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THE COMPLEX-TYPE FIBONACCI p-SEQUENCES IN FINITE GROUPS

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Abstract. *In this paper, we study the complex-type Fibonacci p-sequence according to modulo m and obtain the periods and the ranks of the complex-type Fibonacci p-sequence. Then, we consider the generating matrix of the complex-type Fibonacci p-sequence when read modulo m and we obtain the cyclic groups. Furthermore, we derive the relationships between the periods of the complex-type Fibonacci p-sequence modulo m and the orders of the cyclic groups produced. Also, we redefine the complex-type Fibonacci p-sequence through the elements of the groups and then examine this sequence in the finite groups. Finally, we obtain the periods of the complex-type Fibonacci 2-sequence in the dihedral group D_{2m} as applications of the results produced.*

Keywords: *Complex-type Fibonacci p-sequence; group; period; rank.*

1. INTRODUCTION

The linear recurrence sequences in groups were firstly studied by Wall [1] who calculated the periods of the Fibonacci sequences in cyclic groups. As a natural generalization of the problem, Wilcox [2] investigated the Fibonacci lengths on Abelian groups. The concept extended to some special linear recurrence sequences by several authors; see for example, [3-19]. Deveci and Shannon [20] defined the complex-type k-Fibonacci orbit of a k-generator group. They proved that the complex-type k-Fibonacci orbit of a k-generator group is periodic if the group is finite. In this paper, we consider the complex-type Fibonacci p-sequence and study this sequence according to modulo m. Then, we give some results concerning the periods and the ranks of the complex-type Fibonacci p-sequence for any p and m. Also, we consider the multiplicative orders of the generating matrix of the complex-type Fibonacci p-sequence when read modulo m, and then we obtain the cyclic groups. Furthermore, we derive the relationships between the periods of the complex-type Fibonacci p-sequence modulo m and the orders of the cyclic groups produced. Finally, we extend the complex-type Fibonacci p-sequence to groups and then we obtain the periods of the complex-type Fibonacci 2-sequence in the dihedral group D_{2m} as applications of the results produced.

In [21], the complex-type Fibonacci p-sequence for any given $p\left(p=2,3,\ldots\right)$ is defined as follows:

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$$F_{p,i}^{*}\left(n+p+1\right)=i^{p+1}\cdot F_{p,i}^{*}\left(n+p\right)+i\cdot F_{p,i}^{*}\left(n\right) \qquad \left(n\geq 0\right) \tag{1}$$

where $F_{p,i}^{*}\left(0\right)=\cdots=F_{p,i}^{*}\left(p-1\right)=0$, $F_{p,i}^{*}\left(p\right)=1$ and $\sqrt{-1}=i$.

Also in [21], they gave the generating matrices of the complex-type Fibonacci p-sequence as shown:

$$C_p^F=\begin{bmatrix} i^{p+1} & 0 & \cdots & 0 & i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}_{(p+1)\times(p+1).}$$

Then, for $n\geq p$, they obtained that

$$\left(C_p^F\right)^n=\begin{bmatrix} F_{p,i}^{*}\left(n+p\right) & iF_{p,i}^{*}\left(n\right) & iF_{p,i}^{*}\left(n+1\right) & \cdots & iF_{p,i}^{*}\left(n+p-1\right) \\ F_{p,i}^{*}\left(n+p-1\right) & iF_{p,i}^{*}\left(n-1\right) & iF_{p,i}^{*}\left(n\right) & \cdots & iF_{p,i}^{*}\left(n+p-2\right) \\ \vdots & \vdots & \vdots & & \vdots \\ F_{p,i}^{*}\left(n+1\right) & iF_{p,i}^{*}\left(n-p+1\right) & iF_{p,i}^{*}\left(n-p+2\right) & \cdots & iF_{p,i}^{*}\left(n\right) \\ F_{p,i}^{*}\left(n\right) & iF_{p,i}^{*}\left(n-p\right) & iF_{p,i}^{*}\left(n-p+1\right) & \cdots & iF_{p,i}^{*}\left(n-1\right) \end{bmatrix}. \tag{2}$$

It is important to note that $\det C_p^F=\left(-1\right)^p i$.

