ORIGINAL PAPER

SOLVABILITY AND ULAM STABILITY FOR A NONLINEAR DIFFERENTIAL PROBLEM INVOLVING *p*-LAPLACIAN OPERATOR AND PHI-CAPUTO SEQUENTIAL DERIVATIVES

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Abstract. *This paper deals with a general class of nonlinear fractional differential equations with p-Laplacian operator that involves some* $\varphi-$ *sequential Caputo derivatives. New criteria on the existence and uniqueness of solutions are established. The stability analysis in the sense of Ulam Hyers is discussed. An illustrative example is presented.*

Keywords: *p-Laplacian operator;* $\varphi-$ *Caputo derivative; existence of solution; fixed point.*

1. INTRODUCTION

The fractional calculus has deep applications in various scientific fields; see for instance [1-5]. Most of these research works have been achieved by means of fractional derivatives of Riemann-Liouville, Hadamard, Katugampola, Grunwald Letnikov, or Caputo. However, fractional derivatives "of a function with respect to another function" [6] is different from the others since their kernel appears in terms of another function that called φ. Recently, some fractional differential results have been considered in [7-8]. In most of the present articles, Schauder, Krasnoselskii, Darbo, or Monch theories have been used to prove existence of solutions of nonlinear fractional differential equations with some restrictive conditions [9-10]. Some authors have worked on the uniqueness of solutions for fractional differential problems involving $p-$Laplacian operators. In fact, we cite [11-13] where the authors have studied nonlinear fractional differential equations $p-$Laplacian operators. Also, in the paper [14], the authors have worked on the following problem:

$$\begin{cases} \mathbf{D}^{r_1}\psi_p\left[\mathbf{D}^{r_2}\left(u(t)-v_1(t,u(t))\right)\right]=-A(t)v_2(t,u(t-\tau)), \\ \psi_p\left[\mathbf{D}^{r_2}\left(u(t)-v_1(t,u(t))\right)\right]\Big|_{t=0}=\psi_p\left[\mathbf{D}^{r_2}\left(u(t)-v_1(t,u(t))\right)'\right]\Big|_{t=0}=0, \\ u(0)=u(1)=0,\left[\mathbf{I}^{2-r_2}\left(u(t)-v_1(t,u(t))\right)\right]\Big|_{t=0}=0, \end{cases}$$

where $0<r_1<1<r_2<2$, and v_1,v_2 are continuous but singular at some points. The fractional

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derivatives D^{r_1} and D^{r_2} are of Caputo and Riemann-Liouville, respectively, and $\psi_p(z)=|z|^{p-1}z$ denotes the $p-$Laplacian operator that satisfies $\frac{1}{p}+\frac{1}{q}=1,\left(\psi_p\right)^{-1}=\psi_q.$

Then, the authors of the paper [15] have investigated the questions of existence and uniqueness of solutions as well as the Ulam stability by considering the following problem of "distinct orders" with ψ_p Laplacian operator:

$$\begin{cases} {}^c\mathbf{D}^{r_1}\psi_p\left[{}^c\mathbf{D}^{r_2}\left(u(t)-\sum_{i=1}^m v_i(t)\right)\right]=-w_2(t,u(t)), t\in(0,1] \\ \psi_p\left[{}^c\mathbf{D}^{r_2}\left(u(t)-\sum_{i=1}^m v_i(t)\right)\right]\Big|_{t=0}=0, \\ u(0)=\sum_{i=1}^m v_i(0), \\ u'(1)=\sum_{i=1}^m v_i'(1), \\ u^j(0)=\sum_{i=1}^m v_i^j(0),\ \text{for } j=2,3,\ldots,n-1, \end{cases}$$

where $0<r_1\leq 1, n-1<r_2\leq n, n\geq 4$, and v_i, w are continuous functions ${}^c\mathbf{D}^{r_1}$ and ${}^c\mathbf{D}^{r_2}$ denotes the derivative of fractional order r_1 and r_2 in Caputo's sense, respectively, and $\psi_p(z)=|z|^{p-1}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1,\left(\psi_p\right)^{-1}=\psi_q.$ Based on the above $p-$Laplacian problems, in the present work we are concerned with the following sequential problem:

$$\begin{cases} \mathbf{D}_{0^+}^{r_1;\varphi}\psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))\right)\right]=h(t,u(t),\mathbf{D}_{0^+}^{\sigma;\varphi}u(t)), t\in J=(0,1] \\ \psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))\right)\right]\Big|_{t=0}=0, \\ u(0)=0,\ \ u(1)=\sum_{i=1}^n\lambda_i u\left(\zeta_i\right),\ \zeta_i\in(0,1] \\ \varphi(1)-\varphi(0)=K>0. \end{cases} \tag{1}$$

