $b^{k}(\mathrm{x})$, it can be rewritten as modified Helmholtz equation as follows:

$$(\Delta - \mu^2)u^{k+1}(\mathrm{x}) = b^{k}(\mathrm{x}). \tag{31}$$

where

$$b^{k}(\mathrm{x}) = -\Delta u^{k} + C_0 u^{k} + C_1 u^{k-1} + C_2 u^{k-2} + \frac{2}{\alpha}G(u^{k}) + F^{k+1}. \tag{31}$$

Notice that in the above equation for u^{-1} and u^{-2} we use Euler scheme and problem conditions as follows:

$$\frac{u^{-1} - u^{0}}{\tau} = g_2(\mathrm{x}).$$

and

$$\frac{u^{-2} - u^{0}}{2\tau} = g_2(\mathrm{x}).$$

that by using initial condition $u^{0} = g_1(\mathrm{x})$ we have:

$$u^{-1} = g_1(\mathrm{x}) - \tau.g_2(\mathrm{x}).$$

and

$$u^{-2} = g_1(\mathrm{x}) - 2\tau.g_2(\mathrm{x}).$$

and finally:

$$b^0(\mathrm{x}) = -\Delta g_1(\mathrm{x}) + C_0 g_1(\mathrm{x}) + C_1\big(g_1(\mathrm{x}) - \tau.\, g_2(\mathrm{x})\big) + C_2\big(g_1(\mathrm{x}) - 2\tau.\, g_2(\mathrm{x})\big) + \frac{2}{\alpha} G\big(g_1(\mathrm{x})\big) + F^1.$$

4. NUMERICAL EXAMPLES

In this section we present four examples that numerical solutions acquired by employing the Singular boundary method for approximated solution of the 2D generalized nonlinear Klein-Gordon problems.

We solve these examples using the SBM mentioned in Section2, and report the numerical results.

The accuracy and convergency of the method shown with two types of error measurements, maximum absolute error ε_∞ and root mean square(RMS) error are used as follows:

$$\varepsilon_\infty(u) = \|u_{ex} - u_{ap}\|_\infty = \max\{|u_{ex}(\mathrm{x}_i.\, t) - u_{ap}(\mathrm{x}_i.\, t)| i = 1.2. \dots . N\}$$

and

$$RMS = \sqrt{\frac{\sum_{i=1}^{N}(\, u_{ex}(\mathrm{x}_i; t) - u_{ap}(\mathrm{x}_i; t))^2}{N}}$$

where $u_{ex}(\mathrm{x}_i.\, t)$ and $u_{ap}(\mathrm{x}_i.\, t)$ denote the exact and numerical approximated solutions, respectively.

In the following examples we employ Polyharmonic splines of order 2, $\phi(r) = r^4 ln(r)$ as radial basis function.

Example 1. On a finite square $\Omega = [0.3] \times [0.3]$, consider Eqs.(1)-(2) for

$$\alpha = 1.\ \beta = 0.\ G(u) = u^2$$

as

$$u_{tt} + u^2 = \Delta u - xycost + x^2y^2cos^2t. \qquad \mathrm{x} \in \Omega.\, t > 0.$$

with initial conditions

$$u(\mathrm{x}.0) = xy. \qquad \mathrm{x} = (x.\, y) \in \Omega$$

$$\frac{\partial u(\mathrm{x}.t)}{\partial t}\Big|_{t=0} = 0. \qquad \mathrm{x} = (x.\, y) \in \Omega.$$

and boundary conditions where the analytical solution is: $u(x.\, y.\, t) = xycost$.

Here, the presented approach is applied for numerical solution of the main problem and two kinds of errors are reported in Table.1 at different times until the desired time T=1. Furthermore, it is clear from Table 1 that RMS of the SBM has no growth whenever the time is increasing, therefore, this fact shows stablility of the method. The graphs of numerical (left) and analytical (right) solutions are presented in Fig. 1. The maximum absolute error of this computed solution obtained by the SBM with $\tau = 0.002$, $\tau = 0.001$ and N=256 at time T=1 are shown in Fig. 2.

Table 1. The $\varepsilon_\infty(u)$ and RMS errors with N=256 and different τ at time t for Example 1.

t	$\tau = 0.002$		$\tau = 0.001$	
	$\varepsilon_\infty(u)$	RMS	$\varepsilon_\infty(u)$	RMS
0.10	7.2037×10^{-5}	3.2289×10^{-5}	4.7441×10^{-6}	5.2173×10^{-6}
0.20	1.7219×10^{-4}	3.2377×10^{-5}	1.0507×10^{-5}	5.2960×10^{-6}
0.30	1.8127×10^{-4}	3.2556×10^{-5}	1.3573×10^{-5}	5.3747×10^{-6}
0.40	1.8292×10^{-4}	3.2921×10^{-5}	1.6639×10^{-5}	5.4534×10^{-6}
0.50	1.9501×10^{-4}	3.6476×10^{-5}	1.9705×10^{-5}	5.5321×10^{-6}
0.60	1.8899×10^{-4}	3.7411×10^{-5}	2.0271×10^{-5}	5.6109×10^{-6}
0.70	1.9944×10^{-4}	4.0064×10^{-5}	2.5836×10^{-5}	5.6896×10^{-6}
0.80	2.0045×10^{-4}	4.3667×10^{-5}	2.8902×10^{-5}	5.7684×10^{-6}
0.90	1.9435×10^{-4}	4.2748×10^{-5}	2.1968×10^{-5}	4.8472×10^{-6}
1.00	1.9221×10^{-4}	4.0671×10^{-5}	2.1534×10^{-5}	4.9260×10^{-6}

