ORIGINAL PAPER

A LARGE CLASS EXTENDING *-PARAHYPONORMAL OPERATORS

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Abstract. *Properties of a large class of (M,k)-*-quasi-parahyponormal operators are established in the present article. It is shown that these operators have finite ascent and the single valued extension property. The matrix representation and the multicyclicity on a separable complex Hilbert space are proved too.*

Keywords: *M-*-parahyponormal operator; k-quasi-*-parahyponormal operator; finite ascent; multicyclic operators.*

1. INTRODUCTION

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be:

- Positive if $\langle Ax, x\rangle \geq 0$ for each $x \in \mathcal{H}$.
- Isometry if $A^{\star}A = I$, where I is the identity operator on $\mathcal{H}$.
- *-paranormal if for all $x \in \mathcal{H}$, $\|A^{\star}x\|^2 \leq \|A^2x\|\|x\|$.
- Quasi-*-paranormal if $\|A^{\star}Ax\|^2 \leq \|A^2Ax\|\|Ax\|$ for all $x \in \mathcal{H}$.
- k-quasi-*-paranormal if $\|A^{\star}A^kx\|^2 \leq \|A^2A^kx\|\|A^kx\|$ for all $x \in \mathcal{H}$.
- *-class $\mathbb{A}$ if $|A^2| - |A^{\star}|^2 \geq 0$.
- k-quasi-*-class $\mathbb{A}$ if $A^{\star k}(|A^2| - |A^{\star}|^2)A^k \geq 0$.

where $|A| = (A^{\star}A)^{\frac{1}{2}}$ denotes the modulus of A. Duggal, Jeon and Kim introduced the *-class $\mathbb{A}$ operators in [1], and proved that they are contained in the class of *-paranormal operators. Also, Mecheri introduced the quasi-*-class $\mathbb{A}$ operators in [2] and showed that a quasi-*-class $\mathbb{A}$ operator is quasi-*-paranormal. Authors in [3] introduced the class of k-quasi-*-paranormal operators and gave several basic and spectral properties for such a class of operators. Senthilkumar and Parvatham in [4] presented the classes of *-parahyponormal and k quasi-*-parahyponormal operators, and showed certain of their properties. In the following, it will be introduced a large class of (M,k)-quasi-*-parahyponormal operators generalizing the above classes. We present their matrix representation, their ascent and the single valued extension property, briefly SVEP. Different related properties are also established.

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2. RESULTS

*2.1. CLASS OF **k**-QUASI-*-PARAHYPONORMAL OPERATORS*

Definition 1. [4] An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *-parahyponormal if

$$(A^\star A)^2 - 2\lambda AA^\star + \lambda^2 \geq 0$$

for all $\lambda > 0$.

Definition 2. [4] An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be k -quasi-*-parahyponormal for some integer k if

$$A^{\star k}((A^\star A)^2 - 2\lambda AA^\star + \lambda^2)A^k \geq 0$$

for all $\lambda > 0$.
This definition is equivalent to

$$\|A^\star A^k x\|^2 \leq \|A^k x\| \|A^\star A^{k+1} x\|$$

for all $x \in \mathcal{H}$.

Proposition 3. Let S be the bilateral weighted shift defined on the usual Hilbert space ℓ_2 by $Se_n = \alpha_n e_{n+1}$, where $(e_n)_n$ is the standard basis, and $(\alpha_n)_n$ is a decreasing complex sequence. Then, S is k -quasi-*-parahyponormal if and only if $|\alpha_{n+k-1}| \leq |\alpha_{n+k}|$ for all n.