We notice that we need to take: $\mathbf{D}_{0^+}^{r_i;\varphi}, i=\overline{1,2}$ as the $\varphi-$Caputo fractional derivatives of orders r_i, $0<r_1<1<r_2<2$, $0<\sigma<r_2$, and $\mathbf{I}_{0^+}^{2\sigma;\varphi}$ the fractional integral of order 2σ, $\lambda_i\in\mathrm{R}_+^*$, and $\varphi: J\to\mathrm{R}$ is an increasing function such that $\varphi'(t)\neq 0$, and $\psi_p(z)=|z|^{p-1}z$ denotes the $p-$Laplacian operator with: $\frac{1}{p}+\frac{1}{q}=1,\left(\psi_p\right)^{-1}=\psi_q.$ For all $t\in J$, $g: J\times\mathbb{R}\to\mathbb{R}$, $h: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given functions satisfying some assumptions that will be specified later.

2. PHI-CAPUTO DERIVATIVES CALCULUS

In this section, we introduce some notations and definitions and present preliminary results needed in our proofs.

Definition 1. [6] *For $\alpha>0$, the left-sided $\varphi-$Riemann Liouville fractional integral of order*

α *for an integrable function* $u : J \to \mathbb{R}$ *with respect to another function* $\varphi : J \to \mathbb{R}$ *that is an increasing differentiable function such that* $\varphi'(t) \neq 0$, *for all* $t \in J$ *is defined as follows:*

$$\mathtt{I}_{a^+}^{\alpha;\varphi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1} u(s)ds. \tag{2}$$

Definition 2. [6] *Let* $n \in \mathbb{N}$ *and let* $\varphi, u \in C^n(J)$ *be two functions such that* φ *is increasing and* $\varphi'(t) \neq 0$, *for all* $t \in J$. *The left-sided* $\varphi-$*Riemann Liouville fractional derivative of a function* u *of order* α *is defined by*

$$\mathtt{D}_{a^+}^{\alpha;\varphi} u(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \mathtt{I}_{a^+}^{n-\alpha;\varphi} u(t)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} u(s)ds,$$

where $n = [\alpha] + 1$.

Definition 3. [6] *Let* $n \in \mathbb{N}$ *and let* $\varphi, u \in C^n(J)$ *be two functions such that* φ *is increasing and* $\varphi'(t) \neq 0$, *for all* $t \in J$. *The left-sided* $\varphi-$*Caputo fractional derivative of a function* u *of order* α *is defined by*

$${}^c\mathtt{D}_{a^+}^{\alpha;\varphi} u(t) = \mathtt{I}_{a^+}^{n-\alpha;\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t),$$

where $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}, n = \alpha$ *for* $\alpha \in \mathbb{N}$.

To simplify notation, we will use the abbreviated symbol

$$u_\varphi^{[n]}(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t).$$

From the above definitions, it is clear that

$${}^c\mathtt{D}_{a^+}^{\alpha;\varphi} u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} u_\varphi^{[n]}(s)ds, & \text{if } \alpha \notin \mathbb{N}, \\ u_\varphi^{[n]}(t) & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{3}$$

Lemma 1. [16] *Let* $\alpha, \beta > 0$, *and* $u \in L^1(J)$. *Then*

$$\mathtt{I}_{a^+}^{\alpha;\varphi} \mathtt{I}_{a^+}^{\beta;\varphi} u(t) = \mathtt{I}_{a^+}^{\alpha+\beta;\varphi} u(t), \quad \text{a.e. } t \in J.$$

Lemma 2. [16] *Let* $\alpha > 0$. *Then we have:*

If $u \in C([a,b])$, *then*

$$^{c}\mathbf{D}_{a^+}^{\alpha;\varphi}\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)=u(t), t\in[a,b].$$

If $u\in C^n(J)$, $n-1<\alpha<n$, *hence*

$$\mathbf{I}_{a^+}^{\alpha;\varphi c}\mathbf{D}_{a^+}^{\alpha;\varphi}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u_{\varphi}^{[k]}(a)}{k!}\left[\varphi(t)-\varphi(a)\right]^k,$$

for all $t\in[a,b]$.

Lemma 3. [16] *Let* $t>a$, $\alpha\geq 0,\beta>0$. *Then*

- $\mathbf{I}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left[\varphi(t)-\varphi(a)\right]^{\beta+\alpha-1}$,
- $^{c}\mathbf{D}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left[\varphi(t)-\varphi(a)\right]^{\beta-\alpha-1}$,
- $^{c}\mathbf{D}_{a^+}^{\alpha;\phi}\left[\phi(t)-\phi(a)\right]^{k}=0,$ *for all* $k\in\{0,...,n-1\}, n\in\mathrm{N}$.

