ORIGINAL PAPER

ANALYSIS OF A NON-LINEAR RECURRENCE FORMULATED USING MATRIX PERMANENTS

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Abstract. *In this study, we focus on reanalyzing and reducing a recursive formula that generates the permanents of a family of matrices called k-tridiagonal Toeplitz. This recursive formula is dependent on two free variables and is in non-linear form. We first propose an alternative proof to the one given in "Recursive and combinational formulas for permanents of general k-tridiagonal Toeplitz matrices". This alternative proof is derived from a detailed examination of the designs of the submatrix blocks resulting from the expansion of permanents. Subsequently, we reduce this recursive formula to a linear form. This linear recursive formula simplifies the computation process by effectively reducing k-tridiagonal Toeplitz matrix permanents to those with a bandwidth of* 1.

Keywords: *Permanent; multivariable recurrence; k-tridiagonal Toeplitz matrix.*

1. INTRODUCTION

Let $A = [a_{i,j}]$ be a matrix of size n by n with entries in a field F. Then the permanent of A is defined by

$$\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_i} \tag{1}$$

where S_n is the symmetric group which consists of all permutations of $\{1,2,\dots,n\}$, and σ is an element of this group where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ [1].

The permanent function is used as an effective tool for solving combinatorial enumerating problems. For instance, the Menage problem has been connected to the permanents of (0,1) matrices [2]. Applications associated with the permanents of some special matrix classes are also encountered in other areas. For example, the permanents of adjacency and distance matrices describing the structures of alkanes were used to determine the complexity order of the alkane isomers [3]. As another example, following a technique that uses object recognition to obtain semantic information from the robot's sensors, the problem of calculating the probability of an observation has been solved by matching with the computing of some permanents [4].

Let $A = [a_{ij}]$ be a square matrix of order n by n , and let some entries of A are $a_{ij} = 0$. If there is an integer $k > 0$ that satisfies the condition $|i - j| > k$ for (i, j) indices of

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these zero elements, then matrix A is called the band matrix. The integer k is called the bandwidth of matrix A. Let us consider the nonzero elements of any square band matrix are positioned in just three bands. One of these nonzero bands is the main diagonal, and the other two are symmetrically located to the main diagonal. Let the upper and the lower nonzero bands be shifted equidistantly to the main diagonal with zero bands to be placed between the two. When the number of the zero bands between the upper and the lower nonzeros is $2k-2$, the matrix is called k-tridiagonal. As an example of the applications of this matrix type, as seen in [5], the matrix representing the state operator of a composite system in quantum mechanics has 2-tridiagonal structure.

A Toeplitz matrix consists of the same constant entries for every diagonal from top left to bottom right. An $n \times n$ Toeplitz matrix $A = \left[a_{i,j}\right]$ has $2n-1$ independent elements defined by $a_{i,j} = a_{i-j}$ where $i, j = 1,2, \dots, n$. Toeplitz matrices have applications in various fields, including time series analysis, probability theory, statistics, filtering theory, and estimation theory. The definitions and properties of Toeplitz and related matrices, along with their applications, are detailed in [6].

Let $T_n^{(k)}(a,b,c) = \left[t_{ij}\right] \in M_n(\mathbb{C})$ be the general k-tridiagonal Toeplitz matrix such that

$$t_{ij} = \begin{cases} a, i-j=0 \\ b, i-j=-k \\ c, i-j=k \\ 0, \text{otherwise} \end{cases} \tag{2}$$

where $n \geq k+1$. For example, general 4-tridiagonal Toeplitz matrix of order 8×8 is

$$T_8^{(4)}(a,b,c) = \begin{bmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ c & 0 & 0 & 0 & a & 0 & 0 & b \\ 0 & c & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & a \end{bmatrix}_{8\times 8} . \tag{3}$$

We note that the set of all $n \times n$ matrices over a field F is denoted by $M_n(F)$. These becomes $M_n(\mathbb{C})$ if $F = \mathbb{C}$, where $\mathbb{C}$ is the set of all complex numbers.

