ORIGINAL PAPER

A GENERALIZED APPROACH TO INTEGRATION PROCESS BY AUXILIARY FUNCTIONS

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Abstract. *Analytical integrals of functions in closed form are rare and one has to resort frequently to numerical integration techniques. In this work, approximate auxiliary functions that can be integrated are replaced by the original function to determine approximate analytical and numerical integrals. Information on one and two points of the function is used in determining the parameters of the auxiliary trial functions. Three alternative methods are proposed. Once the function without an available closed-form integral is replaced with an integrable one, analytical and numerical integration results can be found. The integration step size can be substantially increased by the employment of the methods.*
Keywords: *Numerical analysis; analytical approximation; integration algorithms.*

1. INTRODUCTION

Integration is one of the most fundamental topics in calculus. Analytical integration techniques such as substitution method, and integration by parts are widely discussed. However, for most of the functions, a closed-form analytical integral is not available and within this context, numerical integration is also discussed. The area approximation under a curve can be calculated by rectangles with height being the function and the width being the step size. As the step size is reduced, the approximate area represents better the exact area under the curve which is the geometrical definition of the integral. The height of the rectangle may be the functional value at the left or at the right of the incremental area. Concerning the reference point, the rectangular sum can be named as left sum or right sum. Such an approximation is classified as a single-point approximation in this work because the integration depends on the numerical value of the function at a single point. The trapezoidal sum on the other hand is calculated by piecewise linear approximation of the original function at each interval and adding them to each other. This approximation can be classified as a two-point approximation because the left and right values of the function are employed in the calculations. Finally, the Simpson method is discussed in calculus textbooks as a numerical integration technique. The method is based on approximating the function by piecewise parabolas concerning three closely spaced points on the function. Simpson's method falls into the category of three-point approximation formulas. See Thomas & Finney [1] and Strang [2] for details of the mentioned methods.

In this work, a generalization of the integration algorithms is proposed for the first time. By employing auxiliary functions which are integrable, analytical and numerical calculations of the integrals are proposed. One improvement for the single-point approximation and two improvements for the two-point approximations are proposed.

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Improvements for three point approximations are left for further studies. The methods not only produce numerical results but also produce approximate analytical results as well. It is shown that the step size can be increased substantially for the improved versions compared to the rectangular sums in reaching a pre-determined precision of the integral.

It is informative to mention some papers which complement the understanding of integration. For error analysis of several numerical methods, see Jones [3], and Weideman [4]. For discussions of the improvements in Simpson's method, see Richert [5], Velleman [6, 7]. In this work, the proper integrals are taken as examples. For evaluation of improper integrals, see Flynn [8]. For employment of auxiliary functions in determining the roots of nonlinear equations, see Pakdemirli [9].

2. THEORY OF THE INTEGRATION ALGORITHMS

A quite original presentation of the topic which does not exist in the literature will be given in this section. The algorithms depend on approximation of the original function f(x), which cannot be analytically integrated, by the auxiliary integrable functions g(x,a,b,..) which approximate piecewise the original function in the whole domain

$$f(x) = \sum_{k=0}^{n-1} g(x, a_k, b_k, \ldots)\gamma[x_k, x_{k+1})(x), \tag{1}$$

with the Gamma interval function possessing the properties

$$\gamma[a,b)(x) = \begin{cases} 1 & a \leq x < b \\ 0 & x < a, x \geq b \end{cases}. \tag{2}$$

The algorithms will be classified as integration algorithms IA(p,q) where p stands for the number of reference points and q stands for the number of parameters in the auxiliary function. For consistency, p≤q always.

2.1. SINGLE POINT ALGORITHMS

The well-known single point algorithm is the rectangular sum. The reference point is the functional value at $x=x_k$ for the left sum, $x=x_{k+1}$ for the right sum. Only one parameter is employed in the approximate representation, either $f(x_k)=a_k$, $f(x_{k+1})=a_k$. In our classification, the rectangular sum is IA(1,1) since, the reference point is single (p=1) with only one parameter (q=1) is involved in the calculations.

