ORIGINAL PAPER

A NOTE ON WEAK STABILITY OF ε-ISOMETRIES ON CERTAIN BANACH SPACES

MINANUR ROHMAN[1], ILKER ERYILMAZ[1]

Manuscript received: 28.04.2022; Accepted paper: 16.05.2023;
Published online: 30.06.2023.

Abstract. *In this paper, we will discuss the weak stability of ε-isometries on certain Banach spaces. Let* $f: X \to Y$ *be a standard ε-isometry. If* Y^* *is strictly convex, then for any* $x^* \in X^*$*, there is* $\varphi \in Y^*$ *that satisfies* $\|\varphi\| \equiv r = \|x^*\|$*, such that*

$$|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \leq 2r\varepsilon, x \in X.$$

Also, we show that if X and Y are both L_P *spaces* $(1 < p < \infty)$*,* $f: X \to Y$ *is a standard ε-isometry, then there exists a linear operator* $T: Y \to X$ *with norm 1 such that*

$$\|Tf(x) - x\| \leq 2\varepsilon, \forall x \in X.$$

Keywords: *ε-isometry; weak stability; strictly convex; reflexive.*
2020 Mathematics Subject Classification: *46B04; 46B10; 46A55.*

1. INTRODUCTION

Often we cannot find an exact solution to a problem in the real world, even though we have made a mathematical model of the problem. There is always a difference between the ideal result and the real result of applying the mathematical model that is made. This problem gave rise to the term isometry as the ideal result and ε-isometry as the real result. Thus, if distances are known imprecisely one may not be able to say whether a mapping is an isometry, then the concept of ε-isometry is useful.

Isometry mapping should be performed on the metric space, and Banach space is one of the special metric spaces. On the other hand, since each metric space can be embedded in a Banach space while keeping the metric unchanged (see [1], Lemma 1.1]), the study of metric-preserving mapping in Banach space does not lose generality. It is said that a mapping U from Banach space X into Banach space Y is a metric-preserved mapping (or called an isometry mapping) if, for all $x, y \in X$ satisfies

$$\|U(x) - U(y)\| = \|x - y\|,$$

That is, the distance between any two points is preserved under this mapping. If U also satisfies $U(0) = 0$, that is, the origin of X is mapped to the origin of Y, then it is said that U is standard. For the sake of brevity, we choose isometry notion than metric-preserved mapping.

[1] Ondokuz Mayıs University, Department of Mathematics, 55139 Samsun, Turkey.
E-mail: minanurrohmanali@gmail.com; rylmz@omu.edu.tr.

ε -isometry mapping is nothing but a mapping that has a difference of ε with the actual value, where the value of ε also applies to true isometry. Therefore, ε-isometry mapping is defined as

$$\|f(x) - f(y)\| - \|x - y\| \,| \leq \varepsilon.$$

By the above definition, we have four possibilities depending on the surjectivity of f and value of ε: (1) f is surjective and ε = 0 (that is f is surjective isometry); (2) f is non-surjective and ε = 0; (3) f is surjective and ε ≠ 0; and (4) f is non-surjective and ε ≠ 0. By these possibilities, the research of isometry and perturbed isometry mapping can be roughly divided into the following stages: surjective and non-surjective isometry, surjective and non-surjective ε-isometry, and coarse isometry.

Mazur-Ulam shows that every isometry mapping is affine [2]. This means that isometry mapping is just translated linear mappings. This result raises a notion of ε-isometry mappings. Hyers-Ulam [3] proposed a problem, "If f is a surjective ε-isometry, then does always there exist an isometry mapping U such that the difference of f and U is bounded by $k\varepsilon$, for any positive number k?". Many experts gave the affirmative answer.

In the Hyers-Ulam problem, the surjective condition of ε-isometry can not be dropped in the norm topology. Therefore, many mathematicians try to avoid the surjective condition via weak topology.

In the case of surjective ε-isometries, we refer to [4-9]. In this paper, we will discuss the interesting topic for non-surjective ε-isometries. Preliminary results in this specific topic can be found in [10-21], and the important theorem that will be used continuously is as follows [12].