Proof: We have

$$\|S^\star S^k e_n\|^2 \leq \|S^k e_n\| \|S^\star S^{k+1} e_n\|$$

Hence, for all n,

$$|\alpha_n|^2 |\alpha_{n+1}|^2 \ldots |\alpha_{n+k-2}|^2 |\alpha_{n+k-1}|^4 \leq |\alpha_n|^2 |\alpha_{n+1}|^2 \ldots |\alpha_{n+k-2}|^2 |\alpha_{n+k-1}|^2 |\alpha_{n+k}|^2$$

Thus,

$$|\alpha_{n+k-1}| \leq |\alpha_{n+k}|$$

for all n. ■

Theorem 4. Unitarily equivalent operators to a k-quasi-*-parahyponormal operator are also k-quasi-*-parahyponormal.

Proof: Let A be a k-quasi-*-parahyponormal operator, and let $B \in \mathcal{B}(\mathcal{H})$ be unitarily equivalent to A. Then, there exists a unitary operator U on H satisfying $B = U^\star AU$. Hence,

$$\begin{aligned} &B^{\star k}((B^\star B)^2 - 2\lambda BB^\star + \lambda^2)B^k = \\ &= U^\star A^{\star k} U (U^\star (A^\star A)^2 U - 2\lambda U^\star AA^\star U + \lambda^2) U^\star A^k U \\ &= U^\star A^{\star k} (A^\star A)^2 A^k U - 2\lambda U^\star A^{\star k} AA^\star A^k U + \lambda^2 U^\star A^{\star k} A^k U \\ &= U^\star A^{\star k}((A^\star A)^2 - 2\lambda AA^\star + \lambda^2) A^k U \geq 0 \end{aligned}$$

Thus, B is k-quasi-*-parahyponormal. ■

Theorem 5. Let $A \in \mathcal{B}(\mathcal{H})$ be a k-quasi-*-parahyponormal operator, and let $S \in \mathcal{B}(\mathcal{H})$ be an isometry. If A commutes with S, then AS is also k-quasi-*-parahyponormal.

Proof: Since A is k-quasi-*-parahyponormal and $\|S^\star\| \leq 1$,

$$\|S^\star A^\star A^k x\|^2 \leq \|S^\star\|^2 \|A^\star A^k x\|^2 \leq \|A^k x\| \|A^\star A^{k+1} x\|$$

Hence,

$$\begin{aligned}&\langle A^{\star k}[(A^\star A)^2 - 2\lambda SAA^\star S^\star + \lambda^2]A^k x, x\rangle = \\ &= \|A^k A^{k+1}x\|^2 - 2\lambda \|S^\star A^\star A^k x\|^2 + \lambda^2 \|A^k x\|^2 \\ &\geq \|A^k A^{k+1}x\|^2 - 2\lambda \|A^k x\| \|A^\star A^{k+1}x\| + \lambda^2 \|A^k x\|^2 \\ &\geq 0\end{aligned}$$

for all $x \in \mathcal{H}$ and all $\lambda > 0$. Thus,

$$A^{\star k}[(A^\star A)^2 - 2\lambda SAA^\star S^\star + \lambda^2]A^k$$

is a positive operator. Therefore,

$$\begin{aligned}&(AS)^{*k}(((AS)^\star AS)^2 - 2\lambda(AS)(AS)^\star + \lambda^2)(AS)^k = \\ &= S^{\star k}A^{\star k}[(A^\star A)^2 - 2\lambda SAA^\star S^\star + \lambda^2]A^k S^k \geq 0\end{aligned}$$

This achieves the proof. ■

Let $A \in \mathcal{B}(\mathcal{H})$. Denote by $\mathcal{R}(\sigma(A))$ for the algebra of all rational functions with poles of σ(A) rather then the set of all rational analytic functions on σ(A).

Definition 6. [5] An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be n-multicyclic, if there exist n (generating) vectors $x_1, x_2, .., x_n$ in $\mathcal{H}$ such that

$$\vee\{g(A)x_i\ , 1 \leq i \leq n\ , g \in \mathcal{R}(\sigma(A))\} = \mathcal{H}$$

Denote by $ker(A)$ and $ran(A)$ respectively for the null space and the range of A. Then, we have :

Theorem 7. If A is an n-multicyclic k-quasi-*-parahyponormal operator, then its restriction on $\overline{ran(A^k)}$ is also n -multicyclic.

