ORIGINAL PAPER

NEW CONCENTRATION INEQUALITIES AND COMPLETE CONVERGENCE FOR ELNQD RANDOM VARIABLES WITH APPLICATION TO LINEAR MODELS GENERATED BY ELNQD ERRORS

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Abstract. *In this paper, we introduce the concept of extended linear negative quadrant dependence (ELNQD, in short). We establish a new concentration inequalities and complete convergence for the distribution of sums of extended linear negative quadrant dependent random variables. Using these inequalities for proved the complete convergence of first autoregressive processes model generated by identically distributed ELNQD errors.*

Keywords: *autoregressive processes; Random variables; ELNQD sequence; complete convergence.*

1. INTRODUCTION

The concept of negative dependence random variable in the bivariate cas was introduced by [1]: two random variables $X_1$and X_2 are said to be negatively quadrant dependent (NQD, in short), if for all real numbers x_1, x_2

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) \leq \mathbb{P}(X_1 \leq x_1)\mathbb{P}(X_2 \leq x_2),$$

or

$$\mathbb{P}(X_1 > x_1, X_2 > x_2) \leq \mathbb{P}(X_1 > x_1)\mathbb{P}(X_2 > x_2).$$

Joag-Dev and Proschan (in [2]) established stronger concept than extended negative quadrant dependence (NQD): A sequence $(X_i, 1 \leq i \leq n)$ of random variables is said to be negatively lower orthant dependent (NLOD, in short) if for all real numbers $x_1,\dots, x_n$

$$\mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) \leq \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i),$$

and it is said to be negatively upper orthant dependent (NUOD, in short) if for all real numbers $x_1,\dots, x_n$

$$\mathbb{P}(X_1 > x_1, \dots, X_n > x_n) \leq \prod_{i=1}^{n} \mathbb{P}(X_i > x_i).$$

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If a sequence $(X_i, 1 \leq i \leq n)$ of random variables is both NLOD and NUOD, then it is said to be negatively orthant dependent (NOD, in short). [3] formulated a nother notion of negative dependence as follows:

– A sequence $(X_i, 1 \leq i \leq n)$ of random variables is said to be linearly negative quadrant dependent (LNQD, in short) if for any disjoint subsets A and B of $\{1, \dots, n\}$ and positive r_j's , $\sum_{i \in A} r_i X_i$, $\sum_{j \in B} r_j X_j$ are NQD.

Next, [4] introduced the concept of extended negative dependence in the multivariate case: A sequence $(X_i, 1 \leq i \leq n)$ of random variables is said to be extended negative lower orthant dependent (ENLOD, in short) if for all real numbers $x_1, \dots, x_n$, there exists a constant $M > 0$ such that

$$\mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) \leq M \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i),$$

and it is said to be extended negative upper orthant dependent (ENUOD, in short) if for all real numbers $x_1, \dots, x_n$, there exists a constant $M > 0$ such that

$$\mathbb{P}(X_1 > x_1, \dots, X_n > x_n) \leq M \prod_{i=1}^{n} \mathbb{P}(X_i > x_i).$$

A sequence $(X_i, 1 \leq i \leq n)$ of random variables is said to be extended negatively orthant dependent (ENOD) if it is both ENUOD and ENLOD. Based on the concept of by NQD and ENOD we consider the concept of extended negative dependence in the bivariate case as following: two random variables $X_1$and X_2 are said to be extended negatively quadrant dependent (ENQD, in short), if there exists a constant $M > 0$ such that for all real numbers x_1, x_2

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) \leq M\mathbb{P}(X_1 \leq x_1)\mathbb{P}(X_2 \leq x_2),$$

or

$$\mathbb{P}(X_1 > x_1, X_2 > x_2) \leq M\mathbb{P}(X_1 > x_1)\mathbb{P}(X_2 > x_2).$$

And by definitions of extended negative quadrant dependence and linear negative quadrant dependence, another notion of extended negative dependence can be formulated as follows:

– A sequence $(X_i, 1 \leq i \leq n)$ of random variables is said to be extended linearly negative quadrant dependent (ELNQD, in short) if for any disjoint subsets A and B of $\{1, \dots, n\}$ and positive r_j's , $\sum_{i \in A} r_i X_i$, $\sum_{j \in B} r_j X_j$ are ENQD.

The purpose of the present paper is to establish a new probability inequality and complete convergence for ELNQD random variables, and to extend and improve the results of [5-7].

In the paper, let $(X_n,\ n \geq 1)$ be a sequence of random variables defined on a probability space (Ω, A, P) and $S_n = \sum_{i=1}^{n} X_n$.