Some studies have aimed to derive the permanent of a matrix using recursive formulas. For instance, in [7], Minc focuses on finding recursive formulas for the permanents of (0,1)-circulant matrices. Similarly, in [8], the permanent of a tridiagonal matrix can be efficiently computed with the help of a recurrence relation. Further exemplary work can be found in studies referenced [9-16].

Building on these explorations, studies on recurrence relations derived from the permanent of a matrix include [17], which presents an inductive proof of Borchardt's theorem. This well-known result in combinatorics expresses the permanent of a Cauchy matrix in terms of determinants. There are also works dealing with recurrence relations obtained using other matrix functions. One such study, [18], focuses on the matrix exponential. In [18], the authors analyzed various methods for solving a linear recurrence relation with two indices, employing both analytical and combinatorial approaches.

The study of linearization is important for both practical and theoretical purposes in the field of logical databases and in the field of recursive rules [19]. Linearization enables complicated recursive relationships to be transformed into more simple forms, which can lead to more efficient computational methods and easier theoretical analysis. The process of linearizing recurrences can significantly reduce computational complexity. For instance, [20] explores the effects of a linearization process. The study highlights that while linearization may not be perfectly suited to the original non-linear problem, it shares many analogous characteristics. The focus is on applying and solving these nonlinear recurrences, demonstrating the relevance of linearization in understanding complex problems.

In this study, our primary focus is to present a new proof that verifies a non-linear formula presented in [10], and subsequently simplify it by reducing it to a linear form.

2. MAIN RESULTS

In this section, we elaborate on the main findings of our study. We enter into the detailed intricacies of the non-linear recursive formula, discussed in our previous work [10]. This recurrence analysed in this work is one of the recurrences given as an example in the paper titled "Inductive proof of Borchardt's theorem" [17] , where it is used to address critical problems in the numerical tractability of computational methods for certain classes of matrices.

2.1. AN ENHANCED PROOF FOR A NON-LINEAR RECURSIVE FORMULA

We present a new proof using a different approach from the one in [10]. This proof is based on analyzing the permanents of $T_n^{(k)}(a,b,c)$ matrices, which are constructed by varying the order n in relation to the bandwidth k.

Theorem 2.1.1. [10] $P_n^{(k)}$ *denote the permanent of general* k*-tridiagonal Toeplitz matrix* $T_n^{(k)}(a,b,c) \in M_n(C)$ *. Then,*

$$P_n^{(k)} = aP_{n-1}^{(k)} + bcP_{\left\lceil\frac{n}{k}\right\rceil-2}^{(1)} P_{n-\left\lceil\frac{n}{k}\right\rceil}^{(k-1)} \tag{4}$$

where $n \geq k+1$ *, and if* $0 \leq s \leq k$ *then* $P_s^{(k)} = a^s$*.*

Proof: The proof will be conducted on k-length integer intervals, which we define as

$$jk + 1 \leq n \leq (j+1)k \tag{5}$$

where $j \in \mathbb{Z}^+$. Let us analyze separately the permanents of $T_n^{(k)}$ matrices appropriate to each of the intervals defined above.

Let $k+1 \leq n \leq 2k$. If the permanent of matrix $T_n^{(k)}$ is expanded by applying the Laplace expansion with respect to the first row, then we get

$$P_n^{(k)} = aP_{n-1}^{(k)} + bPer\left[\begin{array}{c|cc|c} & \multicolumn{2}{c|}{A_{k-1}} & \begin{array}{c} B_{n-k-1} \\ \hline [0]_{2k-n,n-k-1} \end{array} \\ \hline c & & & \\ \hline & C_{n-k-1} & [0]_{n-k-1,2k-n} & A_{n-k-1} \end{array}\right]_{n-1} \tag{6}$$

where the empty blocks denote that all entries are zero. Also, $[0]_{p,r} \in M_{p,r}(F)$ denotes a matrix whose all elements are zero. Note that we do not use the symbol $[0]_{p,r}$ when $p = 0$ or $r = 0$, and instead leave the block empty.