Lemma 4. [16] *Let* $\alpha>0, n\in\mathbb{N}$*; such that* $n-1<q\leq n$. *Then we have*

- $^{c}\mathbf{D}_{a^+}^{q;\varphi}\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)={}^{c}\mathbf{D}_{a^+}^{q-\alpha;\varphi}u(t);$ *if* $q>\alpha$.
- $^{c}\mathbf{D}_{a^+}^{q;\varphi}\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)u(t)=\mathbf{I}_{a^+}^{\alpha-q;\varphi}u(t);$ *if* $\alpha>q$.

Lemma 5. [16] *Let us consider a function* $u\in C^n\left[a,b\right]$ *and* $0<q<1$. *Hence, we have*

$$\left|\mathbf{I}_{a^+}^{q;\varphi}u(t_2)-\mathbf{I}_{a^+}^{q;\varphi}u(t_1)\right|\leq\frac{2\|u\|}{\Gamma(q+1)}\left(\varphi(t_2)-\varphi(t_1)\right)^q.$$

Lemma 6. [16] *For the operator* ψ_p, *the following assertions are valid:*

(1) If $|\delta_1|,|\delta_2|\geq\rho>0,\ 1<p\leq 2,\delta_1\delta_2>0,$ *then*

$$\left|\psi_p(\delta_1)-\psi_p(\delta_2)\right|\leq(p-1)\rho^{p-2}|\delta_1-\delta_2|.$$

(2) If $p>2,\ |\delta_1|,|\delta_2|\leq\rho_*>0,$ *then*

$$\left|\psi_p(\delta_1)-\psi_p(\delta_2)\right|\leq(p-1)\rho_*^{p-2}|\delta_1-\delta_2|.$$

Lemma 7. *Suppose* $h\in C(J,\mathbb{R})$. *Hence, the unique solution of the problem*

$$\begin{cases}\mathbf{D}_{0^+}^{r_1;\varphi}\psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))\right)\right]=h(t), t\in J=(0,1]\\ \psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-g(t,u(t))\right)\right]\Big|_{t=0}=0,\\ \mathrm{u}(0)=0, u(1)=\sum_{i=1}^{n}\lambda_i u(\zeta_i),\\ \varphi(1)-\varphi(0)=K>0\end{cases}$$

is given by

$$
\begin{aligned}
u(t)
&=\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h(e)de\right]ds \\
&-(\varphi(t)-\varphi(0))\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h(e)de\right]ds \qquad (4)\\
&+\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{2\sigma-1}}{\Gamma(2\sigma)}g(s,u(s))ds+\left(\sum_{i=1}^{n}\frac{\lambda_i}{K}u(\zeta_i)-\frac{g(0,0)}{K}\right)(\varphi(t)-\varphi(0)).
\end{aligned}
$$

Proof: For $0<r_1<1<r_2<2$, Lemma 2 allows us to obtain

$$\psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))\right)\right]=\mathbf{I}_{0^+}^{r_1;\varphi}h(t)+c_1.$$

Using the fact that $\psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-g(t,u(t))\right)\right]\Big|_{t=0}=0,$ we get $c_1=0.$ Then, we have

$$\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))\right)\right]=\psi_q\left[\mathbf{I}_{0^+}^{r_1;\varphi}h(t)\right].$$

Consequently,

$$u(t)=\mathbf{I}_{0^+}^{r_2;\varphi}\left[\psi_q\left[\mathbf{I}_{0^+}^{r_1;\varphi}h(t)\right]\right]+\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))+c_2\left(\varphi(t)-\varphi(0)\right).$$

The conditions $u(0)=0$, and $u(1)=\sum_{i=1}^{n}\lambda_i u(\zeta_i)$ allow us to get

$$c_2=\sum_{i=1}^{n}\frac{\lambda_i}{K}u(\zeta_i)-\frac{g(0,0)}{K}-\mathbf{I}_{0^+}^{r_2;\varphi}\left[\psi_q\left[\mathbf{I}_{0^+}^{r_1;\varphi}H(t)\right]\right]\Big|_{t=1}. \qquad (5)$$

This completes the proof.

3. MAIN RESULTS

Let us consider the Banach space $C_\sigma=\left\{u\ :\ u\in C[0,1],\mathbf{D}_{0^+}^{\sigma;\varphi}u\in C[0,1]\right\}$ with the norm:

$$\|u\|_{C_\sigma}=\max\left\{\|u\|_C,\left\|\mathbf{D}_{0^+}^{\sigma;\varphi}u\right\|_C\right\}, \qquad (6)$$

where, we define:

$$\|u\|_C=\sup_{t\in[0,1]}|u(t)|,\ \text{ and }\ \left\|\mathbf{D}_{0^+}^{\sigma;\varphi}u\right\|_C=\sup_{t\in[0,1]}\left|\mathbf{D}_{0^+}^{\sigma;\varphi}u(t)\right|.$$

Now, wee need to consider the following assumptions:

H_1) The functions g,h are continuous.