2. MATERIALS AND METHODS

Lemma 2.1. [4] Let two random variables X and Y be ENQD, then:

i. If f and g are both nondecreasing (or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are ENQD,

ii. If X and Y are nonnegative random variables , then there exists a constant $M > 0$ such that $\mathbb{E}(XY) \leq M\mathbb{E}(X)\mathbb{E}(Y)$,

iii. Especially, there exists a constant $M > 0$ such that for any real number h, $\mathbb{E}(e^{h(X+Y)}) \leq M\mathbb{E}(e^{hX})\mathbb{E}(e^{hY})$.

Corollary 2.1. [6] Let $(X_n, n \geq 1)$ be a sequence of ELNQD random variables and t >0. Then for each n≥ 1, there exists a constant M > 0 such that

$$\mathbb{E}\left(\prod_{i=1}^{n} e^{t\mathrm{Xi}}\right) \leq M \prod_{i=1}^{n}\left(\mathbb{E}e^{t\mathrm{Xi}}\right) \tag{1}$$

Lemma 2.2. [5] For any x ∈ ℝ, 0<α≤1 , we have

$$e^{x} \leq 1 + x + |x|^{1+\alpha}\, e^{2|x|} \tag{2}$$

Lemma 2.3. Let $(X_n, n \geq 1)$ be a sequence of ELNQD random variables with $\mathbb{E}(X_n) = 0$, for eash $n \geq 1$. If there exists a sequence of positive numbers $(a_n, n \geq 1)$ such that $|X_n| \leq a_i$, and $\max_{i\geq 1} a_i$ exists for each $i \geq 1$, then for any λ> 0,

$$\mathbb{E}\left(e^{\lambda S_n}\right) \leq Mexp\left(\lambda^{1+\alpha}\ e^{2\lambda \max\limits_{i\geq 1} a_i} \sum_{i=1}^{n} \mathbb{E}|X_i|^{1+\alpha}\right) \tag{3}$$

Proof: By lemma 2.2, $\mathbb{E}(X_n) = 0$, $|X_n| \leq a_i$, for each $i \geq 1$ and $1 + x \leq e^{x}$, we have

$$\begin{aligned} \mathbb{E}\left(e^{\lambda X_i}\right) &\leq\ \mathbb{E}\left(1 + \lambda X_i + \lambda^{1+\alpha}|X_i|^{1+\alpha}\, e^{2\lambda|X_i|}\right) \\ &= 1 + \lambda\mathbb{E}(X_i) + \lambda^{1+\alpha}\mathbb{E}(\ |X_i|^{1+\alpha}\, e^{2\lambda|X_i|}) \\ &\leq \exp(\lambda^{1+\alpha}\mathbb{E}(\ |X_i|^{1+\alpha}\, e^{2\lambda|X_i|}) \\ &\leq \exp(\lambda^{1\,+\alpha}\, e^{2\lambda \max_{i\geq 1} a_i}\mathbb{E}\ |X_i|^{1+\alpha}\ ). \end{aligned} \tag{4}$$

By corollary 2.1 and (4) we have can see that

$$\begin{aligned} \mathbb{E}\left(e^{\lambda \sum_{i=1}^{n} X_i}\right) &= \mathbb{E}\left(\prod_{i=1}^{n} e^{\lambda X_i}\right) \\ &\leq\ M \prod_{i=1}^{n}\left(\mathbb{E}e^{\lambda X_i}\right) \\ &\leq\ M \prod_{i=1}^{n} \exp\left(\lambda^{1\,+\alpha}\, e^{2\lambda \max\limits_{i\geq 1} a_i}\mathbb{E}\ |X_i|^{1+\alpha}\ \right) \\ &\leq\ M\exp\left(\lambda^{1+\alpha}\ e^{2\lambda \max\limits_{i\geq 1} a_i} \sum_{i=1}^{n} \mathbb{E}|X_i|^{1+\alpha}\right) \blacksquare \end{aligned}$$

Lemma 2.4. Let $(X_n, n \geq 1)$ be a sequence of ELNQD random variables with $\mathbb{E}(X_n) = 0$, for eash $n \geq 1$, if there exists a sequence of positive numbers $(a_n, n \geq 1)$ such that $|X_n| \leq a_i$, for each $i \geq 1$, then for any λ> 0 and ε>0

$$\mathbb{P}(|S_n| \geq \varepsilon) \leq 2M\, e^{-\lambda\varepsilon} exp\left(\lambda^{1+\alpha}\ e^{2\lambda \max_{i\geq 1} a_i} \sum_{i=1}^{n} \mathbb{E}|X_i|^{1+\alpha}\right) \tag{5}$$