When the Laplace expansion is applied to the permanent of the partitioned matrix in Eq.(6) along its first column, it becomes

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_{k-1}} & \begin{array}{c} B_{n-k-1} \\ \hline [0]_{2k-n,n-k-1} \end{array} \\ \hline C_{n-k-1} & [0]_{n-k-1,2k-n} & A_{n-k-1} \end{array}\right]_{n-2} \tag{7}$$

which corresponds to the matrix $T_{n-2}^{(k-1)}(a, b, c)$. So, we have

$$P_n^{(k)} = aP_{n-1}^{(k)} + bcP_{n-2}^{(k-1)}. \tag{8}$$

Considering $P_0^{(1)} = 1$ from the initial conditions of Eq.(4), we can use

$$P_n^{(k)} = aP_{n-1}^{(k)} + bcP_0^{(1)}P_{n-2}^{(k-1)} \tag{9}$$

instead of Eq.(8).

Let $2k + 1 \leq n \leq 3k$. If the permanent of matrix $T_n^{(k)}$ is expanded by applying the Laplace expansion with respect to the first row, then we get

$$P_n^{(k)} = aP_{n-1}^{(k)} + b\mathrm{Per}(M) \tag{10}$$

where

$$M = \left[\begin{array}{c|c|cc|c|c} & A_{k-1} & \multicolumn{2}{c|}{B_{k-1}} & & [0]_{3k-n,n-2k-1} \\ \hline c & & & & b & \\ \hline & C_{k-1} & \multicolumn{2}{c|}{A_{k-1}} & & \begin{array}{c} B_{n-2k-1} \\ \hline [0]_{3k-n,n-2k-1} \end{array} \\ \hline & & & & a & \\ \hline & [0]_{n-2k-1,3k-n} & C_{n-2k-1} & [0]_{n-2k-1,3k-n} & & A_{n-2k-1} \end{array}\right]_{n-1} \tag{11}$$

If the Laplace expansion is applied along the first column of the permanent of partitioned matrix in Eq.(11), and then applied again along the $(2k-1)^{\text{th}}$ row of the $(n-2)$ −order partitioned matrix which is emerged after the first expansion step, thus the following matrix is obtained.

$$M' = \left[\begin{array}{c|cc|c} A_{k-1} & \multicolumn{2}{c|}{B_{k-1}} & [0]_{3k-n,n-2k-1} \\ \hline C_{k-1} & \multicolumn{2}{c|}{A_{k-1}} & \begin{array}{c} B_{n-2k-1} \\ \hline [0]_{3k-n,n-2k-1} \end{array} \\ \hline [0]_{n-2k-1,3k-n} & C_{n-2k-1} & [0]_{n-2k-1,3k-n} & A_{n-2k-1} \end{array}\right]_{n-3} \tag{12}$$

The matrix M' corresponds to the matrix $T_{n-3}^{(k-1)}(a,b,c)$. So, we get

$$P_n^{(k)} = aP_{n-1}^{(k)} + bcaP_{n-3}^{(k-1)}. \tag{13}$$

Considering the initial condition $P_1^{(1)} = a$ from Theorem 2.1.1, it follows that

$$P_n^{(k)} = aP_{n-1}^{(k)} + bcP_1^{(1)}P_{n-3}^{(k-1)} \tag{14}$$

Let $3k+1 \le n \le 4k$. If the permanent of $T_n^{(k)}$ is expanded by applying the Laplace expansion with respect to the first row, then we get