Instead of the constant function employed in rectangular sums, one may propose a variable approximate function having two parameters with equal functional values and its derivatives at $x=x_k$, i.e.,

$$g(x_k, a_k, b_k) = f(x_k) \tag{3}$$

$$g'(x_k, a_k, b_k) = f'(x_k) \qquad k = 0,1,2,..n-1 \tag{4}$$

where n is the number of intervals. Parameters a_k and b_k are solved from above for each interval and the approximation of the function would then be given by (1). $g(x_k,a_k,b_k)$ should be chosen as an easily integrable function with inspiration from the original function and assume that $\int g(x, a_k, b_k)dx = G\,(x, a_k, b_k)$. The numerical integral would then be

$$\int_{x_0}^{x_n} f(x) \cong \sum_{k=0}^{n-1} G(x_{k+1}, a_k, b_k) - G(x_k, a_k, b_k) \tag{5}$$

Since the auxiliary function is integrable, one also has an approximate analytical solution for the integral of an intermediate value of $[x_i, x_{i+1}]$, $(0 \le i \le n-1)$

$$\int_{x_0}^{x} f(x) \cong G(x, a_i, b_i) - G(x_i, a_i, b_i) + \sum_{k=0}^{i-1} G(x_{k+1}, a_k, b_k) - G(x_k, a_k, b_k). \tag{6}$$

The algorithm (3)-(6) can be classified as IA(1,2) since only one reference point x_k is employed with two parameters in the trial function.

2.2. DOUBLE POINT ALGORITHMS

Two different double point algorithms will be presented, namely the IA(2,2) and IA(2,3) algorithms. The former incorporates two parameters whereas the latter incorporates three parameters in the calculations.

i) IA(2,2) algorithm

Two parameters are involved in this algorithm with reference to two points $x=x_k$ and $x=x_{k+1}$. Parameters are determined by the algebraic equations

$$g(x_k, a_k, b_k) = f(x_k) \tag{7}$$

$$g(x_{k+1}, a_k, b_k) = f(x_{k+1}) \tag{8}$$

If $g(x,a_k,b_k)$ is a linear function, then the trapezoidal solution will be retrieved as a special case of the general IA(2,2) algorithm. The numerical and analytical integrals are then given by (5) and (6).

ii) IA(2,3) algorithm

To increase the precision, one may add an additional parameter with reference to two points $x=x_k$ and $x=x_{k+1}$ again. The three parameters can be determined by the algebraic equations

$$g(x_k, a_k, b_k, c_k) = f(x_k) \tag{9}$$

$$g'(x_k, a_k, b_k, c_k) = f'(x_k) \tag{10}$$

$$g(x_{k+1}, a_k, b_k, c_k) = f(x_{k+1}). \tag{11}$$

where one equates the functional values at the left and right and the slopes at the left for the original function and the trial function. Two reference points with three parameters makes the algorithm an IA(2,3) algorithm. The numerical and analytical integrals are then given by (5) and (6) with inclusion of the parameter c_k.

Higher order algorithms are always possible which are not examined in this work. For example, an IA(2,4) algorithm can be constructed by inclusion of the slope equivalency at the right hand or alternatively by inclusion of the second derivative at the left hand. Simpson's method is in fact a special case of an IA(3,3) algorithm since the three parameters in the parabolic function $g(x, a_k, b_k, c_k) = a_k x^2 + b_k x + c_k$ are determined by equating the functional values at three reference points x_k , x_{k+1} and x_{k+2}. If $g(x, a_k, b_k, c_k)$ is an arbitrary integrable function, then the Simpson's algorithm is generalized.

3. NUMERICAL RESULTS

In this section, the algorithms will be applied to three sample problems. Numerical as well as approximate analytical solutions will be given for each problem.

Example 1. The integral $\int_0^3 e^{-x^2} dx$ does not have a closed form solution in terms of the simple analytical functions. Four algorithms IA(1,1), IA(1,2), IA(2,2) and IA(2,3) will be compared with each other.