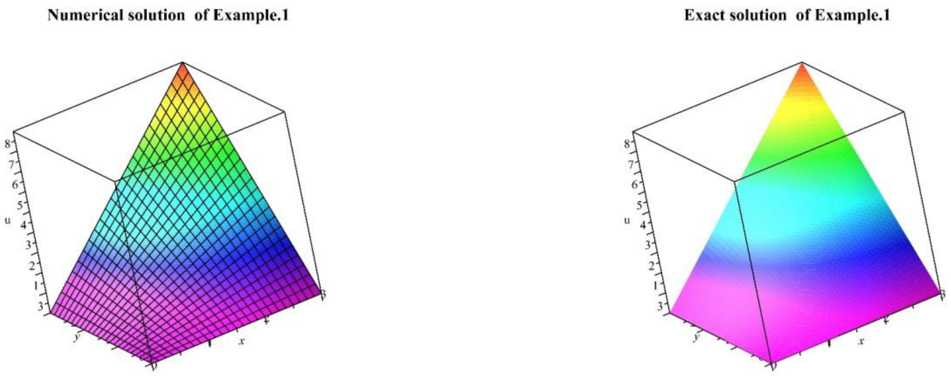

Figure 1. Approximate solution by SBM (left) with $\tau = 0.001$ and N=256 and analytical solution (right) at time T=1 for Example 1.

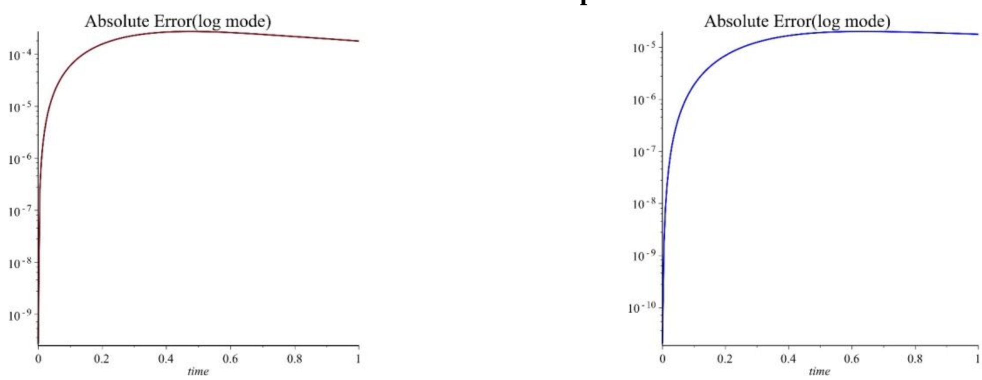

Figure 2. Logplot of the behaviour of $\varepsilon_\infty(u)$ at different time $T = 1$ with $\tau = 0.002$ (left), $\tau = 0.001$ (right) and N=256 for Example 1.

Example 2. In this example, for domain $\Omega = [0.3] \times [0.3]$, consider Eqs.(1)-(2) for

$$\alpha = 1.\ \beta = 0.\ G(u) = u^3$$

as

$$u_{tt} + u^3 = \Delta u + cosx\ cosy\ sint\ + cos^3x\ cos^3y\ sin^3t\ . \qquad \mathrm{x} \in \Omega.\ t > 0$$

with initial conditions

$$u(\mathrm{x}.0) = 0. \qquad \mathrm{x} = (x.y) \in \Omega.$$

$$\frac{\partial u(\mathrm{x}.t)}{\partial t}\Big|_{t=0} = cosx\ cosy\ . \qquad \mathrm{x} = (x.y) \in \Omega.$$

and boundary conditions where the analytical solution is:

$$u(x.y.t)\ = cosx\ cosy\ sint.$$

Here, the presented approach is applied for numerical solution of the main problem and two kinds of errors are reported in Table 2 at different times until the desired time T=1. Furthermore, it is clear from Table 2 that RMS of the SBM has no growth whenever the time is increasing, therefore, this fact shows that the method is stable. The graphs of numerical (left) and analytical (right) solutions are presented in Fig. 3 . The maximum absolute error of this computed solution obtained by the SBM with $\tau = 0.002$, $\tau = 0.001$ and N=256 at time T=1 are shown in Fig. 4.

