ORIGINAL PAPER

THE GEODESIC CURVES ON THE OSCILLATOR GROUP OF DIMENSION FOUR

DERKAOUI RAFIK[1], MEDJAHDI BRAHIM[1]

Manuscript received: 07.08.2022; Accepted paper: 19.05.2023;
Published online: 30.06.2023.

Abstract. *Our work is study the geometry of oscillator groups; they are the only non commutative simply connected solvable Lie groups which have a biinvariant Lorentzian metric. The oscillator group has been generalized to one dimension equals* $2n \geq 4$*, and several aspects of its geometry have been intensively studied, both in differential geometry and in mathematical physics. In this paper, we find geodesic curves on the oscillator group* (G_λ, g_a) *of dimension four.*

Keywords: *Oscillator group; Lorentz metric; Levi-Civita connection; Geodesic curves*

1. INTRODUCTION

In mathematics, differential geometry is the application of the tools of differential calculus to the study of geometry. The basic objects of study are the differential varieties, sets having a sufficient regularity to consider the notion of derivation, and the functions defined on these manifold. Differential geometry finds its main physical application in the theory of general relativity where it allows a modeling of a curvature of space-time.

Riemannian geometry is a branch of differential geometry named after the mathematician Bernhard Riemann, who introduced the founding concept of geometric variety. It extends the methods of analytical geometry by using local coordinates to carry out the study of curved spaces on which exist notions of angle and length.

The most notable concepts of Riemannian geometry are the curvature of the studied space and geodesics, curves solving a shortest path problem on this space. More generally, Riemannian geometry aims at the local and global study of Riemannian manifolds. In mathematics, and more precisely in geometry, a Riemannian manifold is a differentiable manifold having an additional structure (a Riemannian metric) allowing to define the length of a path between two points of the manifold i.e. the differentiable manifolds equipped with a Riemannian metric.

Our work is devoted to the study of the Lorentzian geometry of the oscillator group (G_λ, g_a) of dimension 4. Connected Lie groups that admit a bi-invariant Lorentzian metric were determined by the first of the authors in [1]. Among them, those that are solvable, non-commutative, and simply connected are called oscillator groups.

We study here the geometry of these groups and their networks, i.e their discrete sub-groups co-compact. If G is an oscillator group, its networks determine compact homogeneous. Lorentz manifolds, on which G acts by isometries.

[1] Oran Ammour Ahmed Higher Normal School, Department of Exact Sciences, Oran, Algeria.
E-mail: rafikderkaoui27@yahoo.com.

Let $H_{2k+1} = \mathbb{R} \times \mathrm{C}^k$ be the Heisenberg group and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ k be strictly positive real numbers. Let the additive group $\mathbb{R}$ act on H_{2k+1} by the action:

$$\rho(t)(u,(z_j)) = (u,(e^{i\lambda_j t} z_j)).$$

The group $G_k(\lambda)$, a semi-direct product of $\mathbb{R}$ by H_{2k+1} following ρ, is provided with a bi-invariant Lorentz metric. Here is how it is built:

$$g = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2k}$$

is the tangent space at the origin. Let us extend the usual scalar product of $\mathbb{R}^{2k}$ into a symmetric bilinear form over g so that the plane $\mathbb{R} \times \mathbb{R}$ is hyperbolic and orthogonal to $\mathbb{R}^{2k}$. This form defines an invariant Lorentz metric on the left on $G_k(\lambda)$, it is also invariant on the right because the adjoint operators on g are antisymmetric [2].Groups $G_k(\lambda)$ are characterized [1] by:

Theorem1.1. The groups $G_k(\lambda)$ are the only groups of Lie simply related, resolvable and noncommutative which admit a bi-invariant Lorentz metric.

Since [3-5], the oscillator group has been generalized to a dimension equal to an even number $2n$ with $n \geq 2,$ plus this provides a known example of homogeneous space-time [6]. For $n = 2$, the oscillator group of dimension 4 admits a Lorentzian metric invariant on the left and on the right (bi-invariant). This bi-invariant metric has been generalized a family g_a, $-1 < a < 1,$ invariant Lorentzian metrics on the left. For a = 0, the metric g_0 become or the only example of Lorentzian bi-invariant metric [7].

The researchers Giovani and Zaeim extracted three vectors feilds from the oscillator group, which are: Killing vector feild, Affine vector feild, parallel vector feild (see [7]). And also Giovani and Zaeim classified the totally geodesic and parallel hypersurfaces of four-dimensional groups (see [5]).Our work based explicitly on the geodesic curves on the oscillator group (G_λ, g_a). We will give you a few reminder about geodisie curves.