The left sum is considered in the rectangular IA(1,1) algorithm. The integral is equivalent to the sum

$$\mathrm{IA}(1,1) = \sum_{k=0}^{n-1} h e^{-x_k^2} \tag{12}$$

where h is the constant step size with $x_{k+1} = x_k + h$ and $n = \frac{x_n - x_0}{h}$.

For IA(1,2), one has to choose an auxiliary function first. The original function is replaced with the approximate analytical expression

$$f(x) = \sum_{k=0}^{n-1} (a_k e^{-x} + b_k) \gamma[x_k, x_{k+1})(x), \tag{13}$$

where the form is chosen with inspiration from the original function. The expression is integrable with a result

$$\mathrm{IA}(1,2) = \sum_{k=0}^{n-1} -a_k (e^{-x_{k+1}} - e^{-x_k}) + b_k h. \tag{14}$$

The continuous integral function for any x value is

$$\begin{aligned} \mathrm{IA}(1,2)(x) = -a_i (e^{-x} - e^{-x_k}) + b_i (x - x_i) \\ + \sum_{k=0}^{i-1} -a_k (e^{-x_{k+1}} - e^{-x_k}) + b_k h \end{aligned} \tag{15}$$

for $[x_i, x_{i+1}]$, $(0 \leq i \leq n-1)$. The coefficients in each interval are calculated from (3) and (4)

$$a_k e^{-x_k} + b_k = e^{-x_k^2} \tag{16}$$

$$a_k e^{-x_k} = 2 x_k e^{-x_k^2} \tag{17}$$

where the first equation states the equivalence of the functional values and the second equation, the derivative values at $x = x_k$. The solution is

$$a_k = 2 x_k e^{x_k - x_k^2} \ , \quad b_k = (1 - 2x_k) e^{-x_k^2} \tag{18}$$

The numerical integral is then given in (14) and the approximate analytical integral in (15). For IA(2,2), the equations to be solved are from (7) and (8)

$$a_k e^{-x_k} + b_k = e^{-x_k^2} \tag{19}$$

$$a_k e^{-x_{k+1}} + b_k = e^{-x_{k+1}^2} \tag{20}$$

which yields

$$a_k = \frac{e^{-x_{k+1}^2} - e^{-x_k^2}}{e^{-x_{k+1}} - e^{-x_k}}\ , \quad b_k = e^{-x_k^2} - a_k e^{-x_k} \tag{21}$$

For these coefficients, (14) and (15) still gives the numerical and approximate integrals. Finally, for the IA(2,3) algorithm, an approximate representation with three parameters can be selected

$$f(x) = \sum_{k=0}^{n-1} (a_k e^{x} + b_k x + c_k)\gamma[x_k, x_{k+1})(x)\ , \tag{22}$$

with the numerical integral being

$$\mathrm{IA}(2,3) = \sum_{k=0}^{n-1} a_k (e^{x_{k+1}} - e^{x_k}) + \frac{b_k}{2}(x_{k+1}^2 - x_k^2) + c_k h.\ , \tag{23}$$

The continuous approximate analytical integral would then be

$$\begin{aligned} \mathrm{IA}(2,3)(x) = {} & a_i (e^{x} - e^{x_i}) + \frac{b_i}{2}(x^2 - x_i^2) + c_i (x - x_i) \\ & + \sum_{k=0}^{i-1} a_k (e^{x_{k+1}} - e^{x_k}) + \frac{b_k}{2}(x_{k+1}^2 - x_k^2) + c_k h, \end{aligned} \tag{24}$$

for $[x_i, x_{i+1}]$, $(0 \le i \le n-1)$. The coefficients are determined from (9)-(11)

$$a_k e^{x_k} + b_k x_k + c_k = e^{-x_k^2}, \tag{25}$$

$$a_k e^{x_k} + b_k = -2x_k e^{-x_k^2} \tag{26}$$

$$a_k e^{x_{k+1}} + b_k x_{k+1} + c_k = e^{-x_{k+1}^2} \tag{27}$$

with solutions

$$a_k = \frac{e^{-x_{k+1}^2} - e^{-x_k^2} + 2hx_k e^{-x_k^2}}{e^{x_{k+1}} - (1+h)e^{x_k}},\ b_k = -2x_k e^{-x_k^2} - a_k e^{x_k},\ c_k = e^{-x_k^2} - a_k e^{x_k} - b_k x_k \tag{28}$$

The numerical solutions are contrasted in Table 1 for various step sizes.