Theorem 1.1. Suppose $f: X \to Y$ is a standard ε-isometry, then $\forall x^* \in X^*$, there is $\varphi \in Y^*$ that satisfies $\|\varphi\| \equiv r = \|x^*\|$, such that

$$|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \leq \kappa r \varepsilon, \forall x \in X, \tag{1.1}$$

where $\kappa = 4$.

Recently, Cheng and Dong [11] further proved that the constant κ = 4 in the above theorem can be optimized to κ = 3 (see also [16]). They corrected the conclusion in [10] that the optimal constant is κ = 2.

Unless specifically noted, the Banach spaces (indicated by X *and* Y) considered here are real Banach spaces, and X^* is the dual space of X. We use B_X and S_X to denote the closed unit ball and unit sphere of X, respectively. $\overline{co}$(A) and $\overline{span}$(A), respectively, represent the closed convex hull and closed linear hull of the set $A \subset X$.

2. WEAK STABILITY OF ε-ISOMETRIES ON STRICTLY CONVEX SPACES

This section proves that when Y^* is strictly convex, the optimal constant in Theorem 1.1 is κ = 2, which is an improvement result of [15].

Proposition 2.1. Suppose $\mathfrak{U}$ is a free ultrafilter on X, and K is a compact Hausdorff space. Then for any mapping $f: X \to K$, an ultrafilter limit, $\lim_{\mathfrak{U}} f$, exists.

Since every closed unit ball is weak* compact (The Banach-Alaoglu Theorem), according to Proposition 2.1 and Theorem 1.1, it is not difficult to verify the following lemma.

Lemma 2.2. Let f: X → Y be a standard ε-isometry, and $\mathfrak{U}$ is the free ultrafilter on ℕ, then

$$\Phi(x) = w^* - \lim_{\mathfrak{U}} \frac{f(nx)}{n}, x \in X$$

is an isometry from X to Y^{**}, where $w^* - \lim_{\mathfrak{U}}$ represents the $\mathfrak{U}$-limit of the ultrafilter in the sense of w^*- topology in Y^{**}.

For the dual space X^* of Banach space X, we use $\mathfrak{R}(X^*)$ to denote the collection generated by all w^*-compact convex subsets in X^*.

Definition 2.3. For the dual space X^* of Banach space X, let $\mathfrak{R}(X^*)$ denote the collection generated by all w^*-compact convex subsets in X^*. Let $f: X \to \mathbb{R}$ be a convex function. Its subdifferential mapping $\partial f: X \to \mathfrak{R}(X^*)$ is defined as

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \forall y \in X\}.$$

In particular, when $f = \frac{1}{2}\|\cdot\|^2$, ∂f is called *dual mapping*; when $f = \|\cdot\|$, ∂f is called *supporting mapping* (See [22] and [23]).

Lemma 2.4. Let $f: X \to Y$ be a standard ε-isometry, $\mathfrak{U}$ be the free ultrafilter on ℕ, and Φ be an isometry from X to Y^{**} defined in Lemma 2.2, $z \in S_X$ is Gateaux smooth point, $x^* = d\|\cdot\|$, then for any $x \in X$, there exists $\varphi, \psi \in \partial\|\Phi(z)\| \cap Y^*$, such that

$$\langle x^*, x\rangle - \langle \varphi, f(x)\rangle \leq 2\varepsilon, \tag{2.1}$$

and

$$\langle x^*, x\rangle - \langle \psi, f(x)\rangle \geq -2\varepsilon. \tag{2.2}$$

Particularly, when Y^* is strictly convex, for z above, there exists a unique $\varphi \in \partial\|\Phi(z)\| \cap Y^*$, such that

$$|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \leq 2\varepsilon \tag{2.3}$$

Proof: Since $x^* = d\|\cdot\|$ $(\in S_{X^*})$, for any $x \in X$, we have

$$\lim_{t\to+\infty}(\|x + tz\| - t) = \lim_{t\to 0^+} \frac{\|z + tx\| - \|z\|}{t} = \langle x^*, x\rangle. \tag{2.4}$$