Proof: Markov's inequality and lemma 2.3, we can see that

$$\begin{aligned} \mathbb{P}(S_n \geq \varepsilon) &= \mathbb{P}(e^{\lambda S_n} \geq e^{-\lambda\varepsilon}) \\ &\leq e^{-\lambda\varepsilon}\mathbb{E}\left(\prod_{i=1}^{n} e^{\lambda X_i}\right) \\ &\leq M\, e^{-\lambda\varepsilon}\exp\left(\lambda^{1+\alpha}\ e^{2\lambda \max_{i\geq 1} a_i} \sum_{i=1}^{n} \mathbb{E}|X_i|^{1+\alpha}\right) \end{aligned} \tag{6}$$

Since $-X_n$'s are still ELNQD by (6) we obtain

$$(S_n \leq -\varepsilon) = \mathbb{P}(-S_n \geq \varepsilon) \leq M\, e^{-\lambda\varepsilon}\exp\left(\lambda^{1+\alpha}\ e^{2\lambda \max_{i\geq 1} a_i} \sum_{i=1}^{n} \mathbb{E}|X_i|^{1+\alpha}\right) \tag{7}$$

By (6) and (7) the desired result (5) follows. ■

3. RESULTS AND DISCUSSION

Theorem 3.1. Let $(X_n, n \geq 1)$ be a sequence of identically distributed ELNQD random variables, $\mathbb{E}(X_i) = 0$ and finite variances. If there exists a sequence of positive numbers $(a_n, n \geq 1)$ such that $|X_n| \leq a_i$, for each $i \geq 1$, then for any $p > 1, \varepsilon > 0$ and $n \geq 1$, then

$$\begin{aligned} &\mathbb{P}\left({S_n}/{n} \geq \varepsilon\right) \\ &\leq M\,K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1-\frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\, n^{\frac{p\alpha}{p(1+\alpha)-1}}\right) \end{aligned} \tag{8}$$

With

$$\mathrm{K} = \exp\left(\frac{1}{q}\ b^{\frac{q}{p}}\ e^{2q\lambda \max_{i\geq 1} a_i} \left(\mathbb{E}|X_1|^{1+\alpha}\right)^q\right)$$

Proof: Markov's inequality and lemma 2.3, we can see that

$$\begin{aligned} I = \mathbb{P}\left({S_n}/{n} \geq \varepsilon\right) &\leq \mathbb{P}\left(e^{\lambda S_n} \geq e^{-\lambda\,\varepsilon n}\right) \\ &\leq M\, e^{-\lambda\varepsilon n}\exp\left(\lambda^{1+\alpha}\ n e^{2\lambda \max_{i\geq 1} a_i}\ \mathbb{E}|X_1|^{1+\alpha}\right) \end{aligned}$$

Let $p > 1$, it is well known that

$$UV = inf_{b>0}\left\{\frac{1}{pb}\ U^p + \frac{1}{q}\ b^{\frac{q}{p}}\ V^q\right\} \text{ for } U > 0, V > 0\ and\ \frac{1}{p} + \frac{1}{q} = 1.$$

This yields the inequality

$$\lambda^{1+\alpha}\, n\, e^{2\lambda \max_{i\geq 1} a_i}\, \mathbb{E}|X_1|^{1+\alpha} \leq \frac{1}{pb}\lambda^{p(1+\alpha)}\, n^p + \frac{1}{q}\, b^{\frac{q}{p}}\, e^{2q\lambda \max_{i\geq 1} a_i}(\mathbb{E}|X_1|^{1+\alpha})^q$$

We can thus conclude that for every $p > 0$, there for all $\lambda > 0$ such that

$$I \leq M \exp\left(-\lambda\varepsilon n + \frac{1}{pb}\lambda^{p(1+\alpha)}\, n^p\right) \exp\left(\frac{1}{q}\, b^{\frac{q}{p}}\, e^{2q\lambda \max_{i\geq 1} a_i}(\mathbb{E}|X_1|^{1+\alpha})^q\right)$$
$$\leq MK \exp\left(-\lambda\varepsilon n + \frac{1}{pb}\lambda^{p(1+\alpha)}\, n^p\right)$$

The equation

$$\frac{\partial}{\partial\lambda}\left[-\lambda\varepsilon n + \frac{1}{pb}\lambda^{p(1+\alpha)}\, n^p\right] = 0,$$

has a unique solution

$$\lambda^* = \left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \varepsilon^{\frac{1}{p(1+\alpha)-1}}\, n^{\frac{1-p}{p(1+\alpha)-1}}$$

which minimizes $(-\lambda\varepsilon n + \frac{1}{pb}\lambda^{p(1+\alpha)}\, n^p)$. Then

$$I \leq M\, K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\, n^{\frac{p\alpha}{p(1+\alpha)-1}}\right)$$