Table 1. Numerical Integrals $\int_0^3 e^{-x^2} dx$ for Various Step Sizes

h	IA(1,1)	IA(1,2)	IA(2,2)	IA(2,3)
0.5	1.1362	0.9323	0.8654	0.8901
0.2	0.9862	0.8932	0.8829	0.8864
0.1	0.9362	0.8879	0.8854	0.8862
0.05	0.9112	0.8866	0.8860	
0.02	0.8962	0.8863	0.8862	
0.01	0.8912	0.8862		
0.001	0.8867			
0.0001	0.8863			
0.00005	0.8862			

From the table, the rectangular sum converged to the 4 digit precision with a very small step size of 0.00005. The convergence is much better for IA(1,2) with h=0.01. IA(2,2) is slightly better than IA(1,2) with a step size of h=0.02. The best is IA(2,3) with a much larger step size of h=0.1.

The precision of each integral can be visualized from Fig. 1, which depicts the function and its various approximations. A large step size of h=0.5 is chosen to distinguish the variations from each other and from the original function. Rectangular approximation is not shown. Among the various approximations, excluding IA(1,1), the worst is the IA(1,2) and the best is the IA(2,3). As the number of reference points and the number of parameters increase, the precision is higher.

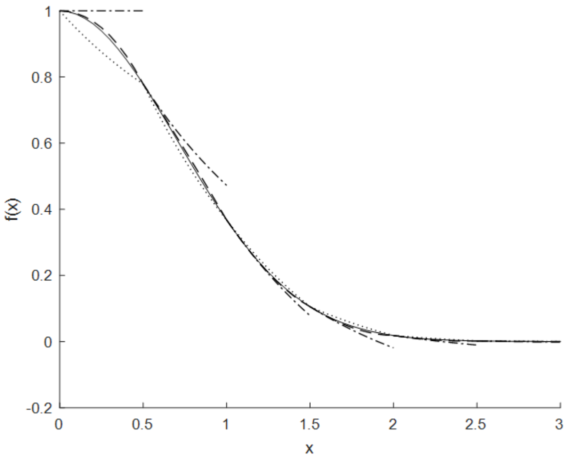

Figure 1. Function (Solid) and its Approximations IA(1,2)(Dashed-dotted), IA(2,2)(dotted) and IA(2,3)(Dashed) (h=0.5).

The continuous integral function is shown in Fig. 2 by plotting the approximate analytical integral IA(2,3) given in (24) for h=0.1.

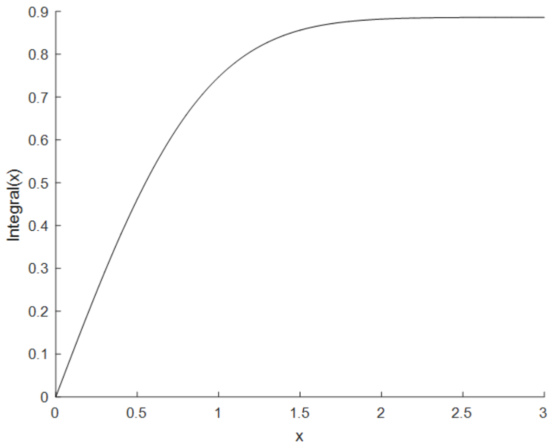

Figure 2. Integral Function (h=0.1)

Example 2. The integral $\int_0^1 \sin(x^2)dx$ does not have a closed form solution in terms of the simple analytical functions. Four algorithms IA(1,1), IA(1,2), IA(2,2) and IA(2,3) will be compared with each other.