$$P_n^{(k)} = aP_{n-1}^{(k)} + b\mathrm{Per}(X) \tag{15}$$

where

$$X = \left[\begin{array}{c|c|c|c|c|c|c}
 & A_{k-1} & B_{k-1} & & [0]_{k-1} & & [0]_{4k-n,n-3k-1} \\ \hdashline
c & & & b & & & \\ \hdashline
 & C_{k-1} & A_{k-1} & & B_{k-1} & & [0]_{4k-n,n-3k-1} \\ \hdashline
 & & & a & & b & \\ \hdashline
 & [0]_{k-1} & C_{k-1} & & A_{k-1} & & B_{n-3k-1} \\ \hdashline
 & & & c & & a & \\ \hdashline
 & [0]_{n-3k-1,4k-n} & [0]_{n-3k-1,4k-n} & & C_{n-3k-1} & & A_{n-3k-1}
\end{array}\right]_{n-1} \tag{16}$$

If the Laplace expansion is applied along the first column of the permanent of matrix X and then applied again along the $(2k-1)^{\text{th}}$ row of the $(n-2)$ −order partitioned matrix which is emerged after the first expansion step, then the following equality is obtained.

$$P_n^{(k)} = aP_{n-1}^{(k)} + bca\mathrm{Per}(Y) + b^2c\mathrm{Per}(Z) \tag{17}$$

where

$$Y = \left[\begin{array}{c|c|c|c|c}
A_{k-1} & B_{k-1} & [0]_{k-1} & & [0]_{4k-n,n-3k-1} \\ \hdashline
C_{k-1} & A_{k-1} & B_{k-1} & & [0]_{4k-n,n-3k-1} \\ \hdashline
[0]_{k-1} & C_{k-1} & A_{k-1} & & B_{n-3k-1} \\ \hdashline
 & & & a & \\ \hdashline
[0]_{n-3k-1,4k-n} & [0]_{n-3k-1,4k-n} & C_{n-3k-1} & & A_{n-3k-1}
\end{array}\right]_{n-3} \tag{18}$$

and

$$Z = \left[\begin{array}{c|c|c|c|c}
A_{k-1} & B_{k-1} & & [0]_{k-1} & [0]_{4k-n,n-3k-1} \\ \hdashline
C_{k-1} & A_{k-1} & & B_{k-1} & [0]_{4k-n,n-3k-1} \\ \hdashline
[0]_{k-1} & C_{k-1} & & A_{k-1} & B_{n-3k-1} \\ \hdashline
 & & c & & \\ \hdashline
[0]_{n-3k-1,4k-n} & [0]_{n-3k-1,4k-n} & & C_{n-3k-1} & A_{n-3k-1}
\end{array}\right]_{n-3}. \tag{19}$$

Finally, if the Laplace expansion is applied to the permanent of matrix Y and the permanent of matrix Z along theirs $(3k-2)^{\text{th}}$ rows, then we get

$$P_n^{(k)} = aP_{n-1}^{(k)} + (bca^2 + b^2c^2)\mathrm{Per}(W) \tag{20}$$

where

$$W=\left[\begin{array}{c|c|c|c} A_{k-1} & B_{k-1} & [0]_{k-1} & [0]_{4k-n,n-3k-1} \\ \hdashline C_{k-1} & A_{k-1} & B_{k-1} & [0]_{4k-n,n-3k-1} \\ \hdashline [0]_{k-1} & C_{k-1} & A_{k-1} & B_{n-3k-1} \\ \hdashline [0]_{n-3k-1,4k-n} & [0]_{n-3k-1,4k-n} & C_{n-3k-1} & A_{n-3k-1} \end{array}\right]_{n-4}. \tag{21}$$

