ORIGINAL PAPER

THE AGREEMENT BETWEEN THE NEW EXACT AND NUMERICAL SOLUTIONS OF THE 3D-FRACTIONAL WAZWAZ-BENJAMIN-BONA-MAHONY EQUATION

AHMET BEKIR[1], EMAD H.M. ZAHRAN[2], MAHA S.M. SHEHATA[3]

Manuscript received: 12.12.2019; Accepted paper: 28.03.2020;
Published online: 30.06.2020.

Abstract. *In this article, we employed the 3D-fractional Wazwaz-Benjamin-Bona-Mahony (3D-FWBBM) equation with its spatial and temporal variables which is stretch for the Korteweg-de-Vries equation that represent the unidirectional propagation of small amplitude long waves on the surface of hydro magnetic and acoustic waves in channel specially for shallow water. New exact soliton solution has been realized using the* $(G'/G)-$ *expansion method. Furthermore, the numerical solution of the suggested equation according to the variational iteration method (VIM) is listed effectively. A good comparison between the obtained exact and numerical solution are successfully demonstrated.*

Keywords: *The 3D-FWBBM equation, the* $(G'/G)-$*expansion method, the variational iteration method (VIM), traveling wave solutions.*

1. INTRODUCTION

The new formulism for the three-dimensional (3D) version of the modified BBM equations is introduced in a sort of coupling and/or generalization of different senses by Wazwaz [1]. This new description of the 3D-Wazwaz-Benjamin-Bona- Mahony equations (3D-WBBM) with dual property of spatial and temporal variables represents the problems in higher dimensions which have more applications in real-life situations, furthermore the obtained solutions of this model help in understanding the physics behind models. Recently, there exists large number of the well-known ansatze approaches methods invented by various authors which used successfully to find the exact solutions of nonlinear problems in different branches of science. Most of these methods are listed in the references [2-15]. Some of these methods are the modified decomposition method, the extended Jacobian elliptic function expansion method, the Riccati-Bernoulli Sub-ODE method, the modified extended tanh-function method, the modified simple equation method, the exp (−φ (ζ))-method, the modified exp (–φ (ζ))-expansion method extended trial equation method,

First integral method and (G′/G)-expansion method. The results obtained by these methods are analogy with that obtained using the reduction methods and the Lui-semi-group methods. The majority of these methods are building on the balance rule in its preparing. One

[1] Neighbourhood of Akcaglan, Imarli Street, Number:28/4, 26030, Eskisehir, Turkey.
E-mail: bekirahmet@gmail.com.
[2] Benha University, Departments of Mathematical and Physical Engineering, Shubra, Egypt.
E-mail: e_h_zahran@hotmail.com.
[3] Zagazig University, Faculty of Science, Departments of Mathematics, Zagazig, Egypt.
E-mail: dr.maha_32@hotmail.com.

of these methods is the well-known (G′/G)-expansion method which realized effectively the exact solutions for the many nonlinear evolution problems.

Also a flux of a numerical methods are invited to find the approximate solutions for the nonlinear phenomena arising in physics and mathematics such as the Adomian decomposition method, Badi approximation method, Rung-Kutta method, finite element method, boundary element method and the Variational iteration method, etc. To realizes the approximate solutions of these problems. There are some tries through different authors that make the study of soliton dynamics possible of models relevant such as the Benney-Luke equation [16], Korteweg-de-Vries equation (KdV) [17], variant Boussinesq equation [18], and Benjamin Bona Mahony (BBM) equation [19]. Seadawy [20] and other discuss a variety of soliton solutions of this converted model using tanh and coth hyperbolic function as well as tan and cot trigonometric functions. The main idea of this article concentrated on finding new exact and the numerical solutions of the suggested equation using the (G′/G)-expansion method [2] and the variation iteration method [21] respectively and making a comparison between these two solutions.