The left rectangular sum of IA(1,1) is equivalent to

$$\mathrm{IA}(1,1) = \sum_{k=0}^{n-1} hsin(x_k^2), \tag{29}$$

where h is the constant step size with $x_{k+1} = x_k + h$ and $n = \frac{x_n - x_0}{h}$.

For IA(1,2), a suitable integrable trial function

$$f(x) = \sum_{k=0}^{n-1} (a_k sinx + b_k)\gamma[x_k, x_{k+1})(x), \tag{30}$$

may be chosen with inspiration from the original function. The integral of the approximation is

$$\mathrm{IA}(1,2) = \sum_{k=0}^{n-1} -a_k(cos(x_{k+1}) - cos(x_k)) + b_k h. \tag{31}$$

The continuous integral function for any x value is

$$\begin{aligned}\mathrm{IA}(1,2)(x) = -a_i(cosx - cosx_i) + b_i(x - x_i)\\ + \sum_{k=0}^{i-1} -a_k(cosx_{k+1} - cosx_k) + b_k h\end{aligned} \tag{32}$$

for $[x_i, x_{i+1}]$, $(0 \le i \le n-1)$. The coefficients in each interval are calculated from (3) and (4)

$$a_k sinx_k + b_k = sinx_k^2 \tag{33}$$

$$a_k cosx_k = 2x_k cosx_k^2 \tag{34}$$

where the first equation states the equivalence of the functional values and the second equation, the derivative values at $x = x_k$. The solutions are

$$a_k = \frac{2x_k cosx_k^2}{cosx_k} \ , \quad b_k = sinx_k^2 - a_k sinx_k \tag{35}$$

The numerical integral is then given in (31) and the approximate analytical integral in (32). For IA(2,2), the equations to be solved are from (7) and (8)

$$a_k sinx_k + b_k = sinx_k^2 \tag{36}$$

$$a_k sinx_{k+1} + b_k = sinx_{k+1}^2 \tag{37}$$

which yields

$$a_k = \frac{sinx_{k+1}^2 - sinx_k^2}{sinx_{k+1} - sinx_k} \ , \quad b_k = sinx_k^2 - a_k sinx_k \tag{38}$$

For the above coefficients, (31) and (32) yields the numerical and approximate integrals. Finally, for the IA(2,3) algorithm, an approximate representation with three parameters can be selected

$$f(x) = \sum_{k=0}^{n-1} (a_k sinx + b_k x + c_k)\gamma[x_k, x_{k+1})(x), \tag{39}$$

with the numerical integral being

$$\mathrm{IA}(2,3) = \sum_{k=0}^{n-1} -a_k(cosx_{k+1} - cosx_k) + \frac{b_k}{2}(x_{k+1}^2 - x_k^2) + c_k h \tag{40}$$

The continuous approximate analytical integral would then be

$$\begin{gathered}\mathrm{IA}(2,3)(x) = -a_i(cosx - cosx_i) + \frac{b_i}{2}(x^2 - x_i^2) + c_i(x - x_i) \\ + \sum_{k=0}^{i-1} -a_k(cosx_{k+1} - cosx_k) + \frac{b_k}{2}(x_{k+1}^2 - x_k^2) + c_k h\end{gathered} \tag{41}$$

for $[x_i, x_{i+1}]$, $(0 \leq i \leq n-1)$.

The coefficients are determined from (9)-(11)

$$a_k sinx_k + b_k x_k + c_k = sinx_k^2 \tag{42}$$

$$a_k cosx_k + b_k = 2x_k cosx_k^2 \tag{43}$$

$$a_k sinx_{k+1} + b_k x_{k+1} + c_k = sinx_{k+1}^2 \tag{44}$$

with solutions

$$\begin{gathered}a_k = \frac{sinx_{k+1}^2 - sinx_k^2 - 2hx_k cosx_k^2}{sinx_{k+1} - sinx_k - hcosx_k},\ b_k = 2x_k cosx_k^2 - a_k cosx_k, \\ c_k = sinx_k^2 - a_k sinx_k - b_k x_k\end{gathered} \tag{45}$$

Table 2 shows the integration results.