It can be seen that matrix W corresponds to the matrix $T_{n-4}^{(k-1)}(a,b,c)$. Thus, instead of Eq.(20), we can write

$$P_n^{(k)}=aP_{n-1}^{(k)}+bcP_2^{(1)}P_{n-4}^{(k-1)}. \tag{22}$$

where $\boldsymbol{P_2^{(1)}}=\boldsymbol{a^2}+\mathbf{bc}$. If we consider Eq.(22), Eq.(14) and Eq.(9), then we have the following generalisation:

$$P_n^{(k)}=aP_{n-1}^{(k)}+bcP_q^{(1)}P_{n-(q+2)}^{(k-1)} \tag{23}$$

such that

$$(q+1)k+1\le n\le(q+2)k. \tag{24}$$

The proof ends by showing that the index q in Eq.(23) equals $\left\lceil\frac{n}{k}\right\rceil-2$. Let us start by dividing each side of the inequality in Eq.(24) by positive integer k . Then, we get

$$q+1+\frac{1}{k}\le\frac{n}{k}\le q+2. \tag{25}$$

Let be applied the ceiling function to both side of Eq.(25), that is

$$\left\lceil q+1+\frac{1}{k}\right\rceil\le\left\lceil\frac{n}{k}\right\rceil\le\lceil q+2\rceil. \tag{26}$$

Considering the fact that $0<\frac{1}{k}\le 1$ and the properties of the ceiling function, Eq.(26) can be rewritten as follows.

$$q+2\le\left\lceil\frac{n}{k}\right\rceil\le q+2 \tag{27}$$

Thus, it is evident from Eq.(27) that $q=\left\lceil\frac{n}{k}\right\rceil-2$.

We now present an alternative recurrence formula that we created by eliminating the term $P_{n-1}^{(k)}$ from the recurrence formula given by Eq.(4). The reduced recurrence formula derived by writing the consecutive terms of the previous recurrence within each other is shown by the following corollary.

Corollary 2.1.1.

$$P_n^{(k)}=a^n+bca^{n-k-1}\sum_{i=0}^{n-k-1}a^{-i}P_{\left\lfloor\frac{i}{k}\right\rfloor}^{(1)}P_{i+k+\left\lfloor-\frac{i+1}{k}\right\rfloor}^{(k-1)} \tag{28}$$

where $n \geq k+1$ *, and the initial conditions are* $P_0^{(k)} = 1$ *and* $P_n^{(0)} = 1$.

Before going on to the proof of Corollary 2.1.1, we provide the following Lemma, which will be utilized in the proof.

Lemma 2.1.1. For every a $\in \mathbb{Q}^+$

$$\lfloor -a \rfloor = -\lceil a \rceil \tag{29}$$

where $\lfloor\ \rfloor$ *and* $\lceil\ \rceil$ *represent the floor and ceiling functions, respectively.*

Proof of Lemma 2.1.1: Let $a = x + r$, where $x \in \mathbb{N}$ and $0 \leq r < 1$.

$$\lfloor -a \rfloor = \lfloor -x - r \rfloor = -x + \lfloor -r \rfloor = -x - 1, \tag{30}$$

and

$$-\lceil a \rceil = -\lceil x + r \rceil = -x - \lceil r \rceil = -x - 1. \tag{31}$$

Since Eq.(30) and Eq.(31) yield the same result, Eq.(29) is valid.

Proof of Corollary 2.1.1.: We prove the corollary by induction on. Let us consider $n = 2$ in Eq.(28). Then, it becomes $k = 1$. In this case, we get

$$P_2^{(1)} = a^2 + bc \sum_{i=0}^{0} a^{-i} P_{\lfloor i \rfloor}^{(1)} P_{i+1+\lfloor -i-1 \rfloor}^{(0)}. \tag{32}$$

Considering the initial conditions of Corollary 2.1.1, the right-hand side of Eq.(32) equals to $a^2 + bc$. So, Eq.(28) is true for $n = 2$, in that

$$P_2^{(1)} = Per \begin{bmatrix} a & b \\ c & a \end{bmatrix} = a^2 + bc. \tag{33}$$

Let us assume that Eq.(28) is held for any integer $n > 2$ and show that Eq.(28) also has true for $n + 1$, i.e.,

$$P_{n+1}^{(k)} = a^{n+1} + bc \sum_{i=0}^{n-k} a^{n-k-i} P_{\left\lfloor \frac{i}{k} \right\rfloor}^{(1)} P_{i+k+\left\lfloor -\frac{i+1}{k} \right\rfloor}^{(k-1)}. \tag{34}$$