Definition 1. A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a_0,a_1,a_2,a_3,a_1,a_2,a_3,a_1,a_2,a_3,\ldots$ is periodic after the initial element a_0 and has period 3. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a_0,a_1,a_2,a_3,a_0,a_1,a_2,a_3,\ldots$ is simply periodic with period 4.

For a finitely generated group $G=\langle A\rangle$, where $A=\{a_1,a_2,\ldots,a_n\}$, the sequence $x_u=a_{u+1}$, $0\leq u\leq n-1$, $x_{n+u}=\prod_{v=1}^{n}x_{u+v-1}$, $u\geq 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted as $F_A\left(G\right)$ in [22]. A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements x_0, x_1, x_2, $\ldots$, x_n, $\ldots$ for which, given an initial (seed) set x_0, x_1, x_2, $\ldots$, x_{j-1}, each element is defined by

$$x_n=\begin{cases} x_0x_1\cdots x_{n-1} & \text{for } j\leq n<k, \\ x_{n-k}x_{n-k+1}\cdots x_{n-1} & \text{for } n\geq k. \end{cases}$$

We also require that the initial elements of the sequence x_0, x_1, x_2, …, x_{j-1} generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G generated by x_0, x_1, x_2, …, x_{j-1} is denoted by $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ in [23].

Notice that the orbit of a k-generated group is a k-nacci sequence. In [20], Deveci and Shannon stated that the following conditions hold for every elements x, y of the group G :

Definition 2. (i) Let $z = a + ib$ such that a and b are integers and let e be the identity of G, then

$$* \;\; x^z \equiv x^{a(mod|x|)+ib(mod|x|)} = x^{a(mod|x|)}x^{ib(mod|x|)} = x^{ib(mod|x|)}x^{a(mod|x|)} = x^{ib(mod|x|)+a(mod|x|)},$$

$$* \; x^{ia} = \left(x^i\right)^a = \left(x^a\right)^i,$$

$$* \;\; e^u = e,$$

$$* \;\; x^{0+i0} = e.$$

(ii) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ such that a_1, b_1, a_2 and b_2 are integers , then $\left(x^{z_1}y^{z_2}\right)^{-1} = y^{-z_2}x^{-z_1}$.

(iii) If $xy \neq yx$, then $x^i y^i \neq y^i x^i$.

(iv) $(xy)^i = y^i x^i$ and $\left(x^i y^i\right)^i = x^{-1}y^{-1}$.

(v) $xy^i = y^i x$ and so $\left(xy^i\right)^i = x^i y^{-1}$ and $\left(x^i y\right)^i = x^{-1}y^i$.

2. THE COMPLEX-TYPE FIBONACCI p-SEQUENCE IN FINITE GROUPS

Reducing the complex-type Fibonacci p-sequence by a modulus m, the following recurrence sequence is obtained

$$\left\{F^*_{m,p,i}(n)\right\} = \left\{F^*_{m,p,i}(0), F^*_{m,p,i}(1), \ldots, F^*_{m,p,i}(j), \ldots\right\}$$

where $F^*_{m,p,i}(n) = F^*_{p,i}(n)(mod\, m)$. It is important to note that the sequence $F^*_{m,p,i}(n)$ has the same recurrence relation as the complex-type Fibonacci p-sequence.

Theorem 1. The sequence $\left\{F^*_{m,p,i}(n)\right\}$ is simply periodic.

Proof: Consider the set $G = \left\{\left(g_0, g_1, \ldots, g_p\right) \mid g_u\text{'s are complex numbers } a_u + ib_u \text{ where } a_u \text{ and } b_u \text{ are integers such that } 0 \le a_u, b_u \le m-1 \text{ and } 0 \le u \le p\right\}$.