Table 2. Numerical Integrals $\int_0^1 \sin(x^2)dx$ for Various Step Sizes

h	IA(1,1)	IA(1,2)	IA(2,2)	IA(2,3)
0.5	0.1237	0.2314	0.3470	0.2932
0.2	0.2298	0.2981	0.3161	0.3089
0.1	0.2691	0.3073	0.3117	0.3101
0.05	0.2895	0.3095	0.3106	0.3102
0.02	0.3019	0.3102	0.3103	0.3103
0.01	0.3061	0.3102		
0.001	0.3098	0.3103		
0.0001	0.3102			
0.00001	0.3103			

From the table, the rectangular sum converged to the 4 digit precision with a very small step size of 0.00001 which requires extremely high computational times. The convergence is much better for IA(1,2) with h=0.001. Although IA(2,2) and IA(2,3) converged with a same step size of h=0.02, the convergence is slightly better for IA(2,3). One can deduce from the results that the best algorithm is IA(2,3) and the next best is IA(2,2).

The approximations of the original function are shown in Fig. 3. A large step size of h=0.2 is chosen to distinguish the variations from each other and from the original function. Rectangular approximation is not shown. Since IA(2,3) is the best approximation with slight discrepancies only for small x, the best numerical integral values can be achieved (Table 2). IA(2,2) better approximates the function than IA(1,2) which results in a relatively higher step size of convergence.

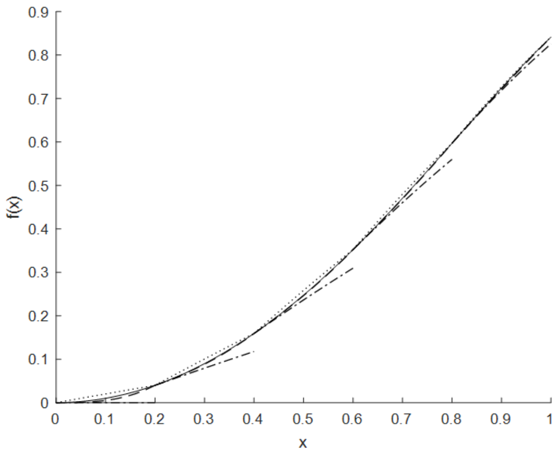

Figure 3. Function (Solid) and its Approximations IA(1,2)(Dashed-dotted), IA(2,2)(dotted) and IA(2,3)(Dashed) (h=0.2)

The continuous integral function is shown in Fig. 4 by plotting the approximate analytical integral IA(2,3) given in (41) for h=0.02.

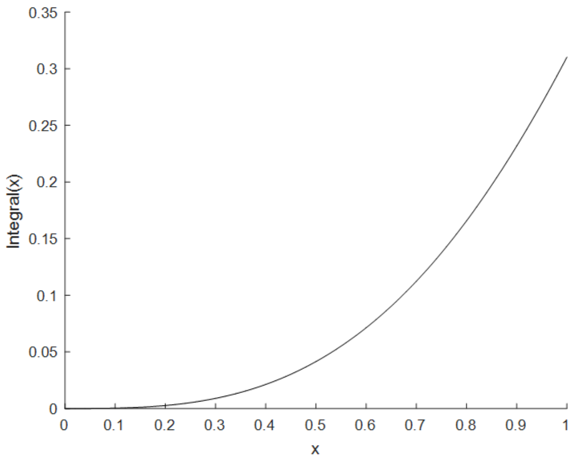

Figure 4. Integral Function (h=0.02)