Starting from the left-hand side of Eq.(34), we will obtain the right-hand side. We will utilize Theorem 2.1.1. to achieve this. Let us begin by substituting $n + 1$ in Eq.(4) for n , that is,

$$P_{n+1}^{(k)} = aP_n^{(k)} + bcP_{\left\lceil \frac{n+1}{k} \right\rceil - 2}^{(1)} P_{n+1-\left\lceil \frac{n+1}{k} \right\rceil}^{(k-1)}. \tag{35}$$

The Eq.(35) is rewritten as below if taking into account our assumption for $P_n^{(k)}$:

$$P_{n+1}^{(k)} = a\left(a^n + bc \sum_{i=0}^{n-k-1} a^{n-k-1-i} P_{\left\lfloor \frac{i}{k} \right\rfloor}^{(1)} P_{i+k+\left\lfloor -\frac{i+1}{k} \right\rfloor}^{(k-1)} \right) + bcP_{\left\lceil \frac{n+1}{k} \right\rceil - 2}^{(1)} P_{n+1-\left\lceil \frac{n+1}{k} \right\rceil}^{(k-1)} \tag{36}$$

Now, let us define the following functions, from ℕ to ℕ , to be used into the indices of permanents in Eq.(36).

$$h(n) = \left\lceil \frac{n+1}{k} \right\rceil, \tag{37}$$

$$f(i) = \left\lfloor \frac{i}{k} \right\rfloor, \tag{38}$$

$$g(i) = i + k + \left\lfloor -\frac{i+1}{k} \right\rfloor \tag{39}$$

where $k \geq 1$. These functions can be typed out as

$$h(n) = \begin{cases} 1 & 0 \leq n < k \\ 2, & k \leq n < 2k \\ 3, & 2k \leq n < 3k \\ \vdots & \vdots \end{cases}, \tag{40}$$

$$f(i) = \begin{cases} 0, & 0 \leq i < k \\ 1, & k \leq i < 2k \\ 2, & 2k \leq i < 3k \\ \vdots & \vdots \end{cases}, \tag{41}$$

$$g(i) = \begin{cases} i+k-1, & 0 \leq i < k \\ i+k-2, & k \leq i < 2k \\ i+k-3, & 2k \leq i < 3k \\ \vdots & \vdots \end{cases}. \tag{42}$$

Using the functions defined above, Eq.(36) can be represented as

$$P_{n+1}^{(k)} = a\left(a^n + bc\sum_{i=0}^{n-k-1} a^{n-k-1-i} P_{f(i)}^{(1)} P_{g(i)}^{(k-1)}\right) + bcP_{h(n)-2}^{(1)} P_{n+1-h(n)}^{(k-1)} \tag{43}$$

The last term on the right-hand side of Eq.(43) needs to be embedded in the summation. To do this, we must demonstrate that the following two conditions are held.

$$f(n-k) = h(n) - 2\,, \tag{44}$$

$$g(n-k) = n + 1 - h(n)\,. \tag{45}$$

By substituting $(n-k)$ into the function $f(i)$ defined by Eq.(38), we obtain

$$f(n-k) = \left\lfloor \frac{n}{k} \right\rfloor - 1 = f(n) - 1 \tag{46}$$

Based on the definitions of the functions f and h , we can write

$$f(n) = h(n) - 1\,. \tag{47}$$

Thus, the condition given by Eq. (44) holds. On the other hand, if we substitute $(n-k)$ into the function $g(i)$ defined by Eq.(39), then we obtain

$$g(n-k) = n + \left\lfloor -\frac{n-k+1}{k} \right\rfloor = n + 1 + \left\lfloor -\frac{n+1}{k} \right\rfloor. \tag{48}$$