Then, it is clear that the set G is finite. Suppose that the notation $|G|$ is the cardinality of the set G. Since there are $|G|$ distinct $p+1$-tuples of the complex-type Fibonacci p-sequence modulo m, at least one of the $p+1$-tuples appears twice in the sequence

$\left\{F^{*}_{m,p,i}(n)\right\}$. Thus, the subsequence following this $p+1$-tuple repeats; that is, $\left\{F^{*}_{m,p,i}(n)\right\}$ is a periodic sequence. Because of the periodicity, for any $v \geq 0$, there exist $w \geq v+p+1$ such that $F^{*}_{m,p,i}(w)=F^{*}_{m,p,i}(v)$, $F^{*}_{m,p,i}(w+1)=F^{*}_{m,p,i}(v+1)$,... , $F^{*}_{m,p,i}(w+p+1)=F^{*}_{m,p,i}(v+p+1)$, then $w \equiv v(mod\ p+1)$. By definition of the sequence $\left\{F^{*}_{p,i}(n)\right\}$, it is clear that

$$F^{*}_{p,i}(n)=(-i)\cdot F^{*}_{p,i}(n+p+1)+i^{p+2}\cdot F^{*}_{p,i}(n+p).$$

So we get $F^{*}_{m,p,i}(w-1)\equiv F^{*}_{m,p,i}(v-1)$, $F^{*}_{m,p,i}(w-2)\equiv F^{*}_{m,p,i}(v-2)$,... , $F^{*}_{m,p,i}(0)\equiv F^{*}_{m,p,i}(w-v)$, which indicates that the $\left\{F^{*}_{m,p,i}(n)\right\}$ is a simply periodic sequence. Let the notation $L_p(m)$ denote the smallest period of the sequence $\left\{F^{*}_{m,p,i}(n)\right\}$.

Given an integer matrix $H=\left[h_{ij}\right]$, $H(mod\ m)$ means that all entries of H are reduced modulo m, that is, $H(mod\ m)=\left(h_{ij}(mod\ m)\right)$. Let us consider the set $\left\langle H\right\rangle_m=\left\{(H)^n(mod\ m) \mid n\geq 0\right\}$ and $\left|\left\langle H\right\rangle_m\right|$ denotes the order of the set $\left\langle H\right\rangle_m$. If $\gcd(m,\det H)=1$, then the set $\left\langle H\right\rangle_m$ is a cyclic group, if $\gcd(m,\det H)\neq 1$, then the set $\left\langle H\right\rangle_m$ is a semigroup. Since $\det C^F_p=(-1)^p i$, it is clear that the set $\left\langle C^F_p\right\rangle_m$ is a cyclic group for every positive integer $m\geq 2$. From (2), it is clear that $L_p(m)=\left|\left\langle C^F_p\right\rangle_m\right|$.

Theorem 2. Let q be a prime and t be the largest positive integer such that $\left|\left\langle C^F_p\right\rangle_q\right|=\left|\left\langle C^F_p\right\rangle_{q^t}\right|$. Then $\left|\left\langle C^F_p\right\rangle_{q^w}\right|=q^{w-t}\cdot\left|\left\langle C^F_p\right\rangle_q\right|$ for every $w\geq t$. In particular, if $\left|\left\langle C^F_p\right\rangle_q\right|\neq\left|\left\langle C^F_p\right\rangle_{q^2}\right|$, then $\left|\left\langle C^F_p\right\rangle_{q^w}\right|=q^{w-1}.\left|\left\langle C^F_p\right\rangle_q\right|$ for every $w\geq 2$.