H_2) There exist two positive constants $\delta_1, \delta_2,$ such that

$$\left|g(t,u)-g(t,v)\right| \le \delta_1\left(\left|u-v\right|\right),$$

and

$$\left|h(t,u,v)-h(t,x,y)\right| \le \delta_2\left(\left|u-x\right|+\left|v-y\right|\right),$$

for all $t\in[0,1],\, u,v,x,y\in\mathbb{R}$.

H_3) There are $\rho_1,\rho_2,\rho_3\in\mathbb{R}_+^*,$ such that

$$\left|g(t,u)\right| \le \rho_1\left|u(t)\right|,$$

and

$$\left|h(t,u,v)\right| \le \rho_2\left|u(t)\right|+\rho_3\left|v(t)\right|.$$

We need now to define the following constants:

$$\mathbf{M}_1=\frac{\left(\rho_2+\rho_3\right)^q (K+1)K^{r_2+qr_1}}{\Gamma\left(r_2+1\right)\left(\Gamma\left(r_2+1\right)\right)^q}$$

$$\mathbf{M}_2=\frac{K^{2\sigma}\rho_1}{\Gamma\left(2\sigma+1\right)}+\sup_{i\in\{1,\dots,n\}} n\lambda_i+\rho_1$$

and

$$\mathbf{M}_3=\frac{\left(\rho_2+\rho_3\right)^q (K+1)K^{r_2-\sigma+qr_1}}{\Gamma\left(r_2-\sigma+1\right)\left(\Gamma\left(r_1+1\right)\right)^q}.$$

We have to prove the following proposition.

Proposition 1. Let $b>0, \varphi, u, v\in C^n\left(\left[0,1\right]\right)$. We have

$$(\varphi(t)-\varphi(0))^b \le K^q,$$

$$\left|\mathbf{I}_{0^+}^{b;\varphi}u(t)\right| \le \frac{K^b\left\|u\right\|_{C_\sigma}}{\Gamma\left(b+1\right)},$$

$$\left|h(s,u(s),\mathbf{D}_{0^+}^{\sigma;\varphi}u(s))\right| \le \left(\rho_2+\rho_3\right)\left\|u\right\|_{C_\sigma},$$

$$\left|h(s,u(s),\mathbf{D}_{0^+}^{\sigma;\varphi}u(s))-h(s,v(s),\mathbf{D}_{0^+}^{\sigma;\varphi}v(s))\right| \le 2\delta_2\left\|u-v\right\|_{C_\sigma}.$$

Proof:

(1) Since $\varphi'(t)>0, t=\left[0,1\right]$, then

$$(\varphi(t)-\varphi(0))^b \le (\varphi(1)-\varphi(0))^b = K^b.$$

(2) We have

$$\left|\mathbf{I}_{0^+}^{b;\varphi}u(t)\right| \le \frac{1}{\Gamma(b)}\int_0^t \varphi'(s)(\varphi(t)-\varphi(s))^{q-1}\left|u(s)\right|ds$$
$$\le \frac{\sup_{s\in[0,1]}\left|u(s)\right|}{\Gamma(b)}\int_0^t \varphi'(s)(\varphi(t)-\varphi(s))^{b-1}ds$$
$$\le \frac{K^b\left\|u\right\|_{C_\sigma}}{\Gamma(b+1)}.$$

(3) By (H_3), we can write

$$\left|h(s,u(s),\mathbf{D}_{0^+}^{\sigma;\varphi}u(s))\right| \le \rho_2 \sup_{s\in[a,b]}\left|u(s)\right| + \rho_3 \sup_{s\in[a,b]}\left|\mathbf{D}_{0^+}^{\sigma;\varphi}u(s)\right|$$
$$\le \rho_2\left\|u\right\|_C + \rho_3\left\|\mathbf{D}_{0^+}^{\sigma;\varphi}u\right\|_C$$
$$\le (\rho_2+\rho_3)\left\|u\right\|_{C_\sigma}.$$

(4) We use (H_2), so we obtain

$$\left|h(s,u(s),\mathbf{D}_{0^+}^{\sigma;\varphi}u(s)) - h(s,v(s),\mathbf{D}_{0^+}^{\sigma;\varphi}v(s))\right|$$
$$\le \delta_2\left(\sup_{s\in[0,1]}\left|u(s)-v(s)\right| + \sup_{s\in[0,1]}\left|\mathbf{D}_{0^+}^{\sigma;\varphi}u(s)-\mathbf{D}_{0^+}^{\sigma;\varphi}v(s)\right|\right)$$
$$\le \delta_2\left(\left\|u-v\right\|_C + \left\|\mathbf{D}_{0^+}^{\sigma;\varphi}(u-v)\right\|_C\right)$$
$$\le 2\delta_2\left\|u-v\right\|_{C_\sigma}.$$

3.1. EXISTENCE OF SOLUTIONS

Theorem 1. Under the hypotheses (H_1)-(H_3), the problem (1) has a solution.