According to Lemma 2.1.1., if $\left(-\left\lceil \frac{n+1}{k} \right\rceil\right)$ is substituted for $\left\lfloor -\frac{n+1}{k} \right\rfloor$ in Eq.(48), then

$$g(n-k) = n + 1 - h(n) \tag{49}$$

is yield. So, the condition given by Eq.(45) is also holds. Therefore, $P^{(1)}_{f(n-k)}$ can be substituted for $P^{(1)}_{h(n)-2}$, and $P^{(k-1)}_{g(n-k)}$ for $P^{(k-1)}_{n-h(n)}$. Thus, if we write Eq.(43) explicitly, we have

$$\begin{aligned} P^{(k)}_{n+1} = a^{n+1} + bc\Big(&a^{n-k} P^{(1)}_{f(0)} P^{(k-1)}_{g(0)} + \ldots + a P^{(1)}_{f(n-k-1)} P^{(k-1)}_{g(n-k-1)} \\ &+ P^{(1)}_{f(n-k)} P^{(k-1)}_{g(n-k)}\Big). \end{aligned} \tag{50}$$

Finally we obtain

$$P^{(k)}_{n+1} = a^{n+1} + bc \sum_{i=0}^{n-k} a^{n-k-i} P^{(1)}_{f(i)} P^{(k-1)}_{g(i)}. \tag{51}$$

2.2. LINEARIZATION PROCESS OF THE NON-LINEAR RECURRENCE

The recurrence relation in Corollary 2.1.1 is non-linear. This recurrence can be transformed into a linear structure. To do this, the following explicit formula, introduced in our previous work [10], can be utilized.

$$P^{(1)}_n = a^{\frac{1-(-1)^n}{2}} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-r}{r} (a^2)^{\left\lfloor \frac{n}{2} \right\rfloor - r} (bc)^r \tag{52}$$

where $P^{(1)}_n$ denotes the permanent of tridiagonal Toeplitz matrix, i.e.,

$$T^{(1)}_n(a,b,c) = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & \ddots & \vdots \\ 0 & c & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & 0 & c & a \end{bmatrix}_{n\times n} . \tag{53}$$

The non-linear recurrence given by Corollary 2.1.1. can be reorganized using the explicit formula in Eq.(52) as follows:

Corollary 2.2.1. *Let $P_n^{(k)}$ denote the permanent of general k-tridiagonal Toeplitz matrix $T_n^{(k)}(a,b,c) \in M_n(\mathbb{C})$. Then,*

$$P_n^{(k)} = a^n + bc \sum_{r=0}^{n-k-1} a^w P^{(k-1)}_{i+k+\left\lfloor -\frac{i+1}{k} \right\rfloor} \sum_{r=0}^{\left\lfloor \frac{i}{2k} \right\rfloor} \binom{\left\lfloor \frac{i}{k} \right\rfloor - r}{r} a^{-2r} (bc)^r \tag{54}$$

where

$$\omega = n - k - 1 - i + 2\left\lfloor \frac{i}{2k} \right\rfloor + \frac{1-(-1)^{\left\lfloor \frac{i}{k} \right\rfloor}}{2} \tag{55}$$

with initial conditions $P_0^{(k)} = 1$ and $P_n^{(0)} = 1$.

Proof: This proof is clear by substituting $\left\lfloor \frac{i}{k} \right\rfloor$ instead of n in Eq.(52), and replacing this in the Corollary 2.1.1. We note that the explicit formula seen by Eq.(52) was proved in [7].

Through this formula, it is possible to directly compute the permanents with bandwidth of k in terms of the permanents with bandwidth of $k-1$.