Proof: Since $L_p(m)=\left|\left\langle C^F_p\right\rangle_m\right|$, we have a positive integer k such that $\left(C^F_p\right)^{L_p\left(q^{k+1}\right)}\equiv I\left(mod\ q^{k+1}\right)$. Then it is clear that $\left(C^F_p\right)^{L_p\left(q^{k+1}\right)}\equiv I\left(mod\ q^{k}\right)$ where I is a $(p+1)\times(p+1)$ identity matrix. Thus we obtain that $L_p\left(q^k\right)$ divides $L_p\left(q^{k+1}\right)$. On the other hand, writing $\left(C^F_p\right)^{L_p\left(q^{k}\right)}=I+\left(c^{(q)}_{i,j}\cdot q^k\right)$, by the binomial theorem, we obtain

$$\left(C^F_p\right)^{L_p\left(q^{k}\right).q}=\left(I+\left(c^{(q)}_{i,j}\cdot q^k\right)\right)^q=\sum_{j=0}^{q}\binom{q}{j}\left(c^{(q)}_{i,j}\cdot q^k\right)^j\equiv I\left(\mathrm{mod}\ q^{k+1}\right).$$

which implies that $L_p\left(q^{k+1}\right)$ divides $L_p\left(q^{k}\right).q$. Therefore, $L_p\left(q^{k+1}\right)=L_p\left(q^{k}\right)$ or $L_p\left(q^{k+1}\right)=L_p\left(q^{k}\right).q$, and the latter holds if and only if there is a $c^{(q)}_{i,j}$ which is not divisible by q. Since $L_p\left(q^{t+1}\right)\neq L_p\left(q^{t}\right)$, there is a $c^{(k+1)}_{i,j}$ which is not divisible by q. This shows that $L_p\left(q^{t+2}\right)\neq L_p\left(q^{t+1}\right)$. So, the proof is complete.

Definition 3. *The rank of the sequence* $\left\{F^{*}_{m,p,i}(n)\right\}$ *is the least positive integer* δ *such that* $F^{*}_{m,p,i}(\delta) \equiv F^{*}_{m,p,i}(\delta+1) \equiv \cdots \equiv F^{*}_{m,p,i}(\delta+p-1) \equiv 0(mod\ m)$, *and we denote the rank of* $\left\{F^{*}_{m,p,i}(n)\right\}$ *by* $ra_p(m)$.

If $F^{*}_{m,p,i}(\delta+p) \equiv 0(mod\ m)$, then the terms of the sequence $\left\{F^{*}_{m,p,i}(n)\right\}$ starting with index $ra_p(m)$, namely $\underbrace{0,0,\ldots,0}_{p},\lambda,\lambda,\ldots,$ are exactly the initial terms of $\left\{F^{*}_{m,p,i}(n)\right\}$ multiplied by a factor λ. The exponents μ for which $\left(C_p^F\right)^{\mu} \equiv I(mod\ m)$ form a simple arithmetic progression. Then we have

$$\left(C_p^F\right)^{\mu} \equiv I(mod\ m) \Leftrightarrow L_p(m) \mid \mu.$$

Similarly, the exponents μ for which $\left(C_p^F\right)^{\mu} \equiv \lambda I(mod\ m)$ for some $\lambda \in \mathbb{C}$ form a simple arithmetic progression, and hence

$$\left(C_p^F\right)^{\mu} \equiv \lambda I(mod\ m) \Leftrightarrow ra_p(m) \mid \mu.$$

So, it is easy to see that $ra_p(m)$ divides $L_p(m)$. The order of the sequence $\left\{F^{*}_{m,p,i}(n)\right\}$ is defined by $\frac{L_p(m)}{ra_p(m)}$ and this order is denoted by $\varphi_p(m)$. Let $\left(C_p^F\right)^{ra_p(m)} \equiv \lambda I(mod\ m)$, then $ord_m(\lambda)$ is the least positive value of w such that $\left(C_p^F\right)^{w.ra_p(m)} \equiv I(mod\ m)$. So it is confirm that $ord_m(\lambda)$ is the least positive integer w with $L_p(m) \mid w.ra_p(m)$. As a result, we obtain $ord_m(\lambda) = w$. Consequently, we may easily conclude that $\varphi_p(m)$ is always a positive integer, and that $\varphi_p(m) = ord_m\left(F^{*}_{m,p,i}\left(ra_p(m)+p\right)\right)$, the multiplicative order of $F^{*}_{m,p,i}\left(ra_p(m)+p\right)$.