For $n \in \mathbb{N}$, take $\varphi_n \in S_{Y^*}$ such that $\langle \varphi_n, f(x + nz)\rangle = \| f(x + nz) \|$, then

$$\begin{aligned}\|f(x + nz)\| &= \langle \varphi_n, f(x + nz)\rangle \\ &= \langle \varphi_n, f(x)\rangle + \langle \varphi_n, f(x + nz) - f(x)\rangle \\ &\leq \langle \varphi_n, f(x)\rangle + \|f(x + nz) - f(x)\| \\ &\leq \langle \varphi_n, f(x)\rangle + (n + \varepsilon)\end{aligned}$$

and then,

$$\lim_{n} \inf(\|f(x + nz)\| - n) \leq \lim_{\mathfrak{U}}\langle \varphi_n, f(x)\rangle + \varepsilon = \langle \varphi, f(x)\rangle + \varepsilon, \tag{2.5}$$

where $\varphi = w^* - lim_{\mathfrak{U}}\, \varphi_n \in B_{Y^*}$. On the other hand,

$$\begin{aligned}\liminf_n (\|f(x+nz)\| - n) &\geq \liminf_n [(\|x+nz\| - \varepsilon) - n] \\ &= \liminf_n \frac{\left\|z + \frac{1}{n}x\right\| - \|z\|}{n^{-1}} - \varepsilon \\ &= \langle x^*, x\rangle - \varepsilon.\end{aligned}$$

Therefore,

$$\liminf_n (\|f(x+nz)\| - n) \geq \langle x^*, x\rangle - \varepsilon. \tag{2.6}$$

Combine (2.5) and (2.6) together

$$\langle x^*, x\rangle - \langle \varphi, f(x)\rangle \leq 2\varepsilon. \tag{2.7}$$

We will prove that $\varphi \in \partial\|\Phi(\mathrm{z})\| \cap Y^*$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} t + \varepsilon &\geq \|f(tz)\| \\ &\geq \langle \varphi_n, f(tz)\rangle \\ &= \langle \varphi_n, f(x+tz)\rangle - \langle \varphi_n, f(x+tz) - f(tz)\rangle \\ &\geq \|f(x+tz)\| - \|f(x+nz) - f(tz)\| \\ &\geq (\|x+nz\| - \varepsilon) - \|x - (n-t)z\| - \varepsilon \\ &\geq t - 2(\|x\| + \varepsilon). \end{aligned}$$

Put n → ∞, then

$$t + \varepsilon \geq \|f(tz)\| \geq \langle \varphi, f(tz)\rangle \geq t - 2(\|x\| + \varepsilon). \tag{2.8}$$

Divide both sides by $t\ (> 0)$, set $t \to +\infty$, and note that $w^* - lim_{\mathfrak{U}} \frac{f(tz)}{t} = \Phi(\mathrm{z})$, then

$$\langle \varphi, \Phi(z)\rangle = \|\Phi(z)\| = 1. \tag{2.9}$$

Therefore, $\varphi \in \partial\|\Phi(\mathrm{z})\| \cap Y^*$. This proves that for any $x \in X$, there exists $\varphi \in \partial\|\Phi(\mathrm{z})\| \cap Y^*$ such that (2.1) holds. On the other hand, for a given $x \in X$, $n \in \mathbb{N}$, let $\psi_n \in S_{Y^*}$ such that

$$\langle \psi_n, f(x+nz) - f(x)\rangle = \|f(x+nz) - f(x)\|,$$

Then

$$\begin{aligned} \|f(x+nz)\| &\geq \langle \psi_n, f(x+nz)\rangle \\ &= \|f(x+nz) - f(x)\| + \langle \psi_n, f(x)\rangle \\ &\geq \|nz - \varepsilon\| + \langle \psi_n, f(x)\rangle \\ &= n - \varepsilon + \langle \psi_n, f(x)\rangle. \end{aligned}$$