2. TECHNIQUE DESCRIPTION OF THE $(G'/G)-$EXPANSION METHOD

To propose the general form of the nonlinear evolution equation let us introduce R as a function of h(x,t) and its partial derivatives as,

$$R(h, h_t, h_x, h_{tt}, h_{xx}, \dots.) = 0, \tag{1}$$

that involve the highest order derivatives and nonlinear terms.

With the aid of the transformation $h(x,t) = h(\xi), \quad \xi = x - ct,$ equation (1) can be reduced to the following ODE:

$$S(h, h', h'', h''', \dots..) = 0, \tag{2}$$

where S is a function in $h(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$

The constructed solution according to this method is:

$$h(\xi) = A_0 + \sum_{k=0}^{m} A_k \left[\frac{G'}{G}\right]^k, A_m \neq 0, \tag{3}$$

where the positive integer m *in E*q. (3) can be located by balancing the highest order derivative term and the nonlinear term, while $G(\xi)$ satisfy the second order different equation such that $G'' + \mu G' + \lambda G = 0$. The solution of this equation admits three forms of solutions depend one of these cases $\mu^2 - 4\lambda \succ 0$, $\mu^2 - 4\lambda \prec 0$ *and* $\mu^2 - 4\lambda = 0$,

Case 1: When $\mu^2 - 4\lambda \succ 0$, the solution is:

$$(\frac{G'}{G}) = \frac{\sqrt{\mu^2 - 4\lambda}}{2}\left(\frac{l_1 \sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}{l_1 cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 \sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}\right) - \frac{\mu}{2} \tag{4}$$

Case 2: When $\mu^2 - 4\lambda \prec 0$, the solution is:

$$(\frac{G'}{G}) = \frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{-l_1 sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}{l_1 cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}\right) - \frac{\mu}{2}, \tag{5}$$

Case 3: When $\mu^2 - 4\lambda = 0$, the solution is:

$$(\frac{G'}{G}) = (\frac{l_2}{l_1 + l_2\xi}) - \frac{\mu}{2}, \tag{6}$$

Substituting about the derivative of the function $h(\xi)$, at the given problem we get a polynomial of $(\frac{G'}{G})^k, (k = 0,1,2,....)$. In this polynomial, adding all terms of the same power of $(\frac{G'}{G})^k$, and let the coefficients of different exponential of $(\frac{G'}{G})^k$, equal to zero, we get a system of algebraic equations which solved by any computer program to find A_k.

3. APPLICATIONS

Before we apply the method mentioned above to get the exact soliton solutions of the 3D-fractional WBBM equation [1], we firstly give some knots about the fractional calculus,

(i) Riemann-Liouville derivatives

$$D_x^\gamma t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\gamma)} t^{r-1},$$

$$D_x^\gamma (h(t)g(t)) = h(t) D_x^\gamma g(t) + g(t) D_x^\gamma h(t),$$

$$D_t^\gamma h(t)[g(t)] = h'[g(t)] D_t^\gamma g(t),$$

and these two well-known forms

$$D^\gamma J^\gamma (h(x)) = h(x) \quad , \quad x \succ 0,$$

and

$$D^\gamma J^\gamma (h(x)) = h(x) - \sum_{k=0}^{m} h^k(0^+)\frac{x^k}{k!} \quad , \quad x \prec 0,$$

this represents derivatives of Caputo operator.

Now, we introduce the general form of the NLFPDE as,

$$p(h, D_x^{\gamma}h, D_t^{\gamma}h, ...) = 0, \quad 0 \prec \gamma \leq 1 \tag{7}$$

where $D_x^{\gamma}h, D_t^{\gamma}h$ are the modified Riemann-Liouville derivatives, and with using these nonlinear transformation,

$$h(x,t) = h(\xi), \quad \xi = \frac{kx^{\gamma}}{\Gamma(1+\gamma)} + \frac{ct^{\gamma}}{\Gamma(1+\gamma)} + \xi_0, \tag{8}$$

while K, C and ξ_0 are constants with $k, C \neq 0$), will transform equation (8) to this ordinary differential equation (ODE) with integer order namely,

$$f(h, h', h'', ...) = 0 \tag{9}$$

where $(' = \frac{d}{d\xi})$.