Example 1. Reducing the complex-type Fibonacci 4-sequence by a modulus 3, we obtain $\left\{F^{*}_{3,4,i}(n)\right\}$ as follows:

$$\left\{\begin{array}{l} 0,0,0,0,1,i,2,2i,1,2i,0,2i,2,0,1,i,0,2i,1,2i,0,0,1,2i,2,2i,1,2i,2,i,0, \\ i,0,2i,0,0,2,2i,2,2i,1,0,1,0,1,2i,1,2i,1,2i,2,0,1,2i,2,i,2,0,1,0,2,i,2, \\ 0,0,2i,0,2i,1,i,0,0,1,2i,0,0,0,i,\underline{0,0,0,0,}2,2i,1,i,2,i,0,i,1,0,2,2i,0,i, \\ 2,i,0,0,2,i,1,i,2,i,1,2i,0,2i,0,i,0,0,1,i,1,i,2,0,2,0,2,i,2,i,2,i,1,0,2, \\ i,1,2i,1,0,2,0,1,2i,1,0,0,i,0,i,2,2i,0,0,2,i,0,0,0,2i,\underline{0,0,0,1,}\ldots \end{array}\right\}.$$

It follows that $ra_4(3) = 78$, $L_4(3) = 156$ and $\varphi_4(3) = 2$.

Theorem 3. Suppose that m_1 and m_2 are positive integers with $m_1, m_2 \geq 2$, then $ra_p\left(lcm[m_1, m_2]\right) = lcm\left[ra_p(m_1), ra_p(m_2)\right]$ for any $p \geq 2$. Similarly, $L_p\left(lcm[m_1, m_2]\right) = lcm\left[L_p(m_1), L_p(m_2)\right]$.

Proof: Let $lcm[m_1, m_2] = m$. Then

$$F^*_{p,i}\left(ra_p(m)\right) \equiv F^*_{p,i}\left(ra_p(m)+1\right) \equiv \cdots \equiv F^*_{p,i}\left(ra_p(m)+p-1\right) \equiv 0 \left(mod\ m\right)$$

and

$$F^*_{p,i}\left(ra_p(m_k)\right) \equiv F^*_{p,i}\left(ra_p(m_k)+1\right) \equiv \cdots \equiv F^*_{p,i}\left(ra_p(m_k)+p-1\right) \equiv 0 \left(mod\ m\right)$$

for $k = 1, 2$. Using the least common multiple operation this implies that $F^*_{p,i}\left(ra_p(m)\right) \equiv F^*_{p,i}\left(ra_p(m)+1\right) \equiv \cdots \equiv F^*_{p,i}\left(ra_p(m)+p-1\right) \equiv 0 \left(mod\ m_k\right)$ for $k = 1, 2$. So we get $ra_p(m_1) \mid ra_p(m)$ and $ra_p(m_2) \mid ra_p(m)$, which means that $lcm\left[ra_p(m_1), ra_p(m_2)\right]$ divides $ra_p\left(lcm[m_1, m_2]\right)$. We also know that

$$\begin{aligned} F^*_{p,i}\left(lcm\left[ra_p(m_1), ra_p(m_2)\right]\right) &\equiv F^*_{p,i}\left(lcm\left[ra_p(m_1), ra_p(m_2)\right]+1\right) \equiv \cdots \\ &\equiv F^*_{p,i}\left(lcm\left[ra_p(m_1), ra_p(m_2)\right]+p-1\right) \equiv 0 \left(mod\ m_k\right) \end{aligned}$$