Therefore,

$$\liminf_n (\|f(x+nz)\| - n) \geq \langle \psi, f(x)\rangle - \varepsilon, \tag{2.10}$$

where $\psi = w^* - lim_{\mathfrak{U}} \psi_n$. Also due to

$$\begin{aligned} \|f(x+nz)\| - n &\leq (\|x+nz\| + \varepsilon) - n \\ &= \frac{\|z + n^{-1}x\|}{n^{-1}} + \varepsilon \to \langle x^*, x\rangle + \varepsilon, \end{aligned}$$

then

$$\liminf_{n}(\|f(x+nz)\|-n) \le \langle x^*, x\rangle - \varepsilon, \tag{2.11}$$

(2.10) and (2.11) give

$$\langle x^*, x\rangle - \langle \psi, f(x)\rangle \ge -2\varepsilon. \tag{2.12}$$

Note that

$$\begin{gathered} t+\varepsilon \ge \|f(tz)\| \ge \langle\, \psi_n, f(tz)\rangle \\ = \langle\, \psi_n, f(x+tz)-f(x)\rangle - \langle\, \psi_n, f(x+tz)-f(tz)\rangle + \langle\, \psi_n, f(x)\rangle \\ \ge \|f(x+tz)-f(x)\| - \|f(x+tz)-f(tz)\| - \|f(x)\| \\ \ge (\|nz\|-\varepsilon) - (\|x+(n-t)z\|-\varepsilon) - (\|x\|+\varepsilon) \\ \ge t - 2\|x\| - 3\varepsilon, \end{gathered}$$

then we get

$$t+\varepsilon \ge \|f(tz)\| \ge \langle \psi, f(tz)\rangle \ge t - 2\|x\| - 3\varepsilon.$$

Therefore,

$$1 = \lim_{t\to+\infty}\left\langle \psi, \frac{f(tz)}{t}\right\rangle = \langle \psi, \Phi(z)\rangle.$$

Furthermore, since $\langle \psi, \Phi(\mathrm{z})\rangle = \| \Phi(\mathrm{z}) \| = 1$, we know that

$$\psi \in \partial\|\Phi(\mathrm{z})\| \cap Y^*.$$

In this way, we prove that for any $x \in X$, there exists $\psi \in \partial\|\Phi(\mathrm{z})\| \cap Y^*$ such that (2.2) is true. In particular, when Y^* is strictly convex, $\partial\|\Phi(\mathrm{z})\| \cap Y^*$ is a single point set. Therefore, $\varphi = \psi$. Combine (2.1) and (2.2) immediately yields (2.3) and the proof is complete.

Using the lemma above, we get the following better estimation when Y^* is strictly convex space.

Theorem 2.6. Let $f: X \to Y$ be a standard ε-isometry. If Y^* is strictly convex, then for any $x^* \in X^*$, there is $\varphi \in Y^*$ that satisfies $\|\varphi\| \equiv r = \|x^*\|$, such that

$$|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \le 2r\varepsilon, x \in X.$$

Proof: Compared with Lemma 3.3 in [16], except for replacing Lemma 3.2 with Lemma 2.4, and the rest is completely the same.

3. WEAK STABILITY OF *ε*-ISOMETRIES ON REFLEXIVE SPACES

This section reviews the series of results of using the weak stability theorem to study ε-isometries, and at the same time gives a new proof. In recent years, the weak stability estimation has played an irreplaceable role in the study of ε-isometry. For example, when ε = 0, it is Figiel theorem (see [24]); when ε = 0 and the mapping is surjective, it is the Mazur-Ulam theorem (see [1]). Proposition 2.1 is a direct inference of the weak stability estimation. It is a result of the following Qian-Šemrl-Väisälä theorem [18].