2. PRELIMINARY RESULT

At the moment we consider on G_λ a family parametre of left-invariantLorentzian metrics g_a. With respect to coordinates (x_1, x_2, x_3, x_4), this metric g_a is explicitly given by

$$g_a = adx_1^2 + 2ax_3 dx_1 dx_2 + (1 + ax_3^2) dx_2^2 + dx_3^2 + 2dx_1 dx_4 + 2x_3 dx_2 dx_4 + adx_4^2,$$

with $-1 < a < 1,$

Note that for $a = 0$ and $\lambda = 1$ we have the bi-invariant metric on the oscillator group G_1 [7]. In all other cases, g_a is only invariant on the left.

The matrix of the metric g_a is given by

$$A_a = \begin{pmatrix} a & ax_3 & 0 & 1 \\ ax_3 & 1+ax_3^2 & 0 & x_3 \\ 0 & 0 & 1 & 0 \\ 1 & x_3 & 0 & a \end{pmatrix}.$$

So

$$A_a^{-1} = \begin{pmatrix} \frac{a}{a^2-1}+x_3^2 & -x_3 & 0 & -\frac{1}{a^2-1} \\ -x_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{a^2-1} & 0 & 0 & \frac{a}{a^2-1} \end{pmatrix}.$$

Levi-Civita connection

Theorem 2.1. Let (M,g) be a pseudo-Riemannian manifold. There is one and only one torsion-free connection ∇ on M for which g is parallel, that is, $\nabla g = 0$. This connection is called the Levi-Civita connection of *(M, g)*.

Unless explicitly stated otherwise, any pseudo-Riemannian manifold will be endowed with its Levi-Civita connection. To say that it is untwisted means that

$$\nabla_X Y - \nabla_Y X = [X,Y],$$

for all $X,Y \in \chi(M)$ (the set of vector fields). For all $X,Y,Z \in \chi(M)$. This condition is also called the compatibility condition of ∇ with metric g.

Proposition 2.1. The Levi-Civita connection on a pseudo Riemannian manifold (M,g) satisfies the following formula:

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[X,Z]) \\ +g(Y,[Z,X]) + g(Z,[X,Y]),$$

for all $X,Y,Z \in \chi(M)$. This is the **Koszul formula**.

Christoffel symbols

Using Koszul formula, a simple calculation shows us that, in a map of local coordinates $(x_1,...,x_n)$, the christoffel symbols Γ_{ij}^k are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} (\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{im} - \frac{\partial}{\partial x_m} g_{ij}),$$

for all $i,j,k \in \{1,...,n\}$ where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and g^{ij} denote the components of the inverse matrix of (g_{ij}) .

Remark 2.1 The condition that ∇ is torsion-free is equivalent to that, for all $1 \le i,j,k \le n$, we have

$$\Gamma_{ij}^{k}=\Gamma_{ji}^{k}.$$

The christoffel symbols on (G_λ, g_a)

The Christoffel symbols $\left\{\Gamma_{ij}^{k}\right\}_{i,j,k=1,..4}$, which are not always zero, are given by

$$\begin{aligned}
&\Gamma_{12}^{3}=-\tfrac{1}{2}a,\ \Gamma_{13}^{1}=-\frac{ax_3}{2},\ \Gamma_{13}^{2}=\tfrac{a}{2},\\
&\Gamma_{22}^{3}=-ax_3,\ \Gamma_{23}^{1}=\frac{1-ax_3^2}{2},\ \Gamma_{23}^{2}=\frac{ax_3}{2},\\
&\Gamma_{34}^{1}=-\frac{x_3}{2},\ \Gamma_{34}^{2}=\tfrac{1}{2},\ \Gamma_{24}^{3}=-\tfrac{1}{2}.
\end{aligned}\tag{1}$$

Definition 2.1. Let (M,∇) be a differentiable manifold endowed with a linear connection. Let $\gamma : I\subset\mathbb{R}\to M$ be a curve on M and let $Y(t)$ be a vector field tangent to M at $\gamma(t)$, for $t\in I$, which varies differentiably witht. We say that $Y(t)$ is parallel along $\gamma(t)$ if

$$\nabla_{\gamma'(t)}Y(t)=0.$$

In local coordinates $x_1, x_2, ..., x_m$ (*m* being the dimension of *M*), if $\gamma'(t)=\gamma'^{i}(t)\frac{\partial}{\partial x_i}_{\gamma(t)}$ and $Y(t)=Y^{j}(t)\frac{\partial}{\partial x_j}_{\gamma(t)}$, the condition of parallelism of Y along γ is written:

$$\frac{d}{dt}Y^{k}(t)+\Gamma_{ij}^{k}\,\gamma'^{i}(t)Y^{j}(t)=0,\ \forall t\in I$$

where Γ_{ij}^{k} are the Christoffel symbols of ∇ .