Proof: Put

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

on the decomposition $\mathcal{H} = \overline{ran(A^k)} \oplus ker(A^{\star k})$. Since $\sigma(A_1) \subset \sigma(A)$ by [4, Theorem 2.1], $\mathcal{R}(\sigma(A_1)) \subset \mathcal{R}(\sigma(A))$. The operator A is n -multicyclic. Then, there exist n generating vectors $x_1, x_2, .., x_n \in \mathcal{H}$ for which

$$\vee\{g(A)x_i, 1 \leq i \leq n, g \in \mathcal{R}(\sigma(A))\} = \mathcal{H}$$

Put $y_i = A^k x_i$, $1 \le i \le n$. Hence,

$$\begin{aligned}\vee\{g(A_1)y_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A))\} &= \vee\{g(A_1)A^k x_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A))\} \\ &= \vee\{g(A)A^k x_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A))\} \\ &= \vee\{A^k g(A) x_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A))\} \\ &= \overline{ran(A^k)}.\end{aligned}$$

But,

$$\vee\{g(A_1)y_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A))\} \subset \vee\{g(A_1)y_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A_1))\}.$$

Thus,

$$\overline{ran(A^k)} \subset \vee\{g(A_1)y_i, 1 \le i \le n, g \in \mathcal{R}(\sigma(A_1))\}$$

Therefore, $\{y_i\}_{i=1}^n$ are n-generating vectors of A_1, and A_1 is then n-muticyclic. The proof is achieved. ■

2.2. CLASS OF $(\boldsymbol{M}, \boldsymbol{k})$-QUASI--PARAHYPONORMAL OPERATORS*

Interesting properties of (M, k)-quasi-*-parahyponormal operators are shown in this section. In particular, the matrix representation, the finite ascent and the SVEP are presented.

Definition 8. [6] An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be M-*-parahyponormal if there exists $M > 0$ satisfying

$$M(A^\star A)^2 - 2\lambda AA^\star + \lambda^2 \ge 0$$

We introduce a new class of (M, k)-quasi-parahyponormal operators as follows:

Definition 9. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be (M, k)-quasi-*-parahyponormal for some integer k, if there exists $M > 0$, satisfying

$$A^{\star k}(M(A^\star A)^2 - 2\lambda AA^\star + \lambda^2)A^k \ge 0$$

This definition is clearly equivalent to

$$\|A^\star A^k x\|^2 \le \sqrt{M}\|A^k x\|\|A^\star A^{k+1} x\| \qquad (1)$$

for all $x \in \mathcal{H}$.

Inequality (1) shows that this class of operators is nested with respect to M, i.e., an (M, k)-quasi-*-parahyponormal operator is (M', k)-quasi-*-parahyponormal for each $0 < M < M'$.

Remark 10. Classes of (M, k)-quasi-*-parahyponormal operators are not identical with respect to k. In fact, let's consider the unilateral weighted right shift on $\ell_2(\mathbb{N})$ defined by

$Ae_1 = 0$, $Ae_{n+1} = \alpha_n e_n$, where $\alpha_1 = \alpha_3 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$, $\alpha_n = 1$, $n \geq 4$, and (e_n) is the standard basis of $\ell_2(\mathbb{N})$. It's easy to show that A is M-*-quasi-parahyponormal. However,

$$\|A^\star e_1\|^2 = \frac{1}{4} > \sqrt{M}\|e_1\|\|A^\star A e_1\| = 0$$

which contradicts the inequality (1). Hence, A is not M-*-parahyponormal. Analogously to proofs of Theorems 4 and 5, we can have:

Theorem 11.

1. A unitarily equivalent operator B to an (M,k)-quasi-*-parahyponormal operator A is also (M,k)-quasi-*-parahyponormal.
2. If A is (M,k)-quasi-*-parahyponormal operator, and S is an isometry that commutes with A, then the product AS is (M,k)-quasi-*-parahyponormal.