Theorem 3.1. Assuming that X and Y are both L_P spaces ($1 < p < \infty$), $f: X \to Y$ is a standard ε-isometry, then there exists a linear operator $T: Y \to X$ with norm 1 such that

$$\|Tf(x) - x\| \leq 2\varepsilon, \forall x \in X. \tag{3.1}$$

Proof: Note that the L_P spaces ($1 < p < \infty$) is reflexive and uniformly smooth and uniformly convex. It is known from Theorem 2.6 that for any $x^* \in X^*$, there is $\varphi \in Y^*$ that satisfies $\|\varphi\| \equiv r = \|x^*\|$, such that

$$|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \leq 2r\varepsilon, x \in X. \tag{3.2}$$

From Lemma 2.2,

$$\Phi(\mathrm{x}) = w^* - \lim_n \frac{f(nx)}{n} = \lim_n \frac{f(nx)}{n}, x \in X \tag{3.3}$$

defines a linear isometry $\Phi: X \to Y$, and $\Phi^*(\varphi) = x^*$. Therefore, $Z \equiv \Phi(X) \subset Y$ is the 1-complemented subspace of Y (see [25]), and $x^* \to \varphi]_Z$ is a linear isometry mapping. Furthermore, $Sx^* = \varphi$ defines a linear isometry mapping, $S: X^* \to Y^*$. Let $T = S^*$, then $T: Y \to X$ is a linear surjective with norm 1. Furthermore,

$$\begin{aligned} 2r\varepsilon &\geq |\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \\ &= |\langle x^*, x\rangle - \langle Sx^*, f(x)\rangle| \\ &= |\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle| \end{aligned}$$

It is easy to see that the inequality above is equivalent to (3.1). To make the discussion of stability consistent, we will first prove a lemma.

Lemma 3.2. Suppose X is an infinite-dimensional separable Banach space, then there is a linearly independent subset $S \subset X$ such that $\overline{S} = X$.

Proof: Arbitrarily take a countable dense subset of X, $(x_n)_{n=1}^{\infty}$, let $X_0 = \mathrm{span}(x_n)_{n=1}^{\infty}$. Let $I_0 \subset (x_n)_{n=1}^{\infty}$ is a linearly independent maximal subset of X_0, then there is a linearly independent subset $I_1 \equiv (z_\alpha) \subset S_X$ of X such that $X_0 \cap X_1 = \{0\}$, and $I = I_0 \cup I_1$ is a linearly independent maximal subset of X, where $X_1 = \mathrm{span}(I_1)$. In this way, we know that $X = X_0 \oplus X_1$ (algebraic direct sum). Take a countable infinite number of disjoint countable infinite subsets from $\left(I_{1,n}\right)_{n=1}^{\infty}$ (where $I_{1,n} = \left(z_{n,j}\right)_{n=1}^{\infty}$), and let $J_n = \left\{\frac{z_{n,j}}{j} : j \in \mathbb{N}\right\}$. Then, let $S = \bigcup_{n\in\mathbb{N}}\{x_n + J_n\}$ is enough.

Theorem 3.3. If Y is reflexive, then the stability of standard ε-isometry $f: X \to Y$ is determined separately, that is, the sufficient and necessary conditions of f to be stable is that there exists $\beta, \gamma > 0$, such that for every separable subspace E of X, a linear operator $T_E: E \to Y$ satisfy

$$\|T_E\| \leq \beta, \|T_E f(x) - (x)\| \leq \gamma\varepsilon, x \in E \tag{3.4}$$

Proof: We only need to prove the sufficient condition. By Lemma 2. 2, for any free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, $\Phi(\mathrm{x}) = w - \lim_{\mathfrak{U}} \frac{f(nx)}{n}$ $(x \in X)$, defines an isometry $\Phi: X \to Y$. According to Figiel theorem [24], there exists a linear operator $F:\overline{\text{span}}(\Phi(X)) \to X$ with the norm 1 such that $F \circ \Phi = id_X$. Let E be a separable subspace of X. According to the assumption, there is a linear operator $T_E: E \to Y$ with a norm of no more than β, $\beta > 0$, such that

$$\|T_E f(x) - (x)\| \leq \gamma\varepsilon, x \in E.$$

We use $\mathfrak{J}$ to represent the family of sets that consists of all finite-dimensional subspace of X. The power set $2^{\mathfrak{J}}$ contains the inclusion relation of subsets forms a filter $\mathfrak{F}$ on $\mathfrak{J}$. We denote the ultrafilter containing the filter $\mathfrak{F}$ as $\mathfrak{U}_{\mathfrak{J}}$, then

$$w - \lim_{\mathfrak{U}_{\mathfrak{J}}} T_F(y),\ y \in Y_f = \overline{\text{span}}[f(x)]$$

defines a linear operator $T: X \to Y$, and satisfies

$$\|T\| \leq \beta, \|Tf(x) - (x)\| \leq \gamma\varepsilon, x \in X$$

which is what desired in 3.4 and the proof is complete.