With

$$\mathrm{K} = \exp\left(\frac{1}{q}\, b^{\frac{q}{p}}\, e^{2q\lambda \max_{i\geq 1} a_i}(\mathbb{E}|X_1|^{1+\alpha})^q\right)$$

That concludes our theorem. ■

Theorem 3.2. Let $(X_n,\ n \geq 1)$ be a sequence of identically distributed ELNQD random variables, $\mathbb{E}(X_i) = 0$ and finite variances. If there exists a sequence of positive numbers $(a_n,\ n \geq 1)$ such that $|X_n| \leq a_i$, for each $i \geq 1$, then for any $p > 1$, $\varepsilon > 0$ and $n \geq 1$,

$$\mathbb{P}(|S_n| \geq \varepsilon) \leq 2M\, K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\, n^{\frac{-p}{p(1+\alpha)-1}}\right)$$

with

$$\mathrm{K} = \exp\left(\frac{1}{q}\, b^{\frac{q}{p}}\, e^{2q\lambda \max_{i\geq 1} a_i}\, (\mathbb{E}|X_1|^{1+\alpha})^q\right)$$

Proof: By hypothesis $\mathbb{E}(X_i) = 0$ and $|X_n| \leq a_i$, for each $i \geq 1$. Hence, by Markov's inequality, lemma 2.4, corollary 2.1 and the fact that $1 + x \leq e^x$, then

$$\mathbb{P}(S_n \geq \varepsilon) \leq \mathbb{P}\left(e^{\lambda S_n} \geq e^{-\lambda\,\varepsilon}\right) \leq M\, e^{-\lambda\,\varepsilon} \exp\left(\lambda^{1+\alpha}\, n\, e^{2\lambda \max_{i\geq 1} a_i}\, \mathbb{E}|X_1|^{1+\alpha}\right)$$
$$\leq M \exp\left(-\lambda\varepsilon + \lambda^{1+\alpha}\, n\, e^{2\lambda \max_{i\geq 1} a_i}\, \mathbb{E}|X_1|^{1+\alpha}\right)$$

Using the same method as proof of the theorem 3.1. We have

$$\mathbb{P}(S_n \geq \varepsilon) \leq M\,K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1-\frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} n^{\frac{-p}{p(1+\alpha)-1}}\right) \tag{9}$$

Since $-X_n$'s are still ELNQD by (9) we obtain

$$\mathbb{P}(S_n \leq -\varepsilon) = \mathbb{P}(-S_n \geq \varepsilon) \leq M\,K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1-\frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} n^{\frac{-p}{p(1+\alpha)-1}}\right) \tag{10}$$

From (9) and (10), we obtain

$$\mathbb{P}(|S_n| \geq \varepsilon) = \mathbb{P}(S_n \geq \varepsilon) + \mathbb{P}(-S_n \geq \varepsilon) \leq 2M\,K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1-\frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} n^{\frac{-p}{p(1+\alpha)-1}}\right)$$

with

$$\mathrm{K} = \exp\left(\frac{1}{q}\, b^{\frac{q}{p}}\, e^{2q\lambda \max_{i\geq 1} a_i} (\mathbb{E}|X_1|^{1+\alpha})^q\right) \qquad \blacksquare$$

Theorem 3.3. Let $(X_n, n \geq 1)$ be a sequence of identically distributed ELNQD random variables, $\mathbb{E}(X_i) = 0$ and finite variances. If there exists a sequence of positive numbers $(a_n, n \geq 1)$ such that $|X_n| \leq a_i$, for each $i \geq 1$, then for any $p > 1, \varepsilon > 0$ and $n \geq 1$, then

$$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq \varepsilon) \leq 2M\,K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1-\frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} n^{\frac{-p}{p(1+\alpha)-1}}\right) \tag{11}$$

where

$$\mathrm{K} = \exp\left(\frac{1}{q}\, b^{\frac{q}{p}}\, e^{4q\lambda \max\limits_{i\geq 1} a_i} (\mathbb{E}|X_1 - \mathbb{E}(X_1)\,|^{1+\alpha})^q\right)$$

Proof: Markov's inequality and theorem 3.2, we obtain that for any $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}(S_n - \mathbb{E}(S_n) \geq \varepsilon) &\leq \mathbb{P}\left(e^{\lambda(S_n - \mathbb{E}(S_n))} \geq e^{-\lambda\varepsilon}\right) \\
&\leq e^{-\lambda\varepsilon}\mathbb{E}[\lambda \sum_{i=1}^{n} (X_i - \mathbb{E}(X_i))] \\
&\leq M\exp\left(-\lambda\varepsilon + \lambda^{1+\alpha}\, e^{4\lambda \max\limits_{i\geq 1} a_i} \sum_{i=1}^{n} \mathbb{E}|X_i - \mathbb{E}(X_i)|^{1+\alpha}\right) \\
&\leq M\,K\exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1-\frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} n^{\frac{-p}{p(1+\alpha)-1}}\right)
\end{aligned} \tag{12}$$

with

$$\mathrm{K} = \exp\left(\frac{1}{q}\ b^{\frac{q}{p}}\ e^{4q\lambda \max_{i\geq 1} a_i}\ (\mathbb{E}|X_1 - \mathbb{E}(X_1)\ |^{1+\alpha})^q \right)$$