Definition 2.2. The curve γ is said to be a geodesic if the velocity vector of γ is parallel along γ (i.e. $\nabla_{\gamma'(t)}\gamma'(t)=0$), or else

$$\frac{d^2\gamma^k}{dt}+\Gamma_{ij}^{k}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}=0,\ \text{for all } i,j,k\in\{1,2,...,m\}$$

it is a system of second-order ordinary differential equations called the geodesic equations.

Proposition Given a pseudo Riemannian manifold (M,g,∇), ∇ being the Riemannian connection and γ a curve on M then γ is a geodesic on (M,g) if and only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_i'}\right)-\frac{\partial L}{\partial x_i}=0,\ \text{for } i\in\{1,2,...,\dim M\},$$

where $L=\frac{1}{2}g_{kj}\gamma'^{k}\gamma'^{j}$ for $k,j\in\{1,2,...,\dim M\}$, the equations $\frac{d}{dt}(\frac{\partial L}{\partial x_i'})-\frac{\partial L}{\partial x_i}=0$ are called Euler equations of the Lagrangian L .

3. RESULTS AND DISCUSSION

In this section, we write explicitly the equations of the geodesics of the oscillator group G_λ, for the left-invariant Lorentzian metric g_a , as well as their resolutions.

Theorem 3.1. Let $\gamma\ :\ t \to \gamma(t)=(x(t),y(t),z(t),w(t))$ be a geodesic curve on the oscillator group of dimension four (G_λ, g_a) and let $\gamma(0)=(x_0,y_0,z_0)$ and $\dot\gamma(0)=(\dot x_0,\dot y_0,\dot z_0)$.
If $ac_1+w_1\neq 0,$ then, any geodesic on (G_λ, g_a) is written in the form:

$$
\begin{aligned}
x(t) &= \frac{(y_0')^2-(z_0')^2}{4\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)^2}\sin\left(2\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)\\
&+\left(\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)\right)\frac{(y_0')^2+(z_0')^2}{2\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)^2}+x_0'+z_0y_0'\right)t\\
&-\frac{(y_0')^2-\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)z_0y_0'}{\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)^2}\sin\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)\\
&-\frac{z_0'y_0'+\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)z_0'z_0}{\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)^2}\cos\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)\\
&+\frac{z_0'y_0'}{\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)^2}\cos^2\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)+x_0+\frac{z_0^2}{a\left(x_0'+z_0y_0'\right)+w_0'},
\end{aligned}
$$

and

$$
\begin{aligned}
y(t) &= \frac{y_0'}{a\left(x_0'+z_0y_0'\right)+w_0'}\sin\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)\\
&+\frac{z_0'}{a\left(x_0'+z_0y_0'\right)+w_0'}\cos\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)+y_0-\frac{z_0'}{a\left(x_0'+z_0y_0'\right)+w_0'},\\
z(t) &= -\frac{y_0'}{a\left(x_0'+z_0y_0'\right)+w_0'}\cos\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)\\
&+\frac{z_0'}{a\left(x_0'+z_0y_0'\right)+w_0'}\sin\left(\left(a\left(x_0'+z_0y_0'\right)+w_0'\right)t\right)+\frac{y_0'}{a\left(x_0'+z_0y_0'\right)+w_0'}+z_0,\\
w(t) &= w_0't+w_0.
\end{aligned}
$$

If $ac_1+w_1=0,$ then, any geodesic on (G_λ, g_a) is written in the form:

$$\begin{aligned} x(t) &= -\frac{y_0' z_0'}{2} t^2 + x_0' t + x_0, \\ y(t) &= y_0' t + y_0, \\ z(t) &= z_0' t + z_0, \\ w(t) &= -a\left(x_0' + z_0 y_0'\right)t + w_0. \end{aligned}$$

Proof: Now let $t \to \gamma(t) = (x(t), y(t), z(t), w(t))$ be a curve on the Oscillator group (G_λ, g_a).