Example 3. Consider the integral $\int_0^1 \frac{sinx}{1+x^2}dx$. The rectangular left sum of IA(1,1) is

$$\mathrm{IA}(1,1) = \sum_{k=0}^{n-1} h \frac{sinx_k}{1+x_k^2} \tag{46}$$

A trial function for IA(1,2) and IA(2,2)

$$f(x) = \sum_{k=0}^{n-1} \frac{a_k x + b_k}{1+x^2} \gamma[x_k, x_{k+1})(x), \tag{47}$$

may be chosen to simplify the integral. The integral of the approximation is

$$\mathrm{IA}(1,2) = \sum_{k=0}^{n-1} \frac{a_k}{2} ln \frac{1+x_{k+1}^2}{1+x_k^2} + b_k(atan\, x_{k+1} - atanx_k)\ . \tag{48}$$

The continuous integral function for any *x* value is

$$\mathrm{IA}(1,2)(x) = \frac{a_i}{2} ln \frac{1+x^2}{1+x_i^2} + b_i(atanx - atanx_i)$$
$$+\sum_{k=0}^{i-1} \frac{a_k}{2} ln \frac{1+x_{k+1}^2}{1+x_k^2} + b_k(atan\, x_{k+1} - atanx_k) \tag{49}$$

for $[x_i, x_{i+1}]$, $(0 \leq i \leq n-1)$. The coefficients in each interval are calculated from (3) and (4) with some simplifications

$$a_k x_k + b_k = sinx_k \tag{50}$$

$$a_k(1+x_k^2) - 2x_k(a_k x + b_k) = (1+x_k^2)cosx_k - 2x_k sinx_k \tag{51}$$

where the first equation states the equivalence of the functional values and the second equation, the derivative values at $x = x_k$. The solutions are

$$a_k = cosx_k \ , \quad b_k = sinx_k - a_k x_k \tag{52}$$

The numerical integral is then given in (48) and the approximate analytical integral in (49). For IA(2,2), the equations to be solved are from (7) and (8)

$$a_k x_k + b_k = sinx_k \tag{53}$$

$$a_k x_{k+1} + b_k = sinx_{k+1} \tag{54}$$

which yields

$$a_k = \frac{sinx_{k+1} - sinx_k}{h} \ , \quad b_k = sinx_k - a_k x_k \tag{55}$$

For the above coefficients, (48) and (49) yields the numerical and approximate analytical integrals. Finally, for the IA(2,3) algorithm, an approximate representation with three parameters can be selected

$$f(x) = \sum_{k=0}^{n-1} \frac{a_k x^2 + b_k x + c_k}{1+x^2} \gamma[x_k, x_{k+1})(x), \tag{56}$$

with the numerical integral being

$$\mathrm{IA}(2,3) = \sum_{k=0}^{n-1} a_k(h - atanx_{k+1} + atanx_k) + \frac{b_k}{2} ln \frac{1+x_{k+1}^2}{1+x_k^2} + c_k(atanx_{k+1} - atanx_k) \tag{57}$$

The continuous approximate analytical integral would then be

$$\mathrm{IA}(2,3)(x) = a_i(x - x_i - atanx + atanx_i) + \frac{b_i}{2} ln \frac{1+x^2}{1+x_i^2} + c_i(atanx - atanx_i)$$
$$+\sum_{k=0}^{i-1} a_k(h - atanx_{k+1} + atanx_k) + \frac{b_k}{2} ln \frac{1+x_{k+1}^2}{1+x_k^2} + c_k(atanx_{k+1} - atanx_k) \tag{58}$$

for $[x_i, x_{i+1}]$, $(0 \leq i \leq n-1)$. The coefficients are determined from (9)-(11)

$$a_k x_k^2 + b_k x_k + c_k = sinx_k \tag{59}$$

$$(2a_k x_k + b_k)(1 + x_k^2) - 2x_k(a_k x_k^2 + b_k x_k + c_k) \\ = (1 + x_k^2)cosx_k - 2x_k sinx_k \tag{60}$$

$$a_k x_{k+1}^2 + b_k x_{k+1} + c_k = sinx_{k+1} \tag{61}$$

with solutions

$$a_k = \frac{sinx_{k+1} - sinx_k - hcosx_k}{h^2}, \;\; b_k = cosx_k - 2x_k a_k, \;\; c_k = sinx_k - a_k x_k^2 - b_k x_k \tag{62}$$

Table 3 shows the integration results.