for $k = 1, 2$. Then we can write

$$\begin{aligned} F^*_{p,i}\left(lcm\left[ra_p(m_1), ra_p(m_2)\right]\right) &\equiv F^*_{p,i}\left(lcm\left[ra_p(m_1), ra_p(m_2)\right]+1\right) \equiv \cdots \\ &\equiv F^*_{p,i}\left(lcm\left[ra_p(m_1), ra_p(m_2)\right]+p-1\right) \equiv 0 \left(mod\ m\right), \end{aligned}$$

and it follows that $ra_p\left(lcm[m_1, m_2]\right)$ divides $lcm\left[ra_p(m_1), ra_p(m_2)\right]$. So, the proof is complete.

The period $L_p(m)$ is proved with a similar proof method. Now we take into account the complex-type Fibonacci p-numbers in groups for any $p \geq 2$.

Let G be a finite p-generator group and let $X = \{(x_1, x_2, \ldots, x_p) \in \underbrace{G \times G \times \cdots \times G}_{p} \mid <\left\{x_1, x_2, \ldots, x_p\right\}>= G\}$. We call $\left(x_1, x_2, \ldots, x_p\right)$ a generating p-tuple for G.

Definition 4. For a generating p-tuple $\left(x_1, x_2, \ldots, x_p\right) \in X$, we define the complex-type Fibonacci p-orbit as follows:

$$a_0 = x_1, a_1 = x_2, \ldots, a_{p-1} = x_p, a_p = x_p, a_{n+p} = \left(a_{n-1}\right)^i \left(a_{n+p-1}\right)^{i^{p+1}} \left(n \geq 1\right).$$

For a p-tuple $(x_1, x_2, \ldots, x_p) \in X$, the complex-type Fibonacci p-orbit is denoted by $F^*_{(x_1, x_2, \ldots, x_p)}(G)$.

Theorem 4. Let G be a p-generator group. If G is finite, then the complex-type Fibonacci p-orbit of G is periodic.

Proof: Let's consider the set

$$K = \left\{ \left((k_1)^{a_1(mod|k_1|)+ib_1(mod|k_1|)}, (k_2)^{a_2(mod|k_2|)+ib_2(mod|k_2|)}, \ldots, (k_p)^{a_p(mod|k_p|)+ib_p(mod|k_p|)} \right) : k_1, k_2, \ldots, k_p \in G \text{ and } a_n, b_n \in Z \text{ such that } 1 \le n \le p \right\}.$$

G is a finite set and therefore K is a finite set. Then for any $u \ge 0$, there exists $t \ge u + v$ such that $a_{u+1} = a_{t+1}$, $a_{u+2} = a_{t+2}$, ... , $a_{u+v} = a_{t+v}$. Because of the repeating, for all generating p-tuples, the sequence $F^*_{(x_1, x_2, \ldots, x_p)}(G)$ is periodic.

We denote the lengths of the periods of the sequence $F^*_{(x_1, x_2, \ldots, x_p)}(G)$ by $HF^*_{(x_1, x_2, \ldots, x_p)}(G)$. It is well-known that the dihedral group D_{2m} of order $2m$ is defined by the presentation

$$D_{2m} = \langle x, y \mid x^m = y^2 = (xy)^2 = e \rangle.$$

We now consider the periods of the complex-type Fibonacci 2-orbit of the dihedral group D_{2m}. Consider a sequence defined as follows:

$$\vartheta_n = \begin{cases} -i\vartheta_{n-1} + i\vartheta_{n-3} & \text{for } n \equiv 0,1,3,8,11,12 \; mod \; 14, \\ i\vartheta_{n-1} + i\vartheta_{n-3} & \text{for } n \equiv 2,4,6,10 \; mod \; 14, \\ i\vartheta_{n-1} - i\vartheta_{n-3} & \text{for } n \equiv 5,7,9,13 \; mod \; 14 \end{cases}$$