2.3. *ILLUSTRATIVE EXAMPLES*

Let us give an example to demonstrate how to apply the linear recursive formula given by Eq.(54). For example, let consider the permanent of 3-tridiagonal Toeplitz matrix with order 15×15. According to the Corollary 2.2.1, we can write

$$P_{15}^{(3)} = a^{15} + bc \sum_{i=0}^{11} \left(a^{\omega} P^{(2)}_{i+3+\left\lfloor -\frac{i+1}{3} \right\rfloor} \sum_{r=0}^{\left\lfloor \frac{i}{6} \right\rfloor} \binom{\left\lfloor \frac{i}{3} \right\rfloor - r}{r} a^{-2r} (bc)^r \right) \tag{56}$$

where $\omega = 11 - i + 2\left\lfloor \frac{i}{6} \right\rfloor + \frac{1-(-1)^{\left\lfloor \frac{i}{3} \right\rfloor}}{2}$ .For clarity, Eq. (56) can be written as

$$\begin{aligned} P_{15}^{(3)} = a^{15} &+ bc\left(a^{11}P_2^{(2)} + a^{10}P_3^{(2)} + 2a^9P_4^{(2)} + a^8P_5^{(2)} + a^7P_6^{(2)}\right) \\ &+ bc(1 + a^{-2}bc)\left(a^7P_6^{(2)} + a^6P_7^{(2)} + a^5P_8^{(2)}\right) \\ &+ bc(2 + a^{-2}bc)\left(a^5P_8^{(2)} + a^4P_9^{(2)} + a^3P_{10}^{(2)}\right). \end{aligned} \tag{57}$$

As another example, let us consider the permanent of 3-tridiagonal Toeplitz matrix of order 8×8 . According to Corollary 2.2.1., we have

$$P_8^{(3)} = a^8 + bc \sum_{i=0}^{4} \left(a^{\omega} P^{(2)}_{i+3+\left\lfloor -\frac{i+1}{3} \right\rfloor} \sum_{r=0}^{\left\lfloor \frac{i}{6} \right\rfloor} \binom{\left\lfloor \frac{i}{3} \right\rfloor - r}{r} a^{-2r} (bc)^r \right) \tag{58}$$

where $\omega = 4 - i + 2\left\lfloor \frac{i}{6} \right\rfloor + \frac{1-(-1)^{\left\lfloor \frac{i}{3} \right\rfloor}}{2}$. Then, we get

$$P_8^{(3)} = a^8 + bc\left(a^4 P_2^{(2)} + a^3 P_3^{(2)} + 2a^2 P_4^{(2)} + a P_5^{(2)}\right). \tag{59}$$

Thus, the permanents of 3-tridiagonal Toeplitz matrices were expressed in terms of the permanents of 2-tridiagonal Toeplitz. According to the initial conditions in Theorem 2.1.1. we have $P_2^{(2)} = a^2$. Besides that, according to both the linear recursive formula seen by Eq.(54) and the explicit formula seen by Eq.(52), we get the following outcomes.

$$P_3^{(2)} = a^3 + bcP_1^{(1)} = a^3 + bca, \tag{60}$$

$$P_4^{(2)} = a^4 + bc\left(aP_1^{(1)} + P_2^{(1)}\right) = a^4 + 2bca^2 + b^2c^2, \tag{61}$$

$$P_5^{(2)} = a^5 + bc\left(a^2P_1^{(1)} + 2aP_2^{(1)}\right) = a^5 + 3bca^3 + 2b^2c^2a. \tag{62}$$

If we plug these outcomes into Eq.(59) then we get

$$P_8^{(3)} = a^8 + 5a^6bc + 8a^4b^2c^2 + 4a^2b^3c^3. \tag{63}$$

3. CONCLUSION

A non-linear recursive formula that yielded the permanents of k-tridiagonal Toeplitz matrix family is the primary focus of this study. This formula has been first reduced by eliminating one term. This reduced recursive formula has been converted into a linear form. Thus, the permanents of k-tridiagonal Toeplitz matrices have been formulated only in terms of the permanents of $(k-1)$-tridiagonal Toeplitz matrices. Thanks to this linear recursive formula for k-tridiagonal Toeplitz matrices, any permanent whose bandwidth is greater than one can be reduced to the permanent whose bandwidth is 1.

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