ORIGINAL PAPER

A SIMPLE FORMULA FOR EVALUATING THE KELLER'S SEQUENCE

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Abstract. *The aim of this paper is to provide a simple formula for evaluating the Keller's sequence.*

Keywords: *Keller sequence; approximations; asymptotic series.*

1. INTRODUCTION

The limit:

$$\lim_{n\to\infty}\left((n+1)\left(1+\frac{1}{n}\right)^{n}-n\left(1+\frac{1}{n-1}\right)^{n-1}\right)=e$$

is known in the literature as the Keller's limit (see [1]), and let us denote by:

$$K_n=(n+1)\left(1+\frac{1}{n}\right)^{n}-n\left(1+\frac{1}{n-1}\right)^{n-1},\ \ n\geq 2$$

the Keller's sequence. Many researchers have provided interesting results on Keller's sequence $(K_n)_{n\geq 2}$ and its limit.

Mortici and Jang [2] and Hu and Mortici [3] have introduced, for every real number c, the following extended sequence

$$K_n(c)=(n+1)\left(1+\frac{1}{n+c}\right)^{n+c}-n\left(1+\frac{1}{n+c-1}\right)^{n+c-1},\ \ n\geq -c+2.$$

Malešević et al. [4] also have presented some inequalities on Keller's sequence, by using the monotonicity of the function $x\mapsto(1+x)^{1/x}$, $x>0$.

Fang et al. [5] have constructed some asymptotic series and contiued fraction associated to the Keller's sequence, by using a method first given by Mortici [6]. This method has a wide range of applications in the problem of estimating Keller's limit and other mathematical constants (see, e.g., [5-15]).

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2. THE RESULTS

The main result in this paper is the next:

Theorem 1. The following approximation formula for Keller's sequence, in terms of middle value $N = n - \frac{1}{2}$, as $n \to \infty$, holds true:

$$(n+1)\left(1+\frac{1}{n}\right)^{n} - n\left(1+\frac{1}{n-1}\right)^{n-1} \sim e\left[\left(2N+\frac{1}{2}\right)\ln\left(1+\frac{1}{2N}\right)+\frac{1}{48N^2}\right]. \qquad (1)$$

Note that there are some difficulties in numerical computation of Keller's sequence, but the approximation (1) offers us a simple expression.

In order to prove this Theorem 1, we use the following formula:

$$(1+x)^{1/x} = e\left(1+\sum_{k=1}^{\infty} e_k x^k\right), \quad x \in (-1,1) \setminus \{0\}, \qquad (2)$$

where

$$e_n = (-1)^n \sum_{k=0}^{n} \frac{(-1)^{n+k} S_1(n+k,k)}{(n+k)!} \sum_{m=0}^{n} \frac{(-1)^m}{(m-k)!};$$

$S_1(p,q)$ is the Stirling's number of the first kind ($p, q \in \mathbb{N}$) (see [16]).

Malešević et al. [4, Rel. 18] have used (2) with $x = 1/n$, $x = 1/(n-1)$ to obtain an asymptotic expansion for the Keller's sequence. The first terms are the following:

$$K_n = e\left(1+\frac{1}{24n^2}+\frac{11}{640n^4}+\frac{5525}{580608n^6}+\frac{1212281}{199065600n^8}+\dots\right). \qquad (3)$$

Proof of Theorem 1: Let us denote the right-hand side expression in (2) by:

$$\rho_n := \left(2\left(n-\frac{1}{2}\right)+\frac{1}{2}\right)\ln\left(1+\frac{1}{2\left(n-\frac{1}{2}\right)}\right)+\frac{1}{48\left(n-\frac{1}{2}\right)^2}.$$

By using the standard power series expansion of the logarithm function:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}, \quad -1 < x \le 1$$

and the binomial expansion

$$\frac{1}{(y-1)^k} = \frac{1}{y^k}\left(1-\frac{1}{y}\right)^{-k} = \sum_{i=0}^{\infty} \frac{\binom{k-1+i}{k-1}}{y^{k+i}}, \quad \frac{1}{y} < 1, k \in \mathbb{N},$$

we get the asymptotic expansion for ρ_n:

$$\rho_n = 1 + \frac{1}{24n^2} + \frac{1}{32n^3} + \frac{13}{640n^4} + \frac{1}{80n^5} + O\left(\frac{1}{n^6}\right). \tag{4}$$