for $n \geqslant 14$, where $\vartheta_0 = 1$, $\vartheta_1 = \vartheta_2 = \vartheta_6 = 0$, $\vartheta_3 = \vartheta_7 = i$, $\vartheta_4 = -1$, $\vartheta_5 = -i$, $\vartheta_8 = 2$, $\vartheta_9 = 2i$, $\vartheta_{10} = -3$, $\vartheta_{11} = 5i$, $\vartheta_{12} = 3$ and $\vartheta_{13} = 6i$.

Reducing this sequence by a modulo m, we can get a repeating sequence, which we denote by:

$$\{\vartheta_n(m)\} = \{\vartheta_0(m), \vartheta_1(m)\}, \ldots, \vartheta_{14}(m) \ldots \vartheta_j(m).$$

It has the same recurrence relation as in (1).

Theorem 5. $\{\vartheta_n(m)\}$ *is a simply periodic sequence.*

Proof: Let $V=\{v_0,v_1,\ldots,v_{13}|\ 0\le v_k\le m-1\}$. Since the order of the set V is m^{14}, the set V is finite. Also, there are m^{14} distinct 14-tuples of elements $\mathbb{Z}_m$. So, the sequence repeats because there are only a finite number of terms possible, and the recurrence of m^{14} term results in the recurrence of all following terms. consequently, the sequence $\{\vartheta_n(m)\}$ is periodic. So if

$$\vartheta_t(m)=\vartheta_s(m),\vartheta_{t+1}(m)=\vartheta_{s+1}(m),\ldots,\vartheta_{t+13}(m)=\vartheta_{s+13}(m)$$

Such that $t>s$, then $t\equiv s(mod\ 14)$. From the defining recurrence relation of the sequence $\{\vartheta_n(m)\}$, we can easily get that

$$\vartheta_{t-1}(m)=\vartheta_{s-1}(m),\vartheta_{t-2}(m)=\vartheta_{s-2}(m),\ldots,\vartheta_{t-s}(m)=\vartheta_0(m)$$

which implies that the $\{\vartheta_n(m)\}$ is a simply periodic sequence.

Suppose that the length of the period of the sequence $\{\vartheta_n(m)\}$ is denoted by $h(m)$. Let

$$A=\begin{bmatrix}-1 & 0 & 2\\ -4i & 1 & 10i\\ -2 & 0 & 5\end{bmatrix}.$$

If we use induction on n, then we derive the following relationships between the elements of the sequence $\{\vartheta_n\}$ and the matrix A:

$$(A)^n=\begin{bmatrix}-\vartheta_{14n-16} & 0 & \vartheta_{14n-6}\\ (\vartheta_{14n-4}-1).i & 1 & (-\vartheta_{14n+4}-1).i\\ -\vartheta_{14n-6} & 0 & (\vartheta_{14n+7}).-i\end{bmatrix} \quad (3)$$

and

$$(A^{-1})^n=\begin{bmatrix}(-1)^{n+1}.(\vartheta_{14n+7}).i & 0 & (-1)^{n+1}.(\vartheta_{14n-6})\\ (-1)^{n+1}.(\vartheta_{14n-9}).i & 1 & (-1)^{n+1}.\left(\vartheta_{14(n-1)}+(-1)^n.3\right).i\\ (-1)^n.(\vartheta_{14n-6}) & 0 & (-1)^{n+1}.(\vartheta_{14n-16})\end{bmatrix}. \quad (4)$$

It is easily seen from equations (3) and (4) that $h(m)=\left|\langle A\rangle_m\right|$.

Corollary 1. Let q be a prime and t be the largest positive integer such that $\left|\langle A\rangle_q\right|=\left|\langle A\rangle_{q^t}\right|$. Then $\left|\langle A\rangle_{q^w}\right|=q^{w-t}\cdot\left|\langle A\rangle_q\right|$ for every $w\ge t$.