(ii) Conformable fractional derivative

If the function $h:[0,\infty) \to \mathbb{R}$, thus the conformable fractional derivative of $u(x)$, $x \succ 0$ is defined as

$$D_x^{\gamma}(u(x)) = \lim_{h \to 0} \frac{u(x + hx^{1-\gamma}) - u(x)}{h}, \ \gamma \in (0,1] \tag{10}$$

and according to [22] it realized all differentiation rules that applied on the ordinary functions.

Thus for the 3D- FWBBM equation [1]

$$D_t^{\gamma}h + D_x^{\gamma}h + D_y^{\gamma}h - D_{xzt}^{3\gamma}h = 0, \tag{11}$$

Under these transformations,

$$h(x,t) = h(\xi), \ \xi = a\frac{x^{\gamma}}{\gamma} + b\frac{y^{\gamma}}{\gamma} + c\frac{z^{\gamma}}{\gamma} - d\frac{t^{\gamma}}{\gamma}, \tag{12}$$

Will changed to,

$$acd\, h''' + b(h^3)' + (-d + a)h' = 0, \tag{13}$$

If we integrate the last equation and neglecting the constant of integration we get,

$$acd\, h'' + bh^3 + (-d + a)h = 0, \tag{14}$$

In this section, we will apply the $(G'/G)-$expansion method as a new technique to realize the exact solution for the 3D- FWBBM equation "in terms of some variables". Hence, we can easily obtain the travelling wave solutions when these variables take specific values.

To apply the proposal method for equation (14), we firstly apply the balance rule between $h'', h^3 \Rightarrow m + 2 = 3m \Rightarrow m = 1$, thus according to the proposed method the solution is

$$h(\zeta) = A_0 + A_1(\frac{G'}{G}), \tag{15}$$

Substitute about h, h^3 and h'''' implies this system of equations,

$$\left(\frac{G'}{G}\right)^3 \quad \Rightarrow \quad 2acd + bA_1^2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 \quad \Rightarrow \quad acd\mu + bA_0A_1 = 0,$$

$$\left(\frac{G'}{G}\right) \quad \Rightarrow \quad \mu^2 acd + 2\lambda acd + 3bA_0^2 + a - d = 0,$$

$$Cons. \Rightarrow \quad \mu A_1 acd + (a-d)A_0 = 0, \tag{16}$$

From which we can easily obtain,

$$A_0 = \pm\frac{\mu}{2}\sqrt{\frac{-2acd}{b}}, \qquad A_1 = \pm\sqrt{\frac{-2acd}{b}}, \tag{17}$$

Thus according to the cases of solution for the proposed method the solutions are,

Case 1: When $\mu^2 - 4\lambda \succ 0$, the solution is:

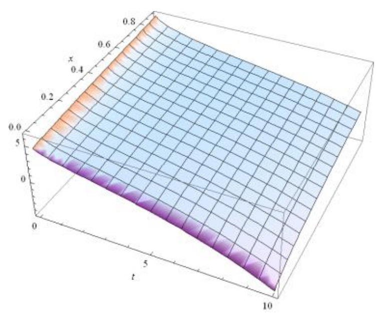
$$h(\xi) = \pm\sqrt{\frac{-2acd}{b}}\left[\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{l_1\sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}{l_1 cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2\sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}\right)\right], \tag{18}$$

Case 2: When $\mu^2 - 4\lambda \prec 0$, the solution is:

$$h(\xi) = \pm\sqrt{\frac{-2acd}{b}}\left\{\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{-l_1 sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}{l_1\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi + l_2 sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\xi}\right)\right\}, \tag{19}$$

Case 3: When $\mu^2 - 4\lambda = 0$, the solution is:

$$h(\xi) = \sqrt{\frac{-2acd}{b}}\left(\frac{l_2}{l_1 + l_2\xi}\right), \tag{20}$$