By substracting (3)-(4), we get:

$$K_n - e\rho_n = e\left(-\frac{1}{32n^3} - \frac{1}{320n^4} - \frac{1}{80n^5} + O\left(\frac{1}{n^6}\right)\right). \tag{5}$$

As a consequence,

$$\lim_{n\to\infty} (K_n - e\rho_n) = 0,$$

so the approximation formula (1) is valid. Moreover,

$$\lim_{n\to\infty} [n^3(K_n - e\rho_n)] = -\frac{e}{32} \neq 0,$$

which means that the speed of convergence (to zero) of the sequence $(K_n - e\rho_n)_{n\geq 2}$ is n^{-3}.

3. CONCLUSIONS AND NUMERICAL COMPARISON

Representation (3) is an useful tool for obtaining further results. Mortici and Jang [2] have used some inequalities to present the speed of convergence of the Keller's sequence:

$$\lim_{n\to\infty} n^2(K_n - e) = \frac{e}{24} \tag{6}$$

and of an extention

$$\lim_{n\to\infty} n^2(K_n(c) - e) = \frac{e}{24}(1-12c); \tag{7}$$

In case $c = 1/12$, they proved:

$$\lim_{n\to\infty} n^3\left(K_n\left(\frac{1}{12}\right) - e\right) = \frac{5e}{144}. \tag{8}$$

Now, (3) is a simple consequence of the expansion (6). Also we have:

$$\lim_{n\to\infty} n^2\left(n^2(K_n - e) - \frac{e}{24}\right) = \frac{11}{640}, \lim_{n\to\infty} n^2\left[n^2\left(n^2(K_n - e) - \frac{e}{24}\right) - \frac{11}{640}\right] = \frac{5525}{580608n^6}.$$

The limits (7)-(8) can be obtained by a suitable construction of the asymptotic expansion of $(K_n(c))_{n\geq 2}$.

Finally, we present the following numerical analysis associated to the approximation formula (1). These computation were performed using Maple software.

n	K_n	$e\rho_n$	$\lvert K_n - e\rho_n \rvert$
2	2.75	2.7622	0.0122
10	2.7194	2.7195	8.6131×10^{-5}
50	2.7183	2.7183	6.8104×10^{-7}
100	2.7183	2.7183	8.5035×10^{-8}
1000	2.7183	2.7183	8.4955×10^{-11}

REFERENCES

[1] Brede, M., *Mathematical Intelligencer*, **27**(3), 6, 2005.
[2] Mortici, C., Jang, X.J., *Filomat*, **7**, 1535, 2015.
[3] Hu, Y., Mortici, C., *Journal of Inequalities and Applications*, **2016**, 4pp, 2016.
[4] Malešević, B., Hu, Y., Mortici, C., *Filomat*, **32**(13), 4673, 2018.
[5] Fang, S., Lai, L., Lu, D., Wang, X., *Results in Mathematics*, **73**(69), 12pp, 2018.
[6] Mortici, C., *American Mathematical Monthly*, **117**(5), 434, 2010.
[7] Chen, C.P., Qi, F., *Tamkang Journal of Mathematics*, **36**(4), 303, 2005.
[8] Chen, C.P., Qi, F., *Proceedings of the American Mathematical Society*, **133**(2), 397, 2005.
[9] Mortici, C., Cristea, V.C., Lu. D, *Applied Mathematics and Computation*, **240**, 168, 2014.
[10] Mortici, C., *Ramanujan Journal*, **38**(3), 549, 2015.
[11] Mortici, C., *Mathematical and Computer Modelling*, **51**(9-10), 1154, 2010.
[12] Mortici, C., *Computational and Applied Mathematics*, **29**(3), 479, 2010.
[13] Mortici, C., *Ramanujan Journal*, **26**(2), 185, 2011.
[14] Mortici, C., *Ramanujan Journal*, **38**(3), 549, 2015.
[15] Qi, F., Mortici, C., *Applied Mathematics and Computation*, **253**, 363, 2015.
[16] Finch, S.R., *Mathematical Constants*, Cambridge University Press, New York, 2003.