Remark 12. Property (1) of Theorem 11 needs not to be in general true if A and B are similar operators, i.e., there exists an invertible operator U for which $AU=UB$, and A and B are not unitarily equivalent. Indeed, let's consider the bilateral weighted shift A defined on the Hilbert space $\ell_2(\mathbb{Z})$ by

$$Ae_n = \begin{cases} e_{n-1}, & n \neq 3 \\ \sqrt{2}e_2, & n = 3 \end{cases}$$

The operator A is 8-*-parahyponormal, and the invertible operator

$$Ue_n = \begin{cases} e_{n+1}, & n \neq 2 \\ \frac{1}{3}e_3, & n = 2 \end{cases}$$

is non unitary. With an easy computation, the operator $B = U^{-1}AU$ satisfies

$$\|B^\star e_1\|^2 = 18 > \sqrt{8}\|e_1\|\|B^\star B e_1\| = \sqrt{8}$$

Thus, B is not 8-*-parahyponormal.

Theorem 13. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H})$ be an (M,k)-quasi-*-parahyponormal operator. If $\overline{\mathrm{ran}(\mathrm{A}^k)} = \mathcal{H}$, then A is M-*-parahyponormal.

Proof: Let $x \in \mathcal{H}$. Since $ran(A^k)$ is dense in $\mathcal{H}$, there exists a sequence $(x_n)_n$ in $\mathcal{H}$ such that $\lim\limits_{n\to\infty} A^k x_n = x$. Since A is (M,k) -quasi-*-parahyponormal,

$$\begin{aligned} \|A^\star x\|^2 = \left\|\lim_{n\to\infty} A^\star A^k x_n\right\|^2 &= \lim_{n\to\infty}\left\| A^\star A^k x_n\right\|^2 \\ &\leq \sqrt{M}\lim_{n\to\infty}\|A^k x_n\|\|A^\star A^{k+1}x_n\| \\ &\leq \sqrt{M}\|x\|\|A^\star A x\| \end{aligned}$$

by the continuity of the inner product. Thus, A is M-*-parahyponormal. ■

Corollary 14. Let A be a nonzero (M,k)- quasi-*-parahyponormal operator but not M-*-parahyponormal. Then, A admits at least a non trivial closed invariant subspace.

Proof: By the absurd, assume that A has no non trivial closed invariant subspace. Since A is not null, $ker(A) \neq \mathcal{H}$ and $\overline{ran(A)} \neq \{0\}$ are non trivial closed invariant subspaces for A. Thus, we must have $ker(A) = \{0\}$ and $\overline{ran(A)} = \mathcal{H}$. By Theorem 13, A is M-*-parahyponormal, which contradicts the hypothesis. ■

Theorem 15. Let $A \in \mathcal{B}(\mathcal{H})$ be an (M,k)- quasi-*-parahyponormal operator. If $ran(A^k)$ is not dense in $\mathcal{H}$, then A admits the matrix representation $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ under the decomposition $\mathcal{H} = \overline{ran(A^k)} \oplus ker(A^{\star k})$. Furthermore, A_1 is M-*-parahyponormal, $A_3^k = 0$, and $\sigma(A) = \sigma(A_1) \cup \{0\}$.

Proof: Since A is (M,k)-quasi-*-parahyponormal,

$$\langle A^{\star k}(M(A^\star A)^2 - 2\lambda AA^\star + \lambda^2)A^k y, y\rangle \geq 0$$

for all $y \in \mathcal{H}$. Hence,

$$\langle (M(A^\star A)^2 - 2\lambda AA^\star + \lambda^2)A^k y, A^k y\rangle \geq 0$$

Thus, for all $x \in \overline{ran(A^k)}$,

$$\langle (M(A^\star A)^2 - 2\lambda AA^\star + \lambda^2)x, x\rangle = \langle (M(A_1^\star A_1)^2 - 2\lambda A_1 A_1^\star + \lambda^2)x, x\rangle \geq 0$$