Let $-\mathrm{S_n} = \mathrm{T_n} = \sum_{\mathrm{i=1}}^{\mathrm{n}}(-\mathrm{X_i})$. Since $(\ -\mathrm{X_n}, \mathrm{n} \geq 1)$ is also a sequence of ELNOD random variables. We also have

$$\mathbb{P}(S_n - \mathbb{E}(S_n) \leq -\varepsilon) = \mathbb{P}(T_n - \mathbb{E}(T_n) \geq\ \varepsilon)$$

$$\leq M\ K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\ n^{\frac{-p}{p(1+\alpha)-1}} \right) \tag{13}$$

By (12) and (13) the desired result (11) follows. ■

Corollary 3.1. Let $(X_n, n \geq 1)$ be a sequence of identically distributed ELNQD random variables, $\mathbb{E}(X_i) = 0$ and finite variances. If there exists a sequence of positive numbers $(a_n, n \geq 1)$ such that $|X_n| \leq a_i$, for each $i \geq 1$, then for any $p > 1, \varepsilon > 0$ and $n \geq 1$, then

$$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq n\ \varepsilon)\ \leq$$

$$2M\ K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\ n^{\frac{p\alpha}{p(1+\alpha)-1}} \right) \tag{14}$$

where

$$\mathrm{K} = \exp\left(\frac{1}{q}\ b^{\frac{q}{p}}\ e^{4q\lambda \max_{i\geq 1} a_i}\ (\mathbb{E}|X_1 - \mathbb{E}(X_1)\ |^{1+\alpha})^q \right)$$

Theorem 3.4. Let $(X_n, n \geq 1)$ be a sequence of identically distributed ELNQD random variables, $\mathbb{E}(X_i) = 0$ and finite variances. If there exists a sequence of positive numbers $(a_n, n \geq 1)$ such that $|X_n| \leq a_i$, for each $i \geq 1$, then for any $p > 1, \varepsilon > 0$ and $n \geq 1$, then

$$n^{-r}\, S_n \underset{n\to\infty}{\longrightarrow} 0 \text{ completely.}$$

Proof: For any $\varepsilon > 0$, by theorem 2.3 we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq\ n^r \varepsilon)$$

$$\leq 2M\ K \sum_{n=1}^{\infty} \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right) (n^r \varepsilon)^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\ n^{\frac{-p}{p(1+\alpha)-1}} \right)$$

$$\leq\ 2M\ K \sum_{n=1}^{\infty} \left\{ \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right) \varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} \right) \right\}^{n^{\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}}}$$

$$\leq 2M\,K \sum_{n=1}^{\infty} \{\exp(-k')\}^{n^{\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}}}$$

where k' is positive number not depending on n. Using the inequality $e^{-y} \leq (\frac{d}{ey})^d$, choosing $d = \frac{2p(1+\alpha)-1}{rp(1+\alpha)-p}$, $d > 0, y > 0$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq n^r\varepsilon) \leq 2M\,K \sum_{n=1}^{\infty} \left(\frac{d}{ek'}\right)^d \left(\frac{1}{\mathrm{n}}\right)^{\left(\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}\right)^d}$$

$$\leq 2MK\left(\frac{d}{ek'}\right)^d \sum_{n=1}^{\infty} \frac{1}{n^{\frac{d(rp(1+\alpha)-p)}{p(1+\alpha)-1}}}$$

$$\leq 2MK\left(\frac{d}{ek'}\right)^d \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\leq 2MK\left(\frac{d}{ek'}\right)^d \frac{\pi^2}{6}$$

$$< \infty \quad (15)$$

■

Theorem 3.5. Let $(X_n, n \geq 1)$ be a sequence of identically distributed ELNQD random variables and $\mathbb{E}(X_i) = 0$. Assume that there exists a positive integer n_0 such that $|X_n| \leq a_n$, for each $1 \leq i \leq n, n \geq n_0$, where $(a_n, n \geq 1)$ is a sequence of positive numbers. Then,

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq n^r\varepsilon) \leq \infty$$