Proof: We define the operator $N : C_\sigma \to C_\sigma$ as follows:

$$\begin{aligned}(Nu)(t)\\ &= \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]ds\\ &-(\varphi(t)-\varphi(0))\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]ds\\ &+\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{2\sigma-1}}{\Gamma(2\sigma)}g(s,u(s))ds + \left(\sum_{i=1}^n \frac{\lambda_i}{K}u(\zeta_i) - \frac{g(0,0)}{K}\right)(\varphi(t)-\varphi(0)),\end{aligned}$$

where

$$h_u(e) = h(s,u(s),\mathbf{D}_{0^+}^{\sigma;\varphi}u(s)).$$

We consider the subset $U_r := \{u \in C_\sigma, \|u\|_{C_\sigma} \le r\}$. We have

$$2\max\{\mathtt{M}_1, \mathtt{M}_3\} r^{q-1} \le 1, \quad \text{and} \quad 2\max\left\{\mathtt{M}_2, \frac{K^\sigma \rho_1}{\Gamma(\sigma+1)}\right\} \le 1.$$

For any $u \in U_r$ and by (H_3), we show that $\mathbb{N}U_r \subset U_r$. Therefore,

$$\begin{aligned}
&\|(Nu)(t)\|_c \\
&\le \sup_{t\in[0,1]}\Bigg| \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma(r_2)} \psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)} h_u(e)de\right] ds \\
&-(\varphi(t)-\varphi(0))\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma(r_2)} \psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)} h_u(e)de\right] ds \\
&+\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{2\sigma-1}}{\Gamma(2\sigma)} g(s,u(s))ds \\
&+\left(\sum_{i=1}^n \frac{\lambda_i}{K} u(\zeta_i) - \frac{g(0,0)}{K}\right)(\varphi(t)-\varphi(0))\Bigg|.
\end{aligned}$$

By Lemma 4 and Proposition 1, we obtain

$$\begin{aligned}
&\|N(u)(t)\|_c \\
&\le \frac{(K+1)K^{r_2}}{\Gamma(r_2+1)} \sup_{s\in[0,1]}\left|\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)} h_u(e)de\right]\right| \\
&\quad + \frac{K^{2\sigma}\rho_1}{\Gamma(2\sigma+1)}\|u\|_{C_\sigma} + \left(\sup_{i\in\{1,\dots,n\}} n\lambda_i + \rho_1\right)\|u\|_{C_\sigma} \\
&\le \frac{(\rho_2+\rho_3)^q (K+1)K^{r_2+qr_1}}{\Gamma(r_2+1)(\Gamma(r_1+1))^q}\|u\|_{C_\sigma}^q + \left(\frac{K^{2\sigma}\rho_1}{\Gamma(2\sigma+1)} + \sup_{i\in\{1,\dots,n\}} n\lambda_i + \rho_1\right)\|u\|_{C_\sigma} \\
&\le \mathtt{M}_1\|u\|_{C_\sigma}^q + \mathtt{M}_2\|u\|_{C_\sigma}.
\end{aligned}$$

Also, we can write

$$\begin{aligned}
&\left\|\mathbf{D}_{0^+}^{\sigma;\varphi}(Nu)(t)\right\|_c \\
&\le \sup_{t\in[0,1]}\Bigg| \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-\sigma-1}}{\Gamma(r_2-\sigma)} \psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)} h_u(e)de\right] ds \\
&+\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\sigma-1}}{\Gamma(\sigma)} g(s,u(s))ds \\
&-(\varphi(t)-\varphi(0))\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-\sigma-1}}{\Gamma(r_2-\sigma)} \psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)} h_u(e)de\right] ds\Bigg|.
\end{aligned}$$

By Lemma 4 and Proposition 1, we obtain

$$
\begin{aligned}
&\left\|\mathbf{D}_{a^+}^{\sigma;\varphi}(Nu)\right\|_c \\
&\leq \frac{(K+1)K^{r_2-\sigma}}{\Gamma(r_2-\sigma+1)}\sup_{s\in[0,1]}\left|\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]\right|+\frac{K^\sigma\rho_1}{\Gamma(\sigma+1)}\|u\|_{C_\sigma} \\
&\qquad\leq \frac{(\rho_2+\rho_3)^q(K+1)K^{r_2-\sigma+qr_1}}{\Gamma(r_2-\sigma+1)\left(\Gamma(r_1+1)\right)^q}\|u\|_{C_\sigma}^q+\frac{K^\sigma\rho_1}{\Gamma(\sigma+1)}\|u\|_{C_\sigma} \\
&\qquad\leq \mathbf{M}_3\|u\|_{C_\sigma}^q+\frac{K^\sigma\rho_1}{\Gamma(\sigma+1)}\|u\|_{C_\sigma}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|Nu\|_{C_\sigma} &= \max\left\{\|(Nu)\|_c,\left\|\mathbf{D}_{a^+}^{\sigma;\varphi}(Nu)\right\|_c\right\} \\
&\leq \max\{\mathbf{M}_1,\mathbf{M}_3\}r^q+\max\left\{\mathbf{M}_2,\frac{K^\sigma\rho_1}{\Gamma(\sigma+1)}\right\}r \\
&\leq \frac{r}{2}+\frac{r}{2}=r.
\end{aligned}
$$