Consequently, A_1 is M-parahyponormal. Let now P be the orthogonal projection on $\overline{ran(A^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in \mathcal{H}$, we have

$$\langle A_3^k x_2, y_2\rangle = \langle A^k(I-P)x, (I-P)y\rangle = \langle (I-P)x, A^{\star k}(I-P)y\rangle = 0$$

Thus, $A_3^k = 0$. Furthermore, $\sigma(A_1) \cup \sigma(A_3) = \sigma(A) \cup \Omega$, where Ω is the union of holes in $\sigma(A)$ which happen to be a subset of $\sigma(A_1) \cap \sigma(A_3)$ by [7], Corollary 7], with $\sigma(A_1) \cap \sigma(A_3)$ has no interior point, and A_3 is nilpotent. Thus, $\sigma(A) = \sigma(A_1) \cup \{0\}$. ■

Corollary 16. Let $A \in \mathcal{B}(\mathcal{H})$ be (M,k) -quasi-*-parahyponormal. If the restriction $A_1 = A_{|\overline{ran(A^k)}}$ is invertible, then A is similar to the direct sum of an M-*-parahyponormal operator and a nilpotent operator.

Proof: Let

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{ran(A^k)} \oplus ker(A^{\star k})$$

Then, A_1 is M-*-parahyponormal by Theorem 15. Since A_1 is invertible, $0 \notin \sigma(A)$. Hence, $\sigma(A_1) \cap \sigma(A_3) = \emptyset$. By Rosenblum's Corollary [8-9], there exists $C \in \mathcal{B}(\mathcal{H})$ for which $A_1 C - C A_3 = A_2$. Thus,

$$A = \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$$

The proof is achieved. ■

Theorem 17. The restriction of an (M, k)-quasi-*-parahyponormal operator $A \in \mathcal{B}(\mathcal{H})$ on a closed invariant subspace $\mathcal{M} \subset \mathcal{H}$ is also (M, k)-quasi-*-parahyponormal.

Proof: Let $A_1 = A_{|\mathcal{M}}$, and let P be the orthorprojector on $\mathcal{M}$. Then, $A^k P = PA^k P$ and $A_1 = PAP_{|\mathcal{M}}$. Hence, the result holds by Theorem 15. ■

Definition 18. [10] An operator A in $\mathcal{B}(\mathcal{H})$ is said to have the Single Valued Extension Property SVEP at a complex number α, if for each open neighborhood V of α, the unique analytic function $f: V \to \mathcal{H}$ satisfying

$$(A - \lambda)f(\lambda) = 0$$

for all $\lambda \in V$ is $f \equiv 0$. Furthermore, A is said to have SVEP if A has SVEP at every complex number.

Definition 19. [10] For $A \in \mathcal{B}(\mathcal{H})$, the smallest integer m such that $ker(A^m) = ker(A^{m+1})$ is said to be the ascent of A, and is denoted by $\alpha(A)$. If no such integer exists, we shall write $\alpha(A) = \infty$.

Example 1. [11] $\alpha(A) = 1$ for a dominant operator $A \in \mathcal{B}(\mathcal{H})$, i.e., $ran(A - \lambda) \subseteq ran(A - \lambda)^\star$ for all $\lambda \in \mathbb{C}$.

Example 2. Author in [12] showed that if A is a k-quasi-M-hyponormal operators, i.e.,

$$\sqrt{M}\|(A - \lambda)A^k x\| \geq \|(A - \lambda)^\star A^k x\|$$

for certain $M > 0$, and all $x \in \mathcal{H}$, then $\alpha(A) = k$ and $\alpha(A - \lambda) = 1$, $(\lambda \in \mathbb{C},\ \lambda \neq 0)$.

Definition 20. [10] The smallest integer m satisfying $ran(A^m) = ran(A^{m+1})$ is said to be the descent of A, and is denoted by $\delta(A)$. If no such integer exists, we set $\delta(A) = \infty$.