Table 2.The $\varepsilon_\infty(u)$ and RMS errors with N=256 and different τ at time t for Example 2.

	$\tau = 0.002$		$\tau = 0.001$	
t	$\varepsilon_\infty(u)$	RMS	$\varepsilon_\infty(u)$	RMS
0.10	7.5723×10^{-6}	1.6435×10^{-6}	5.2050×10^{-7}	4.3219×10^{-7}
0.20	2.5940×10^{-5}	1.9141×10^{-6}	1.4721×10^{-6}	4.6242×10^{-7}
0.30	2.8575×10^{-5}	2.4643×10^{-6}	1.7988×10^{-6}	5.4516×10^{-7}
0.40	3.2876×10^{-5}	2.9797×10^{-6}	2.1780×10^{-6}	6.8696×10^{-7}
0.50	3.0156×10^{-5}	2.6513×10^{-6}	2.6452×10^{-6}	7.6165×10^{-7}
0.60	9.2368×10^{-6}	2.4470×10^{-6}	3.2344×10^{-6}	7.1741×10^{-7}
0.70	8.5188×10^{-6}	2.4593×10^{-6}	6.1214×10^{-7}	6.7511×10^{-7}
0.80	7.8477×10^{-6}	2.6429×10^{-6}	5.8860×10^{-7}	5.7524×10^{-7}
0.90	7.2538×10^{-6}	2.1034×10^{-6}	4.9654×10^{-7}	5.1394×10^{-7}
1.00	6.7718×10^{-6}	1.9625×10^{-6}	3.2333×10^{-7}	4.6154×10^{-7}

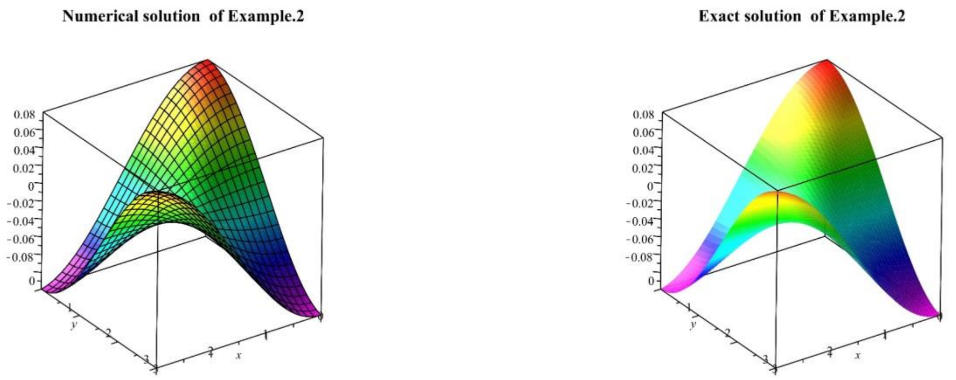

Figure 3. Approximate solution by SBM (left) with $\tau\ = 0.001$ and N=256 and analytical solution (right) at time T=1 for Example 2.

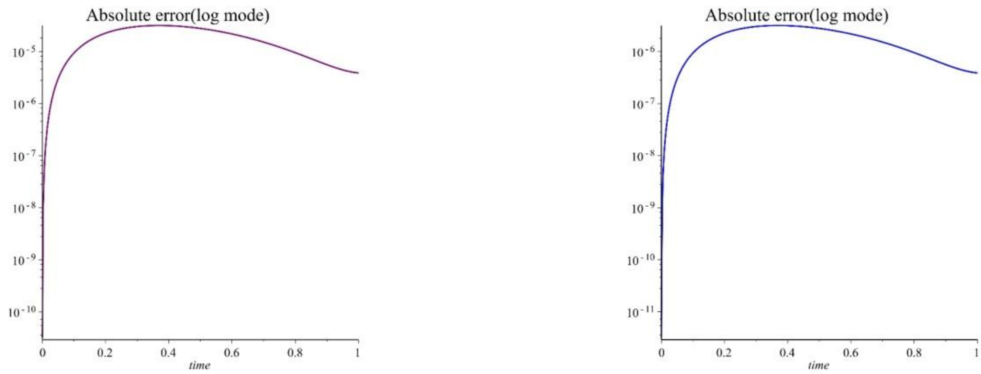

Figure 4. Logplot of the behaviour of $\varepsilon_\infty(u)$ at different time $T = 1$ with $\tau = 0.002$ (left), $\tau = 0.001$ (right) and N=256 for Example 2.

5. CONCLUSION

In this article, singular boundary method (SBM) was employed for solving two-dimensional nonlinear Klein-Gordon equation. A time discretization was applied to approximate the time derivatives. Also, to illustrate the accuracy and efficiency of this method, some numerical examples with different domains have been investigated. Through numerical experiments, we find that numerical results achieved by the SBM are in a good accordance with the exact analytical solutions. The results show the stability and convergency for this meshless technique has been considered that this technique is stable and furthermore, it is convergence. As illustrated by the computational results, implementation of the proposed method is very easy for similar problems.

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