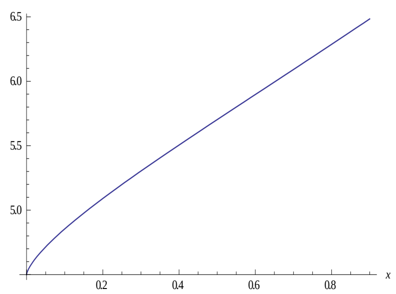


Figure 1. The plot of Eq.(18) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 3, \lambda = 1.25,$ $\gamma = 0.7, a = 0.3, b = 0.001, c = -0.3, \ d = 0.2,$ **20 < x < 1, 0 < t < 10.**

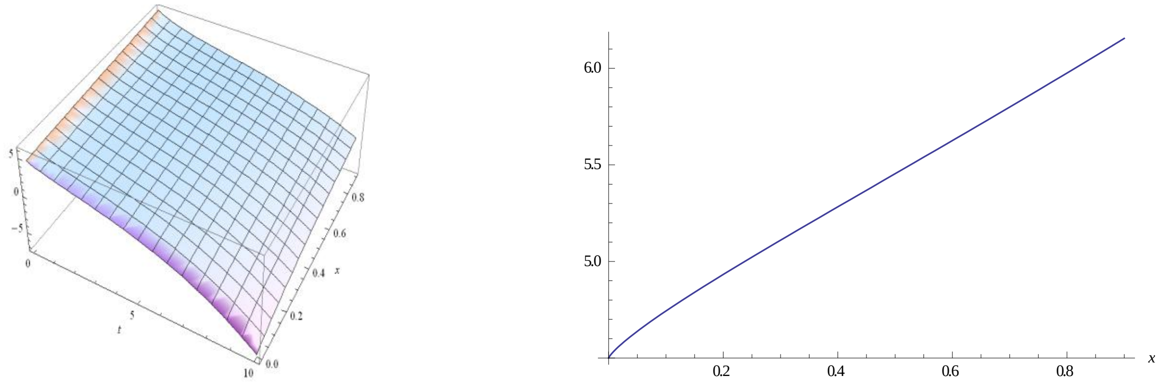


Figure 2. The plot of Eq.(18) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 3, \lambda = 1.25,$ $\gamma = 0.8, a = 0.3, b = 0.001, c = -0.3, \ d = 0.2,$ **20 < x < 1, 0 < t < 10.**

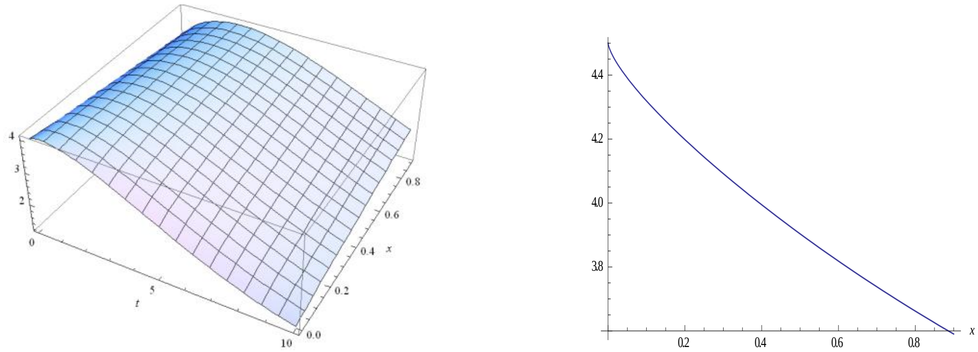


Figure 3. The plot of Eq.(19) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 2, \lambda = 2,$ $\gamma = 0.7, a = 0.3, b = 0.001, c = -0.3, \ d = 0.2,$ **20 < x < 1, 0 < t < 10.**