Table 3. Numerical Integrals $\int_0^1 \frac{sinx}{1+x^2} dx$ for Various Step Sizes

h	IA(1,1)	IA(1,2)	IA(2,2)	IA(2,3)
0.5	0.1918	0.3309	0.3149	0.3206
0.2	0.2759	0.3236	0.3207	0.3217
0.1	0.2998	0.3223	0.3215	0.3218
0.02	0.3175	0.3218	0.3218	
0.01	0.3197			
0.001	0.3216			
0.0002	0.3218			

From the table, the rectangular sum converged to the 4 digit precision with a very small step size of 0.0002. The convergence is much better for IA(1,2) and IA(2,2) with h=0.02. The best is IA(2,3) with a relatively large step size of h=0.1.

The approximations of the original function are shown in Fig. 5. A very large step size of h=0.5 is chosen to distinguish the variations from each other and from the original function. Although the step size is very large, IA(2,3) approximation is perfect. Excluding IA(1,1), the worse approximation is due to IA(1,2). If the step size is reduced, then the approximations better represent the real function hence resulting in a higher precision of the integral.

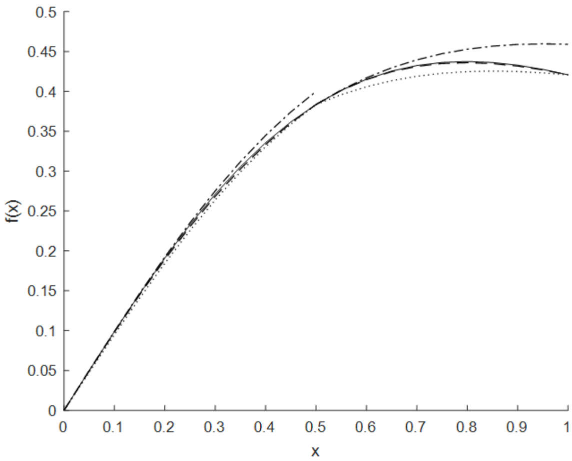

Figure 5. Function (Solid) and its Approximations IA(1,2)(Dashed-dotted), IA(2,2)(dotted) and IA(2,3)(Dashed) (h=0.5)

The continuous integral function is shown in Fig. 6 by plotting the approximate analytical integral IA(2,3) given in (58) for h=0.1.

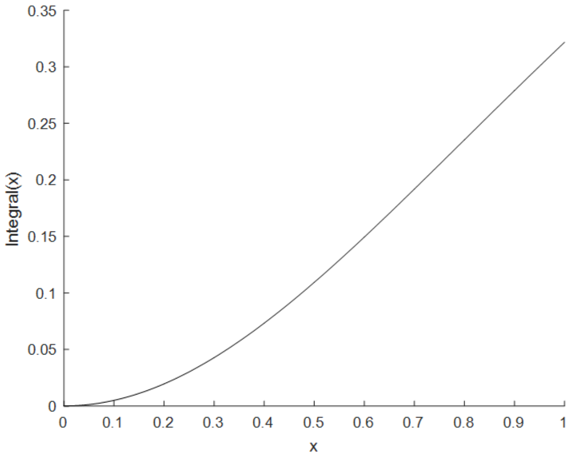

Figure 6. Integral Function (h=0.1)

4. CONCLUDING REMARKS

A unified and generalized classification is proposed for the integration algorithms. The following conclusions can be drawn:
1) Integration precision can be increased by selecting a suitable auxiliary function;
2) Increasing the reference points, for which the integration formula is based, increases the precision;
3) Increasing the parameters in the auxiliary function increases the precision;
4) Depending on the specific choice of auxiliary function, number of reference points and the parameters involved, infinitely many integration algorithms can be developed;
5) Besides the numerical calculations, approximate analytical expressions of integral functions can be derived using the given algorithms;

The work can be extended in the future to include higher order IA(p,q) algorithms.

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