A direct calculation shows us that γ is a geodesic on (G_λ, g_a) if and only if, the following system is satisfied

$$\begin{cases} x''(t) - az(t)x'(t)z'(t) + \left(1 - az^2\right)y'(t)z'(t) - z(t)z'(t)w_1 = 0, \\ y''(t) + ax'(t)z'(t) + az(t)y'(t)z'(t) + w_1 z'(t) = 0, \\ z''(t) - ax'(t)y'(t) - az(t)\left(y'(t)\right)^2 - w_1 y'(t) = 0, \\ w(t) = w_1 t + w_2, \end{cases} \tag{2}$$

where $w_1, w_2 \in \mathbb{R}$.

The first and second equations of the system (2) imply $x' + zy' + y'z' = 0$. Integrating by parts, we get

$$x' = -zy' + c_1, \; c_1 \in \mathbb{R}. \tag{3}$$

Thus (2) reduces to

$$\begin{cases} z(t)\left(y''(t) + \left(ac_1 + w_1\right)z'(t)\right) = 0, \\ y''(t) + \left(ac_1 + w_1\right)z'(t) = 0, \\ z''(t) - \left(ac_1 + w_1\right)y'(t) = 0, \\ w(t) = w_1 t + w_2, \end{cases} \tag{4}$$

From the second equation of the system (4), we deduce

$$y' = -\left(ac_1 + w_1\right)z + c_2, \; c_2 \in \mathbb{R}. \tag{5}$$

It follows that

$$z'' + \left(ac_1 + w_1\right)^2 z - c_2\left(ac_1 + w_1\right) = 0.$$

Therefore, assuming that $ac_1 + w_1 \neq 0,$ we show that

$$z(t) = c_3 \cos\left(\left(ac_1 + w_1\right)t\right) + c_4 \sin\left(\left(ac_1 + w_1\right)t\right) + \frac{c_2}{ac_1 + w_1},$$

where $c_3, c_4 \in \mathbb{R}$.

On the other hand, by replacing the expression of $z(t)$ in (5) we will have

$$y'(t)=-c_3(ac_1+w_1)\cos((ac_1+w_1)t)-c_4(ac_1+w_1)\sin((ac_1+w_1)t).$$

We then have

$$y(t)=-c_3\sin((ac_1+w_1)t)+c_4\cos((ac_1+w_1)t)+\alpha,\ \alpha\in\mathbb{R}.$$

So (3) becomes

$$x'(t)=\lambda c_3^2\cos^2((ac_1+w_1)t)+\lambda c_4^2\sin^2((ac_1+w_1)t)+c_2c_3\cos((ac_1+w_1)t)$$
$$+c_2c_4\sin((ac_1+w_1)t)+2\lambda c_3c_4\cos((ac_1+w_1)t)\sin((ac_1+w_1)t)+c_1.$$

So

$$x(t)=\frac{c_3^2-c_4^2}{4}\sin(2(ac_1+w_1)t)+\left((ac_1+w_1)\frac{c_3^2+c_4^2}{2}+c_1\right)t$$

$$-\frac{c_2c_4}{ac_1+w_1}\cos((ac_1+w_1)t)-c_3c_4\cos^2((ac_1+w_1)t)+$$

$$+\frac{c_2c_3}{ac_1+w_1}\sin((ac_1+w_1)t)\beta,$$

where $\beta\in\mathbb{R}$.

It remains to determine the constants $c_1,..,c_4,\alpha,\beta$. Let us then set

$$\gamma(0)=(x_0,y_0,z_0,w_0)\text{ and }\gamma'(0)=(x_0',y_0',z_0',w_0').$$

From the equation (3), we will have $c_1=x_0'+z_0y_0'$, $w(0)=w_0=w_2$ and $w_1=w_0'$. Moreover, as

$$y'(t)=-(ac_1+w_1)z(t)+c_2,$$

we will have

$$c_2=y_0'+(a(x_0'+z_0y_0')+w_0')z_0.$$

Moreover, we have

$$y'(0)=y_0'=-c_3(ac_1+w_1)\text{ and }z'(0)=z_0'=c_4(ac_1+w_1).$$

We then have

$$c_3=-\frac{y_0'}{a(x_0'+z_0y_0')+w_0'}\text{ and }c_4=\frac{z_0'}{a(x_0'+z_0y_0')+w_0'}.$$

On the other hand, $y(0)=y_0=c_4+\alpha$ and $x(0)=x_0=-\frac{c_2c_4}{ac_1+w_1}-c_3c_4+\beta$. Subsequently,

$$\alpha = y_0 - \frac{z_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'} \text{ and } \beta = x_0 + \frac{z_0^2}{a\left(x_0' + z_0 y_0'\right) + w_0'}.$$