Theorem 3.6. Let $(X_n, n \geq 1)$ be a sequence of identically distributed ELNQD random variables and $\mathbb{E}(X_i) = 0$.If there exists a positive number a such that $|X_i| \leq a_{ni}$, for each $i \geq 1$, then for any $r > 0$

$$n^{-r}\left(S_n - \mathbb{E}(S_n)\right) \underset{n\to\infty}{\longrightarrow} 0 \text{ completely.}$$

Proof: For any $\varepsilon > 0$, it follows from corollary 3.1 that

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq n^r\varepsilon)$$

$$\leq 2M\,K \exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right)(n^r\varepsilon)^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} n^{\frac{-p}{p(1+\alpha)-1}}\right) \quad (16)$$

$$\leq 2M\,K\sum_{n=1}^{\infty}\left\{\exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}}\left(1-\frac{1}{p(1+\alpha)}\right)\varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\right)\right\}^{n^{\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}}}$$

$$\leq 2M\,K\sum_{n=1}^{\infty}\{\exp(-k')\}^{n^{\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}}}$$

$$\sum_{n=1}^{\infty}\mathbb{P}(|S_n-\mathbb{E}(S_n)|\geq n^r\varepsilon)$$

$$\leq 2M\,K\exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}}\left(1-\frac{1}{p(1+\alpha)}\right)(n^r\varepsilon)^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}n^{\frac{-p}{p(1+\alpha)-1}}\right)$$

$$\leq 2M\,K\sum_{n=1}^{\infty}\left\{\exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}}\left(1-\frac{1}{p(1+\alpha)}\right)\varepsilon^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\right)\right\}^{n^{\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}}}$$

$$\leq 2M\,K\sum_{n=1}^{\infty}\{\exp(-k')\}^{n^{\frac{rp(1+\alpha)-p}{p(1+\alpha)-1}}}$$

By this result we get (15).

4. APPLICATION TO THE RESULTS TO LINEAR MODELS GENERATED BY ELNQD ERRORS

The basic object of this section is applying the results to first-order autoregressive processes (AR(1)) model. We consider an autoregressive time series of first order AR(1) defined by

$$X_{n+1}=\theta X_n+\zeta_{n+1}\;n=1,2\,...,\tag{17}$$

where θ is a parameter with $|\theta|<1$, and where $\{\zeta_n\,,n\geq 0\}$ is a sequence of identically distributed ELNQD random variables with $\zeta_0=X_0=0, 0<\mathbb{E}\zeta_k^4<\infty, k=1,2,....$ Here, we can rewrite X_{n+1} in (17) as follows:

$$X_{n+1}=\theta^{n+1}X_0+\theta^n\zeta_1+\theta^{n-1}\zeta_2+\cdots+\zeta_{n+1}.\tag{18}$$

The coefficient θ is fitted least squares, giving the estimator θ_n, for all $n\geq 1$, by

$$\widehat{\theta_n}=\frac{\sum_{j=1}^{n}X_jX_{j-1}}{\sum_{j=1}^{n}X_{j-1}^2}\tag{19}$$

It immediately follows from (18) and (19) that

$$\widehat{\theta_n} - \theta = \frac{\sum_{j=1}^{n} \zeta_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2} \tag{20}$$

Theorem 4.1: Let the condition of theorem 3.3 be satisfied then for any $\frac{(\mathbb{E}\zeta_1^2)^{\frac{1}{2}}}{\rho^2} < \xi$ positive, we have

$$\begin{aligned} &\mathbb{P}\left(\sqrt{n}\left|\widehat{\theta_n} - \theta\right| > \rho\right) \\ &\leq 2M\,K\,exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}}\right. \\ &\left.\times \left(1 - \frac{1}{p(1+\alpha)}\right)(\rho^2\xi^2 - \mathbb{E}\zeta_1^2)^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\, n^{\frac{p(1+\alpha)-p}{p(1+\alpha)-1}}\right) \\ &+\ exp\left(-\frac{1}{2}\,n\frac{(T_1 - n\xi^2)^2}{T_2}\right) \end{aligned}$$

where $\mathrm{K} = \exp\left(\frac{1}{\mathrm{q}}\ \mathrm{b}^{\frac{\mathrm{q}}{\mathrm{p}}}\ \mathrm{e}^{4\lambda \mathrm{q} \max_{\mathrm{i}\geq 1} \mathrm{a_i}}\ (\mathbb{E}|\zeta_1^2 - \mathbb{E}\zeta_1^2|^{1+\alpha})^q\right), T_1 = \mathbb{E}(\,X_j^2)$ and $T_2 = \mathbb{E}(\,X_j^4)$.