Acknowlegment: *The authors thank OMU Functional Analysis and Function Theory Research Group (OFAFTReG) for helpful discussions. The authors also thank to Prof. Cenap Duyar, Prof. Birsen Duyar, and Dr. Halil Mutuk for their meaningful suggestions. The authors also thank the reviewers for their comments and suggestions in this article.*

REFERENCES

[1] Benyamini, Y., Lindenstrauss, J., *Geometric nonlinear Functional Analysis I*, Colloquium publications 48, American Mathematical Society, Providence, 11, 2000.
[2] Mazur, S., Ulam S., *Comptes Rendus de I'Académie des Sciences-Series I-Mathematics*, **194**, 946, 1932.
[3] Hyers, D. H., Ulam, S. M., *Bulletin of the American Mathetemtaical Society*, **51,** 288, 1945.
[4] Bhatia, R., Semrl, P., *American Mathematical Monthly*, **104**, 497, 1997.
[5] Bourgin, D. G., *Bulletin of the American Mathetemtaical Society*, **52**, 704, 1946.
[6] Gevirtz, J., *Proceedings of the American Mathetemtaical Society*, **89**, 633, 1983.
[7] Gruber, P. M., *Transactions of the American Mathetemtaical Society*, **245**, 263, 1978.
[8] Omladic, M., Semrl, P., *Mathematische Annalen*, **303**, 617, 1995.
[9] Zhou, Y., Zhang, Z., Liu, C., *Studia Mathematica*, **230**, 31, 2015.
[10] Cheng, L., Cheng, Q., Tu, K., Zhang, J., *Journal of Functional Analysis*, **269**, 199, 2015.
[11] Cheng, L., Dong, Y., *Proceedings of the American Mathetemtaical Society*, **148**, 4837, 2020.
[12] Cheng, L., Dong, Y., Zhang, W., *Journal of Functional Analysis*, **264**, 713, 2013.
[13] Cheng, L., Fang, Q., Kato, M., Sun, L., *Pure and Applied Functional Analysis*, **5**(3), 555, 2020.
[14] Qian, S., *Proceedings of the American Mathetemtaical Society*, **123**, 1797, 1995.
[15] Rohman, M., *Cauchy: Jurnal Matematika Murni dan Aplikasi*, **4**, 167, 2017.

[16] Rohman, M., Eryılmaz, İ., *Avrupa Bilim ve Teknoloji Dergisi*, **34**, 110, 2022.
[17] Rohman, M., Wibowo, R. B. E., Marjono, *Australian Journal of Mathematical Analysis and Applications*, **13**, 1, 2016.
[18] Šemrl, P., Väisälä, J., *Journal of Functional Analysis*, **198**, 268, 2003.
[19] Väisälä, J., *The American Mathematical Monthly*, **110**, 633, 2003.
[20] Vestfrid, I. A., *Journal of Functional Analysis*, **269**, 2165, 2015.
[21] Vestfrid, I. A., *Proceedings of the American Mathetemtaical Society*, **145**, 2481, 2017.
[22] Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V., *Banach Space Theory : The Basis for Linear and Nonlinear Analysis*, Springer, New York, 337-343, 2010.
[23] Megginson, R. E., *An Introduction to Banach Space Theory*, Springer, New York, 280, 1991.
[24] Figiel, T., *Bulletin L'Acadé Polonaise des Science, Série des Sciences Mathématiques, Astronomiques et Physiques*, **16**, 185, 1968.
[25] Lacey, H. E., *The Isometric Theory of Clasical Banach Spaces*, Springer, New York, 162, 1974.