Therefore, when $a\left(x_0' + z_0 y_0'\right) + w_0' \neq 0$, any geodesic on $\left(G_\lambda, g_a\right)$ will be given by

$$x(t) = \frac{\left(y_0'\right)^2 - \left(z_0'\right)^2}{4\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)^2} \sin\left(2\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right)$$

$$+\left(\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)\right) \frac{\left(y_0'\right)^2 + \left(z_0'\right)^2}{2\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)^2} + x_0' + z_0 y_0'\right)t$$

$$-\frac{\left(y_0'\right)^2 - \left(a\left(x_0' + z_0 y_0'\right) + w_0'\right) z_0 y_0'}{\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)^2} \sin\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right)$$

$$-\frac{z_0' y_0' + \left(a\left(x_0' + z_0 y_0'\right) + w_0'\right) z_0' z_0}{\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)^2} \cos\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right)$$

$$+\frac{z_0' y_0'}{\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)^2} \cos^2\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right) + x_0 + \frac{z_0^2}{a\left(x_0' + z_0 y_0'\right) + w_0'},$$

and

$$y(t) = \frac{y_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'} \sin\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right)$$

$$+\frac{z_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'} \cos\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right) + y_0 - \frac{z_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'},$$

$$z(t) = -\frac{y_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'} \cos\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right)$$

$$+\frac{z_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'} \sin\left(\left(a\left(x_0' + z_0 y_0'\right) + w_0'\right)t\right) + \frac{y_0'}{a\left(x_0' + z_0 y_0'\right) + w_0'} + z_0,$$

$$w(t) = w_0' t + w_0.$$

If $ac_1 + w_1 = 0,$ the system (4) reduces to

$$\begin{cases} y''(t) = 0, \\ z''(t) = 0, \\ w(t) = w_1 t + w_2. \end{cases}$$

It results

$$y(t) = a_1 t + a_2,\ a_1, a_2 \in \mathbb{R},$$

$$z(t) = b_1 t + b_2,\ b_1, b_2 \in \mathbb{R}.$$

From the equation we show

$$x(t) = -\frac{a_1 b_1}{2} t^2 + \left(c_1 - a_1 b_2\right) t + \mu,\ c_1, \mu \in \mathbb{R}.$$

It remains to determine the constants a_1, a_2, b_1, b_2, c_1 for the initial conditions $\gamma(0) = (x_0, y_0, z_0, w_0)$ and $\gamma'(0) = (x_0', y_0', z_0', w_0')$. We then have,

$$x(0) = x_0 = \mu \text{ and } c_1 = x_0' + z_0 y_0',$$

$$y(0) = y_0 = a_2 \text{ and } y'(0) = y_0' = a_1,$$

$$z(0) = z_0 = b_2 \text{ and } z'(0) = z_0' = b_1.$$

Therefore, when $a\left(x_0' + z_0 y_0'\right) + w_0' = 0$, any geodesic on $\left(G_\lambda, g_a\right)$ will be given by

$$x(t) = -\frac{y_0' z_0'}{2} t^2 + x_0' t + x_0,$$

$$y(t) = y_0' t + y_0,$$

$$z(t) = z_0' t + z_0,$$

$$w(t) = -a\left(x_0' + z_0 y_0'\right) t + w_0.$$

4. CONCLUSION

In this work we study the Lorentzian geometry of the oscillator group of dimension 4 and their geodesic curves. This group can be described as a semi-direct product $\mathbb{R} \times H$, H being the three-dimensional Heisenberg group. We found two cases in our solution to the system which gives us all the geodesic curves on this group $\left(G_\lambda, g_a\right)$.

REFERENCES

[1] Medina A., *Seminary. Differential Geometry,* 44-51, 1982.
[2] Milnor J., *Advances in Mathematics*, **21**, , 283-329, 1976.
[3] Streater R.F., *Communications in Mathematical Physics*, **4**, 217-236, 1967.
[4] Boucetta, M., Medina, A., *Journal of Geometry and Physics,* **61**, 2309-2320, 2011.
[5] Bromberg, S.,Medina, A., *Geometriae Dedicata*, **106,** 97-111, 2004.
[6] Calvaruso, G., Van der Veken, J.,*Results Mathematics*, **64**, 135-153, 2013.
[7] Duran Diaz, R., Gadea, P.M., Oubina, J.A., *LIBERTAS MATHEMATICA*., **19**, 9-18, 1999.
[8] Gadea, P.M., Oubina, J.A.,*Archiv der Mathematik*, **73,** 311-320, 1999.
[9] Calvaruso, G., Zaeim, A., *Collectanea Mathematica*, **68**, 51, 2017.