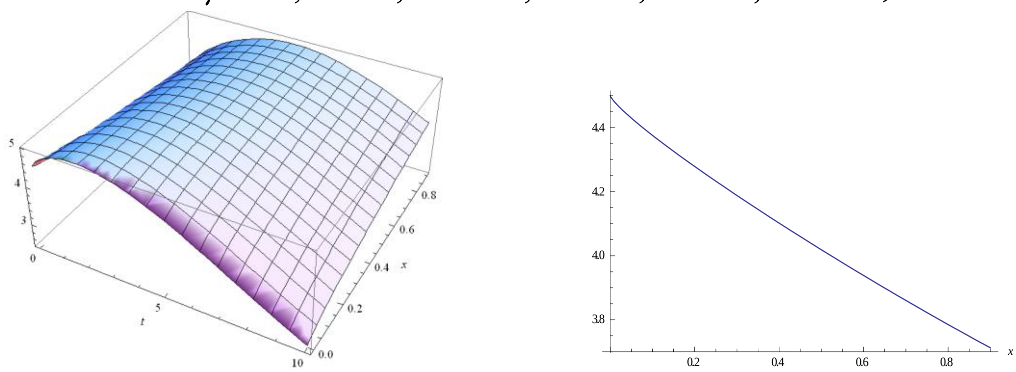


Figure 4. The plot of Eq.(19) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 2, \lambda = 2,$ $\gamma = 0.8, a = 0.3, b = 0.001, c = -0.3, \ d = 0.2,$ **20 < x < 1, 0 < t < 10.**

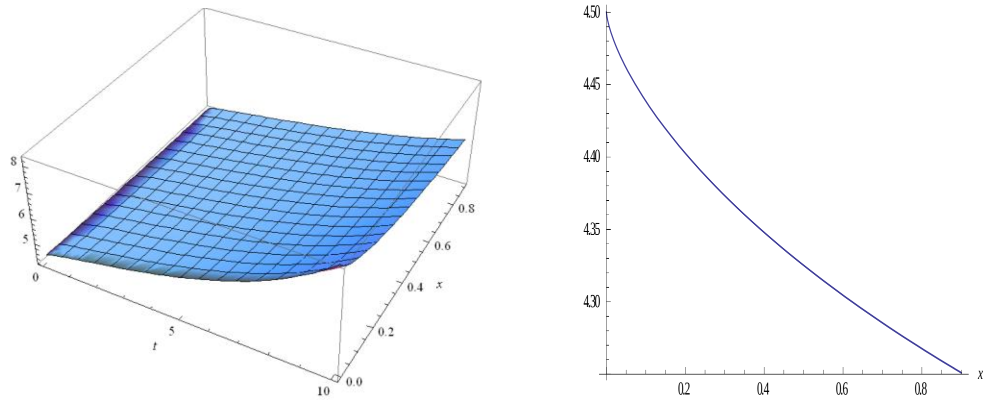


Figure 5. The plot of Eq.(20) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 3,$
$\gamma = 0.7, a = 0.3, b = 0.001, c = -0.3, \; d = 0.2,$ **0 < x < 1, 0 < t < 10.**