Proof: Let

$$\widehat{\theta_n} - \theta = \frac{\sum_{j=1}^{n} \zeta_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2}.$$

It follows that

$$\mathbb{P}\left(\sqrt{n}\left|\widehat{\theta_n} - \theta\right| > \rho\right) \leq \mathbb{P}\left(\left|\frac{\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \zeta_j X_{j-1}}{\frac{1}{n}\sum_{j=1}^{n} X_{j-1}^2}\right| > \rho\right).$$

By virtue of the probability properties and Hölder's inequality, we have for any ξ positive

$$\begin{aligned} \mathbb{P}\left(\sqrt{n}\left|\widehat{\theta_n} - \theta\right| > \rho\right) &\leq \mathbb{P}\left(\frac{1}{n}\sum_{j=1}^{n} \zeta_j^2 \geq \rho^2\xi^2\right) + \mathbb{P}\left(\frac{1}{n^2}\sum_{j=1}^{n} X_{j-1}^2 \leq \xi^2\right) \\ &= \mathbb{P}\left(\sum_{j=1}^{n} \zeta_j^2 \geq (\rho^2\xi^2)n\right) + \mathbb{P}\left(\sum_{j=1}^{n} X_{j-1}^2 \leq n^2\xi^2\right) \\ &= I_{1n} + I_{2n} \end{aligned}$$

Next we estimate I_{1n} and I_{2n}

$$I_{1n} = \mathbb{P}\left(\sum_{j=1}^{n} \zeta_j^2 \geq (\rho^2\xi^2)n\right) \tag{21}$$

$$= \mathbb{P}\left(\sum_{j=1}^{n}(\zeta_j^2 - \mathbb{E}\zeta_j^2 + \mathbb{E}\zeta_j^2) \geq (\rho^2\xi^2)n\right)$$
$$= \mathbb{P}\left(\sum_{j=1}^{n}(\zeta_j^2 - \mathbb{E}\zeta_j^2) \geq (\rho^2\xi^2 - \mathbb{E}\zeta_1^2)n\right)$$
$$\leq \mathbb{P}\left(\left|\sum_{j=1}^{n}(\zeta_j^2 - \mathbb{E}\zeta_j^2)\right| \geq (\rho^2\xi^2 - \mathbb{E}\zeta_1^2)n\right)$$

By using the theorem (3.3), we have

$$I_{1n} = \mathbb{P}\left(\sum_{j=1}^{n}\zeta_j^2 \geq (\rho^2\xi^2)n\right)$$
$$\leq 2M\,K\,exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}}\left(1 - \frac{1}{p(1+\alpha)}\right)((\rho^2\xi^2 - \mathbb{E}\zeta_1^2)n)^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\, n^{\frac{-p}{p(1+\alpha)-1}}\right) \quad (22)$$

where $\mathrm{K} = \exp\left(\frac{1}{\mathrm{q}}\ \mathrm{b}^{\frac{\mathrm{q}}{\mathrm{p}}}\ \mathrm{e}^{4\lambda \mathrm{q} \max_{\mathrm{i}\geq 1} \mathrm{a_i}}\, (\mathbb{E}|\zeta_1^2 - \mathbb{E}\zeta_1^2|^{1+\alpha})^{q}\right)$.

Now, we limite the probability I_{2n}, using the Markov's inequality, we have for any λ positive

$$I_{2n} = \mathbb{P}\left(\sum_{j=1}^{n}X_{j-1}^2 \leq n^2\xi^2\right)$$
$$= \mathbb{P}\left(n^2\xi^2 - \sum_{j=1}^{n}X_{j-1}^2 \geq 0\right)$$
$$= \mathbb{E}\left(\mathbb{I}_{\left\{n^2\xi^2 - \sum_{j=1}^{n}X_{j-1}^2 \geq 0\right\}}\right)$$
$$\leq \mathbb{E}\left(e^{\lambda\left(n^2\xi^2 - \sum_{j=1}^{n}X_{j-1}^2\right)}\right), \lambda > 0$$
$$\leq e^{\lambda n^2\xi^2}\,\mathbb{E}\left(e^{-\lambda\sum_{j=1}^{n}X_{j-1}^2}\right)$$
$$\leq e^{\lambda n^2\xi^2}\prod_{j=1}^{n}\mathbb{E}\left(e^{-\lambda X_{j-1}^2}\right).$$

Since

$$I_{2n} \leq e^{\lambda n^2\xi^2}\prod_{j=1}^{n}\mathbb{E}\left(e^{-\lambda X_{j-1}^2}\right)$$

We take the inequality

$$e^{-x} \leq 1 - x + \frac{1}{2}\, x^2 \tag{23}$$
.

To show this let $\psi(x) = e^{-x}$, $\phi(x) = 1 - x + \frac{1}{2}\, x^2$ and recall that for every x,

$$e^{x} \geq 1 + x, \forall x \in \mathbb{R} \tag{24}$$
.