According to [10], if $\alpha(A)$ and $\delta(A)$ are finite, then they are equal. More information on these notions can be found in [10] and in [13-14]. We've then,

Theorem 21. If $A \in \mathcal{B}(\mathcal{H})$ is (M, k)-quasi-*-parahyponormal, then $(A - \mu)$ has finite ascent for all complex scalar μ. Furthermore, $\alpha(A) \leq k$ and $\alpha(A - \mu) = 1$.

Proof: ***Case (i).*** $\mu = 0$. Since A is (M, k)-quasi-*-parahyponormal operator,

$$\|A^\star A^k x\|^2 \leq \sqrt{M}\|A^k x\|\|A^\star A^{k+1} x\|$$

for certain $M > 0$. Let $x \in ker(A^{k+1})$. Then, $A^\star A^k x = 0$. Hence, for all $z \in \mathcal{H}$

$$\langle A^\star A^k x, z\rangle = 0$$

i.e.,

$$\langle A^k x, Az\rangle = 0$$

For all $z \in H$. Thus, $A^k x \in \left(ran(A)\right)^\perp$.

Since $ran(A^k) \subset ran(A)$, $\left(ran(A)\right)^{\perp} \subset \left(ran(A^k)\right)^{\perp}$. Therefore,

$$A^k x \in (ranA^k)^{\perp} \cap ran(A^k) = \{0\}$$

and so $x \in ker(A^k)$.

Case (ii). $\mu \neq 0$. Since $ker(A-\mu) \subset ker(A-\mu)^{\star}$ by , $ker(A-\mu)$ reduces A. Then,

$$\mathcal{H} = ker(A-\mu)^{\perp} \oplus ker(A-\mu)$$

Let then $x = x_1 + x_2 \in \mathcal{H}$ with $x_1 \in ker(A-\mu)^{\perp}$ and $x_2 \in ker(A-\mu)$. Hence,

$$\begin{aligned} x \in ker(A-\mu)^2 &\Rightarrow (A-\mu)^2 x = 0 = (A-\mu)^2 x_1 \\ &\Rightarrow (A-\mu)x_1 \in ker(A-\mu) \\ &\Rightarrow (A-\mu)x_1 \in ker(A-\mu)^{\star} \cap ran(A-\mu) = \{0\} \\ &\Rightarrow x_1 \in ker(A-\mu) \\ &\Rightarrow x_1 = 0 \Rightarrow x = x_2 \in ker(A-\mu) \end{aligned}$$

Therefore, $ker(A-\mu)^2 \subset ker(A-\mu)^2$, and $\alpha(A-\mu) = 1$. ■

Corollary 22. (M,k)-quasi-parahyponormal operators have SVEP.

Proof: An immediate consequence of [10, Theorem 3.8] and Theorem 21. ■

Definition 23. [10] For an operator $A \in \mathcal{B}(\mathcal{H})$, the local resolvent set of A at a vector $x \in \mathcal{H}$, denoted by $\rho_A(x)$, is defined to consist of complex elements z_0 such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, for which $(A-z)f(z) = x$.

Definition 24. [10] The set $\mathbb{C}\backslash\rho_A(x)$ is called the local spectrum of A at a vector x in $\mathcal{H}$.

We've then the following important result:

Theorem 25. Let $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ be an (M,k)-quasi-*-parahyponormal operator with respect to the decomposition $\mathcal{H} = \overline{ran(A^k)} \oplus ker(A^{\star k})$. Then, for all $= x_1 + x_2 \in \mathcal{H}$:

a. $\sigma_{A_3}(x_2) \subset \sigma_A(x_1 + x_2)$

b. $\sigma_{A_1}(x) = \sigma_{A_1}(x_1 + 0)$

Proof: *a*. Let $z_0 \in \rho_A(x_1 + x_2)$. Then, there exists a neighborhood U of z_0 and an analytic function $f(z)$ defined on U, with values in $\mathcal{H}$, for which