Now we pass to prove that the operator N is completely continuous.The functions φ,u,g,h are continuous, thus, N is continuous.On the other hand, for any $u\in U_r$ and $t_1,t_2\in[0,1]$ such that $t_1<t_2$, by Proposition 1, and Lemma 5 we have

$$
\begin{aligned}
&\left|(Nu)(t_2)-(Nu)(t_1)\right| \\
&=\left|\int_0^{t_2}\frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s\frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]ds\right. \\
&-\int_0^{t_1}\frac{\varphi'(s)(\varphi(t_1)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s\frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]ds \\
&-(\varphi(t_2)-\varphi(t_1))\int_0^1\frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s\frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]ds \\
&+\int_0^{t_2}\frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{2\sigma-1}}{\Gamma(2\sigma)}g(s,u(s))ds-\int_0^{t_1}\frac{\varphi'(s)(\varphi(t_1)-\varphi(s))^{2\sigma-1}}{\Gamma(2\sigma)}g(s,u(s))ds \\
&\left.+\left(\sum_{i=1}^n\frac{\lambda_i}{K}u(\zeta_i)-\frac{g(0,0)}{K}\right)(\varphi(t_2)-\varphi(t_1))\right|
\end{aligned}
$$

$$\leq \frac{\left(\varphi(t_2)-\varphi(t_1)\right)^{r_2}}{\Gamma\left(r_2+1\right)} \sup_{s\in[0,1]}\left|\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)} h_u(e)de\right]\right|$$
$$+\frac{K^{r_2}\left(\varphi(t_2)-\varphi(t_1)\right)}{\Gamma\left(r_2+1\right)} \sup_{s\in[0,1]}\left|\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)} h_u(e)de\right]\right|$$
$$+\frac{\rho_1 r}{\Gamma\left(2\sigma+1\right)}(\varphi(t_2)-\varphi(t_1))^{2\sigma}+\left(n\sup_{i\in\{1,\dots,n\}}\lambda_i+\rho_1\right)\frac{r}{K}\left(\varphi(t_2)-\varphi(t_1)\right),$$

so

$$\begin{aligned}
&\left|(Nu)(t_2)-(Nu)(t_1)\right| \\
&\leq \frac{\left(\rho_2+\rho_3\right)^q K^{qr_1} r}{\Gamma\left(r_2+1\right)\left(\Gamma\left(r_1+1\right)\right)^q}\left(\varphi(t_2)-\varphi(t_1)\right)^{r_2} \\
&+\frac{\rho_1 r}{\Gamma\left(2\sigma+1\right)}(\varphi(t_2)-\varphi(t_1))^{2\sigma} \\
&+\left[\frac{\left(\rho_2+\rho_3\right)^q K^{qr_1+r_2} r}{\Gamma\left(r_2+1\right)\left(\Gamma\left(r_1+1\right)\right)^q}+\frac{r}{K}\left(\sup_{i\in\{1,\dots,n\}} n\lambda_i+\rho_1\right)\right]\left(\varphi(t_2)-\varphi(t_1)\right).
\end{aligned} \tag{7}$$

We have also

$$\begin{aligned}
&\left|\mathbf{D}_{a^+}^{\sigma;\varphi}(Nu)(t_2)-\mathbf{D}_{a^+}^{\sigma;\varphi}(Nu)(t_1)\right| \\
&=\left|\int_0^{t_2}\frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma\left(r_2-\sigma\right)}\psi_q\left[\int_0^s\frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_u(e)de\right]ds\right. \\
&+\int_0^{t_2}\frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{\sigma-1}}{\Gamma\left(\sigma\right)}g(s,u(s))ds-\int_0^{t_1}\frac{\varphi'(s)(\varphi(t_1)-\varphi(s))^{\sigma-1}}{\Gamma\left(\sigma\right)}g(s,u(s))ds \\
&\left.-\int_0^{t_1}\frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma\left(r_2-\sigma\right)}\psi_q\left[\int_0^s\frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_u(e)de\right]ds\right| \\
&\leq\frac{\left(\varphi(t_2)-\varphi(t_1)\right)^{r_2}}{\Gamma\left(r_2-\sigma+1\right)}\sup_{s\in[0,1]}\left|\psi_q\left[\int_0^s\frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_u(e)de\right]\right| \\
&+\frac{\rho_1 r}{\Gamma\left(\sigma+1\right)}(\varphi(t_2)-\varphi(t_1))^{\sigma} \\
&\leq\frac{\left(\rho_2+\rho_3\right)^q K^{qr_1} r}{\Gamma\left(r_2-\sigma+1\right)\left(\Gamma\left(r_1+1\right)\right)^q}\left(\varphi(t_2)-\varphi(t_1)\right)^{r_2}+\frac{\rho_1 r}{\Gamma\left(\sigma+1\right)}(\varphi(t_2)-\varphi(t_1))^{\sigma}.
\end{aligned} \tag{8}$$