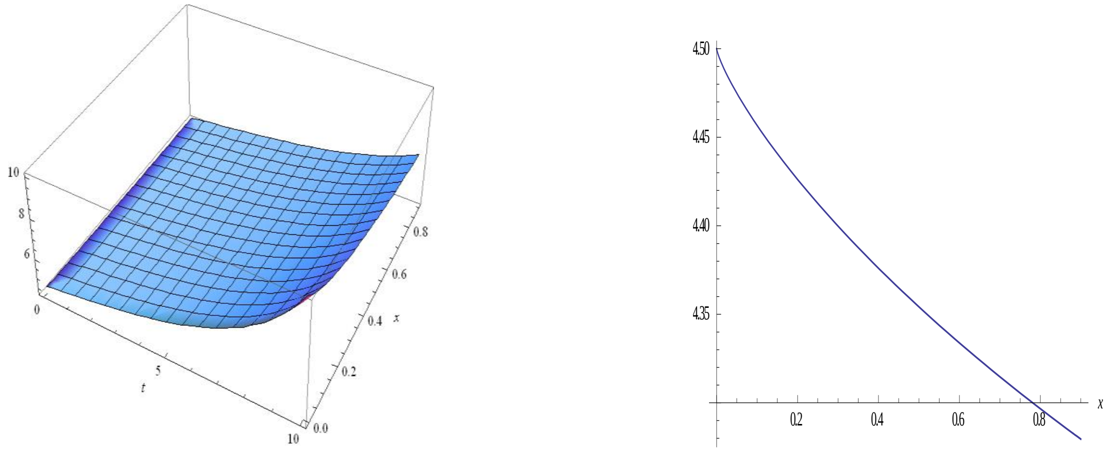


Figure 6. The plot of Eq.(20) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 3,$
$\gamma = 0.8, a = 0.3, b = 0.001, c = -0.3, \; d = 0.2,$ **20 < x < 1, 0 < t < 10.**

4. THE VARIATIONAL ITERATION METHOD

Consider the differential equation with inhomogeneous term $f(\xi)$ and R, S the linear and the nonlinear operators respectively as:

$$Lh + Nh = f(\xi), \tag{21}$$

The VIM proposes a correction functional for equation (21) to be:

$$h_{m+1}(\xi) = h_m(\xi) + \int_0^{\xi} \lambda(t)(Lh_m(t) + N\tilde{h}_m(t) - g(t))dt, \tag{22}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{h}_m$ as a restricted variation which means $\delta\tilde{h}_m$. The Lagrange multiplier λ is crucial and critical in the method, and it can be a constant or a function. Having λ determined, an iteration formula should be used for the determination of the successive approximations $h_{m+1}(\xi)$ *;* $n \geq 0$ of the solution $h(\xi)$. The zeros approximation h_0 can be any selective function. However, using the initial values $h(0)$ *;* $h'(0)$ are preferably used for the selective zeros approximation u_0 as will be seen later. Consequently, the solution is given by $h(\xi) = \lim_{\xi \to \infty} h_m(\xi)$ It is interesting to point out that we formally derived the distinct forms of the Lagrange multipliers λ in [21], hence we skip details. We only set a summary of the obtained results,

It is important to give briefly the significant forms of Eq. (22) according to the Lagrange multipliers in these results,

For the 1st order ODE in the form,

$$h' + q(\xi)h = p(\xi), \; h(0) = \rho, \tag{23}$$

We find that $\lambda = -1$, and the correction function give the iteration formula:

$$h_{m+1}(\xi) = h_m(\xi) - \int_0^{\xi} (h'_m(t) + q(t)h_m(t) - p(t))dt. \tag{24}$$

The 2nd order ODE in the form

$$h''(\xi) + ch'(\xi) + dh(\xi) = g(\xi), \quad h(0) = \rho, h'(0) = \eta, \tag{25}$$

we find that $\lambda = t - x$, and the correction function give the iteration formula:

$$h_{m+1}(\xi) = h_m(\xi) + \int_0^{\xi} (t-x)(h''_m(t) + ch'_m(t) + dh_m - g(t))dt. \tag{26}$$

The 3th order ODE in the form,

$$h'''(\xi) + ch''(\xi) + dh'(\xi) + eh(\xi) = g(\xi), \;\; h(0) = \rho, h'(0) = \eta, h''(0) = \sigma, \tag{27}$$

We find that $\lambda = -\frac{1}{2!}(t-x)^2$, and the correction function give the iteration formula

$$h_{m+1}(\xi) = h_m(\xi) - \frac{1}{2!}\int_0^{\xi} (t-x)^2 (h'''_m(t) + ch''_m(t) + dh'_m(t) + eh_m - g(t))dt, \tag{28}$$