So that

$$\psi(x)' = -e^{-x} \leq -1 + \ x^2 = \phi(x)'.$$

Since $\psi(0) = \phi(0) = 1$this implies $\psi(x) \leq \phi(x)$ for all $x \geq 0$ and (23) is claimed. By (23) and (24), we can write that any for λ positive

$$\begin{aligned}
e^{\lambda n^2 \xi^2} \prod_{j=1}^{n} \mathbb{E}\left(e^{-\lambda X_{j-1}^2}\right) &\leq e^{\lambda n^2 \xi^2} \prod_{j=1}^{n} \mathbb{E}\left(1 - \lambda X_{j-1}^2 + \frac{1}{2}\left(\lambda X_{j-1}^2\right)^2\right) \\
&= e^{\lambda n^2 \xi^2} \prod_{j=1}^{n}\left(1 - \lambda \mathbb{E} X_{j-1}^2 + \frac{\lambda^2}{2}\, \mathbb{E}\left(X_{j-1}^4\right)\right) \\
&\leq e^{\lambda n^2 \xi^2}\left(1 - \lambda \mathbb{E} X_{j-1}^2 + \frac{\lambda^2}{2}\, \mathbb{E}\left(X_{j-1}^4\right)\right)^n \\
&\leq e^{\lambda n^2 \xi^2}\left(exp\left(-\lambda \mathbb{E} X_{j-1}^2 + \frac{\lambda^2}{2}\, \mathbb{E}\left(X_{j-1}^4\right)\right)\right)^n \\
&\leq e^{\lambda n^2 \xi^2}\, exp\left(-\lambda n \mathbb{E} X_{j-1}^2 + \frac{\lambda^2}{2} n\, \mathbb{E}\left(X_{j-1}^4\right)\right)
\end{aligned}$$

where $T_1 = \mathbb{E}(\,X_j^2) < \infty, T_2 = \mathbb{E}(\,X_j^4) < \infty$, we have

$$\begin{aligned}
I_{2n} = \mathbb{P}\left(\sum_{j=1}^{n} X_{j-1}^2 \leq n^2 \xi^2\right) &\leq exp\left(\lambda(n^2\xi^2 - nT_1) + \frac{\lambda^2}{2} n\, T_2\right) \\
&\leq \inf_{\lambda>0} exp\left(\lambda(n^2\xi^2 - nT_1) + \frac{\lambda^2}{2} n\, T_2\right), \forall \lambda > 0
\end{aligned}$$

The equation

$$\frac{\partial}{\partial \lambda}\left[\lambda(n^2\xi^2 - nT_1) + \frac{\lambda^2}{2} n\, T_2\right] = 0,$$

has a unique solution

$$\lambda = \frac{T_1 - n\xi^2}{T_2}$$

which minimizes $\left(\lambda(n^2\xi^2 - nT_1) + \frac{\lambda^2}{2} n\, T_2\right)$. Hence

$$I_{2n} = \mathbb{P}\left(\sum_{j=1}^{n} X_{j-1}^2 \leq n^2\xi^2\right) \leq exp\left(-\frac{1}{2}\, n \frac{(T_1 - n\xi^2)^2}{T_2}\right). \tag{25}$$

Then for any $\rho > 0, T_1 < \infty, T_2 < \infty$, and by (22), (25), we have

$$\mathbb{P}\left(\sqrt{n}\left|\widehat{\theta_n} - \theta\right| > \rho\right)$$

$$\leq 2M\, K\, exp\left(-\left(\frac{b}{1+\alpha}\right)^{\frac{1}{p(1+\alpha)-1}} \times \left(1 - \frac{1}{p(1+\alpha)}\right) (\rho^2\xi^2 - \mathbb{E}\zeta_1^2)^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\, n^{\frac{p(1+\alpha)-p}{p(1+\alpha)-1}}\right)$$

$$exp\left(-\frac{1}{2}\, n \frac{\left(T_1 - n\xi^2\right)^2}{T_2}\right).$$

This completes the proof. ■

Corollary 4.1.The sequence $(\hat{\theta}_n)_{n\in\mathbb{N}}$ defined in (19) converge complete to the parameter θ of the order 1 autoregressive process. Then we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sqrt{n}\left|\hat{\theta}_n - \theta\right| > \rho\right) \leq \infty. \tag{26}$$

Proof: By using theorem 3.6 and $\mathbb{E}(\, X_j^2) < \infty, \mathbb{E}(\, X_j^4) < \infty$, we get the result of (26) immediately. ■

5. CONCLUSION

We are interested in this work to establish some new exponential type inequalities and complete convergence for distribution of sums of extended linear negative quadrant dependence (ELNQD, in short) random variables. We applied these inequalities and proved the complete convergence of first-order autoregressive processes AR(1) with identically distributed ELNQD errors.

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