So (7) and (8) are close to zero when $t_2\to t_1$. Then, $N\left(U_r\right)$ is equi-continuous. The Arzela-Ascoli theorem implies that $\overline{N\left(U_r\right)}$ is compact. Hence, $N:U_r\to U_r$ is completely continuous. Therefore, by Schauder fixed-point theorem, we confirm that (1) has a solution.

3.2. UNIQUENESS OF SOLUTIONS

Theorem 2. If the three hypotheses (H_1 - H_3) are satisfied, and the quantity $\vartheta < 1$,

$$\vartheta := \max\left\{\vartheta_1, \vartheta_2\right\} \tag{9}$$

then, problem (1.1) has a unique solution,where

$$\vartheta_1 = \frac{2\delta_2\left(q-1\right)\Delta^{q-1}(K+1)K^{r_2+r_1}}{\Gamma\left(r_2+1\right)\Gamma\left(r_1+1\right)} + \frac{\delta_1 K^{2\sigma}}{\Gamma\left(2\sigma+1\right)} + \sup_{i\in\{1,\dots,n\}} n\lambda_i$$

$$\vartheta_2 = \frac{2\delta_2\left(q-1\right)\Delta^{q-1}K^{r_2-\sigma+r_1}}{\Gamma\left(r_2-\sigma+1\right)\Gamma\left(r_1+1\right)} + \frac{\delta_1 K^{\sigma}}{\Gamma\left(\sigma+1\right)}.$$

Proof: Fo r any $u, v \in U$, $t \in [0,1]$ and (H_2), we give

$$\begin{aligned}
&\left\|(Nu)(t)-(Nv)(t)\right\|_c \\
&\leq \sup_{t\in[0,1]}\left|\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma\left(r_2\right)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_u(e)de\right]ds\right. \\
&-\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma\left(r_2\right)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_v(e)de\right]ds \\
&+\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{2\sigma-1}}{\Gamma\left(2\sigma\right)}g(s,u(s))ds - \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{2\sigma-1}}{\Gamma\left(2\sigma\right)}g(s,v(s))ds \\
&-\left(\varphi(t)-\varphi(0)\right)\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma\left(r_2\right)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_u(e)de\right]ds \\
&+\left(\varphi(t)-\varphi(0)\right)\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma\left(r_2\right)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_v(e)de\right]ds \\
&\left.+\left(\varphi(t)-\varphi(0)\right)\sum_{i=1}^n \frac{\lambda_i}{K}\left(u\left(\zeta_i\right)-v\left(\zeta_i\right)\right)\right|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left\|(Nu)(t)-(Nv)(t)\right\|_c \\
&\leq \left(q-1\right)\Delta^{q-1}\frac{(K+1)K^{r_2}}{\Gamma\left(r_2+1\right)}\sup_{s\in[0,1]}\left|\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_u(e)de\right. \\
&\left.-\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma\left(r_1\right)}h_v(e)de\right| + \frac{\delta_1 K^{2\sigma}}{\Gamma\left(2\sigma+1\right)}\left\|u-v\right\|_{C_\sigma} + \sup_{i\in\{1,\dots,n\}} n\lambda_i\left\|u-v\right\|_{C_\sigma} \qquad (10) \\
&\leq \left(\frac{2\delta_2\left(q-1\right)\Delta^{q-1}(K+1)K^{r_2+r_1}}{\Gamma\left(r_2+1\right)\Gamma\left(r_1+1\right)} + \frac{\delta_1 K^{2\sigma}}{\Gamma\left(2\sigma+1\right)} + \sup_{i\in\{1,\dots,n\}} n\lambda_i\right)\left\|u-v\right\|_{C_\sigma} \\
&\leq \vartheta_1\left\|u-v\right\|_{C_\sigma},
\end{aligned}$$

where

$$\left|\psi_q(x)-\psi_q(y)\right| \le (q-1)\Delta^{q-1}\left|x-y\right|, \text{ and } \Delta>0.$$