Corollary 2. Suppose that m_1 and m_2 are positive integers with $m_1, m_2 \geq 2$, then $\left|\langle A\rangle_{lcm[m_1,m_2]}\right| = lcm\left[\left|\langle A\rangle_{m_1}\right|, \left|\langle A\rangle_{m_2}\right|\right]$.

Theorem 6. For $m \geq 2$,

$$HF^{*}_{(x,y)}\left(D_{2m}\right) = h(m) = 14.\left|\langle A\rangle_m\right|.$$

Proof: The sequence $F^{*}_{(x,y)}\left(D_{2m}\right)$ is

$$\begin{gathered} x, y, y, x^i y^i, x^{-1} y^{i+1}, x^{-i} y, y^{i+1}, x^i, x^2 y^i, x^{2i} y^i, x^{-3} y, x^{5i} y^{i+1}, \\ x^3 y^i, x^{6i} y^{i+1}, x, x^{2i} y, x^{-8} y, x^{9i} y^i, x^{-11i} y^{i+1}, x^{-3i} y, x^{-6} y^{i+1}, x^{5i}, \\ x^8 y^i, x^{14i} y^i, x^{-19} y, x^{27i} y^{i+1}, x^{13} y^i, x^{32i} y^{i+1}, x^5, x^{8i} y, x^{-40} y, \\ x^{45i} y^i, x^{-53} y^{i+1}, x^{-13} y, x^{-32} y^{i+1}, x^{21i}, x^{34} y^i, x^{66i} y^i, x^{-87} y, \\ x^{1219i} y^{i+1}, x^{55} y^i, x^{142i} y, x^{21}, x^{34i} y, x^{-176} y, \ldots \end{gathered}$$

It can be clearly said that the complex-type Fibonacci 2-orbit from layers of fourteen. Using the above, the orbit becomes:

$$\begin{gathered} a_{14k} = x^{\vartheta_{14k}}, a_{14k+1} = x^{\vartheta_{14k+1}} y, a_{14k+2} = x^{\vartheta_{14k+2}} y, \\ a_{14k+3} = x^{\vartheta_{14k+3}} y^i, a_{14k+4} = x^{\vartheta_{14k+4}} y^{i+1}, a_{14k+5} = x^{\vartheta_{14k+5}} y, \\ a_{14k+6} = x^{\vartheta_{14k+6}} y^{i+1}, a_{14k+7} = x^{\vartheta_{14k+7}}, a_{14k+8} = x^{\vartheta_{14k+8}} y^i, \\ a_{14k+9} = x^{\vartheta_{14k+9}} y^i, a_{14k+10} = x^{\vartheta_{14k+10}} y, a_{14k+11} = x^{\vartheta_{14k+11}} y^{i+1}, \\ a_{14k+12} = x^{\vartheta_{14k+12}} y^i, a_{14k+13} = x^{\vartheta_{14k+13}} y^{i+1}, \ldots \end{gathered}$$

Since the period of the sequence $\{\vartheta_n(m)\}$ is $h(m)$ and the order of the element x is m, we obtain $HF^{*}_{(x,y)}\left(D_{2m}\right) = lcm\left[14, h(m)\right] = lcm\left[14, 14.\left|\langle A\rangle_m\right|\right] = 14.\left|\langle A\rangle_m\right| = h(m)$.

3. CONCLUSION

In this paper, the complex-type Fibonacci p-sequence according to modulo m was studied. Also, the periods and the ranks of the complex-type Fibonacci p-sequence were obtained. Then, the generating matrix of the complex-type Fibonacci p-sequence when read modulo m was considered. Also, the complex-type Fibonacci p-sequence by means of the elements of the groups was redefined. Finally, the periods of the complex-type Fibonacci 2-sequence in the dihedral group D_{2m} were examined.

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