$$(A-z)f(z) = x,\ z \in U \qquad (2)$$

Let $f = f_1 + f_2$ where f_1, f_2 are in the spaces $O(U, \overline{ran(A^k)})$ and $O(U, ker(A^{\star k}))$ respectively, consisting of analytic functions on U with values in $\mathcal{H}$, with respect to the uniform topology [1]. Equality (2) can then be written

$$\begin{pmatrix} A_1 - z & A_2 \\ 0 & A_3 - z \end{pmatrix}\begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then

$$(A_3 - z)f_2(z) = x_2, z \in U$$

Hence, $z_0 \in \rho_{a_3}(x_2)$. Thus, (*a*) holds by passing to the complement.

b. If $z_1 \in \rho_A(x_1 + 0)$, then, there exists a neighborhood V_1 of z_1 and an analytic function g defined on V_1 with values in $\mathcal{H}$ verifying

$$(A - z)f(z) = x_1 + 0, z \in V_1 \tag{3}$$

Let $g = g_1 + g_2$, where $g_1 \in O\left(V_1, \overline{ran(A^k)}\right)$, $g_2 \in O\left(V_1, ker(A^{\star k})\right)$ are as in (*a*). From equation (3) we obtain

$$(A_1 - z)g_1(z) + A_2 g_2(z) = x_1$$

and

$$(A_3 - z)g_2(z) = 0, z \in V_1$$

Since A_3 is nilpotent by Theorem 15, A_3 has SVEP by [10]. Thus, $g_2(z) = 0$. Consequently, $(A_1 - z)g_1(z) = x_1$. Therefore, $z_1 \in \rho_{A_1}(x_1)$, and then $\rho_A(x_1 + 0) \subset \rho_{A_1}(x_1)$. Thus, $\sigma_{A_1}(x) = \sigma_{A_1}(x_1 + 0)$.

Now, if $z_2 \in \rho_{A_1}(x_1)$, then, there exists a neighborhood V_2 of z_2 and an analytic function h from V_2 onto $\mathcal{H}$, such that $(A_1 - z)h(z) = x_1$, for all $\in V_2$. Thus,

$$(A - z)(h(z) + 0) = (A_1 - z)h(z) = x_1 = x_1 + 0$$

Hence, $z_2 \in \rho_A(x_1 + 0)$. ■

Remarks 1. Note that if A is (M, k)-quasi-*-parahyponormal, then for all $\alpha \in \mathbb{C}$ and all $x \in \mathcal{H}$,

$$\begin{aligned}\|(\alpha A^\star)(\alpha A)^k x\|^2 &= |\alpha|^{2k+2}\|A^\star A^k x\|^2 \le |\alpha|^{2k+2}\sqrt{M}\|A^k x\|\|A^\star A^{k+1}x\| \\ &= \sqrt{M}\|(\alpha A)^k x\|\|(\alpha A)^\star(\alpha A)^{k+1}x\|\end{aligned}$$

Hence, αA is also (M, k)-quasi-*-parahyponormal.

Remarks 2. On the other hand, operators $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are 2-*-parahyponormal. However, for $S = \frac{1}{2}(A + B)$ and $x = (0,1) \in \mathbb{C}^2$, we get

$$\|S^\star x\|^2 = \frac{1}{4} > \sqrt{2}\|x\|\|S^\star S x\| = 0$$

This contradicts the inequality (1). Hence, S is not 2-*-parayponormal. Thus, the above class is not convex.

Remarks 3. Also, for the previous operator $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, the operator $A - I$ satisfies

$$\|(A-I)(0,1)\|^2 = 1 > \sqrt{2}\|(0,1)\|\|(A-I)(A-I)^{\star}(0,1)\| = 0$$

Hence, $A - I$ is not 2-*-parahyponormal. This shows that the considered class is not translation invariant.

3. CONCLUSION

We've shown certain fundamental properties of the considered class of operators. A matrix representation, the SVEP, the finite ascent as well as different imoprtant properties have been established.

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