We have also

$$\begin{aligned}
&\left\|\mathbf{D}_{0^+}^{\sigma;\varphi}(Nu)(t)-\mathbf{D}_{0^+}^{\sigma;\varphi}(Nv)(t)\right\|_c \\
&\le \sup_{t\in[0,1]}\left|\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-\sigma-1}}{\Gamma(r_2-\sigma)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_u(e)de\right]ds\right. \\
&\quad +\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\sigma-1}}{\Gamma(\sigma)}g(s,u(s))ds-\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\sigma-1}}{\Gamma(\sigma)}g(s,v(s))ds \\
&\quad \left.-\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-\sigma-1}}{\Gamma(r_2-\sigma)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h_v(e)de\right]ds\right| \\
&\le \left(\frac{2\delta_2(q-1)\Delta^{q-1}K^{r_2-\sigma+r_1}}{\Gamma(r_2-\sigma+1)\Gamma(r_1+1)}+\frac{\delta_1K^{\sigma}}{\Gamma(\sigma+1)}\right)\|u-v\|_{C_\sigma} \qquad (11) \\
&\le \vartheta_2\|u-v\|_{C_\sigma}.
\end{aligned}$$

Using (9) and (10), we obtain

$$\|Nu-Nu\|_{C_\sigma} \le \vartheta\|u-v\|_{C_\sigma}.$$

Hence, N is a contraction operator. The contraction mapping principle implies that (1) has a unique solution.

3.3. ULAM STABILITIES ANALYSIS

Definition 4. The problem (1) is Ulam-Hyers stable if for every $\varepsilon\in\mathbb{R}_+^*$, there is a constant $\Upsilon>0$, such that if one has

$$\left\|\mathbf{D}_{0^+}^{r_1;\varphi}\psi_p\left[\mathbf{D}_{0^+}^{r_2;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{2\sigma;\varphi}g(t,u(t))\right)\right]-h(t,u(t),\mathbf{D}_{0^+}^{\sigma;\varphi}u(t))\right\|_{C_\sigma}<\varepsilon, \qquad (12)$$

then, $\exists v\in C_\sigma$ satisfying:

$$\begin{aligned}
&v(t) \\
&=\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h(e)de\right]ds \\
&-(\varphi(t)-\varphi(0))\int_0^1 \frac{\varphi'(s)(\varphi(1)-\varphi(s))^{r_2-1}}{\Gamma(r_2)}\psi_q\left[\int_0^s \frac{\varphi'(e)(\varphi(s)-\varphi(e))^{r_1-1}}{\Gamma(r_1)}h(e)de\right]ds \qquad (13) \\
&+\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{2\sigma-1}}{\Gamma(2\sigma)}g(s,v(s))ds+\left(\sum_{i=1}^n \frac{\lambda_i}{K}v(\zeta_i)-\frac{g(0,0)}{K}\right)(\varphi(t)-\varphi(0)),
\end{aligned}$$

such that, the inequality

$$\|u-v\|_{C_\sigma} \leq \Upsilon\varepsilon, \tag{14}$$

holds.

Theorem 3. If the conditions of Theorem 2 are satisfied, then problem (1) is Ulam-Hyers stable.

Proof: Let $u \in C_\sigma$ be the solution of (1.1), and $v(t)$ is an approximate solution and satisfying (13). Then, we have

$$\|u(t)-v(t)\|_{C_\sigma} \leq \vartheta\|u-v\|_{C_\sigma}, \tag{15}$$

where ϑ is in (9).Hence by definition, we state that problem (1) is Hyers-Ulam stable.

3.4. AN EXAMPLE

Consider the following problem:

$$\begin{cases} \mathbf{D}_{0^+}^{0.5;\varphi}\psi_p\left[\mathbf{D}_{0^+}^{1.5;\varphi}\left(u(t)-\mathbf{I}_{0^+}^{1.5;\varphi}g(t,u(t))\right)\right]=h(t,u(t),\mathbf{D}_{0^+}^{0.75;\varphi}u(t)), t\in(0,1] \\ p=4, \text{ and } q=\frac{4}{3} \\ \varphi(1)-\varphi(0)=1, \end{cases} \tag{16}$$

with:

$$g(t,u)=\left(e^{\frac{1}{1+t^2}}\right)u,$$

$$h(t,u,v)=\left(e^{\frac{1}{1+t}}\right)u+\left(e^{\frac{1}{\sqrt{1+t^2}}}\right)v.$$

We can easy verify that the condition of Theorems 1 and 2 are satisfied. Hence, we state that problem (16) has a unique solution.

4. CONCLUSION

The nonlinear fractional differential equations with p-Laplacian operator and some sequential Caputo derivatives are a general class discussed in this paper. There are new standards for the existence and originality of solutions. and stability analysis in the Ulam Hyers meaning was spoken about. Our effort was ended with an illustration.

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