Consecountly,for the general form of ODE

$$h^{(m)} + f(h', h'', h''' ...., h^{(m-1)}) = g(\xi), h(0) = \rho_0, h'(0) = \rho_1, h''(0) = \rho_2 ....., h^{m-1}(0) = \rho_{m-1}, \tag{29}$$

The lagrange multiplier λ take the general form $\lambda = \frac{(-1)^m}{(m-1)!}(t-x)^{m-1}$, while the general form of iteration rule become,

$$h_{m+1}(\xi) = h_m(\xi) + \frac{(-1)^m}{(m-1)!}\int_0^{\xi} (t-x)^{m-1} \left(h^{(m)} + f(h', h'', h''', ..., h^{(m-1)}) - g(t) \right) dt, \tag{30}$$

Furthermore the zeros approximation $h_0(\xi)$ can be selected perfectly to be,

$$h_0(\xi) = h_0(0) + h'(0)\xi + \frac{1}{2!}h''(0)\xi^2 + \frac{1}{3!}h'''(0)\xi^3 .... + \frac{1}{(m-1)!}h^{m-1}(0)\xi^{m-1}, \tag{31}$$

where m is the order of the ODE.

4.1 APPLICATIONS OF METHOD

For the second order differential equation (14) mentioned above,

$$acd\,h'' + bh^3 + (-d + a)h = 0,$$

with the initial condition

$$h(0) = \sqrt{\frac{-2acd}{b}}\left(\frac{l_2}{l_1}\right) - \frac{\mu}{2}, \qquad h'(0) = \sqrt{\frac{-2acd}{b}}\left(-\frac{l_2^2}{l_1^2}\right). \tag{32}$$

According to the variational iteration method the first and the second iteration is,

$$h_0(\xi) = h(0) + \xi h'(0), \qquad h_0(\xi) = 3(1 - \frac{\xi}{2}), \tag{33}$$

Using the fact that the exact solution is obtained by using $h(\xi) = \lim_{\xi \to \infty} h_m(\xi)$

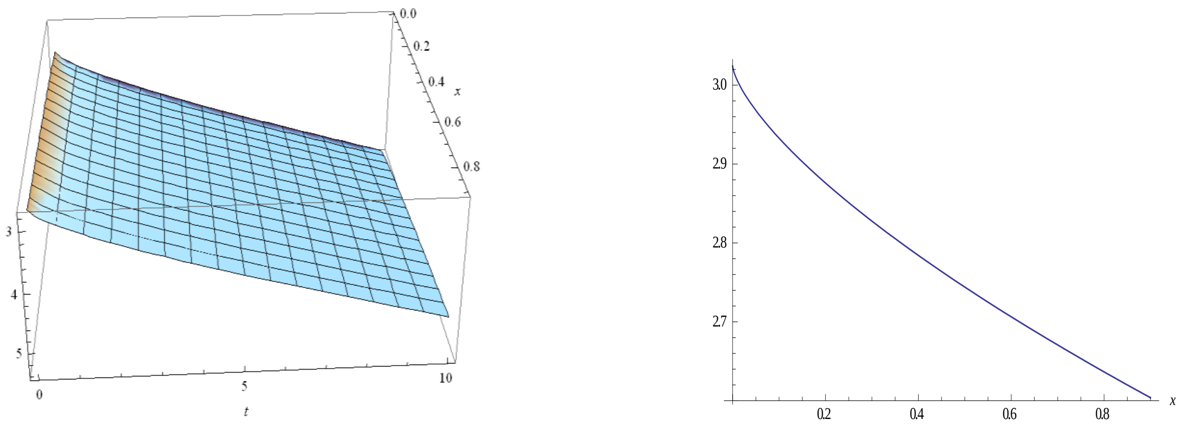


Figure 7. the plot of Eq.(33) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 3,$
$\gamma = 0.7, a = 0.3, b = 0.001, c = -0.3, \; d = 0.2,$ **0 < x < 1, 0 < t < 10,**

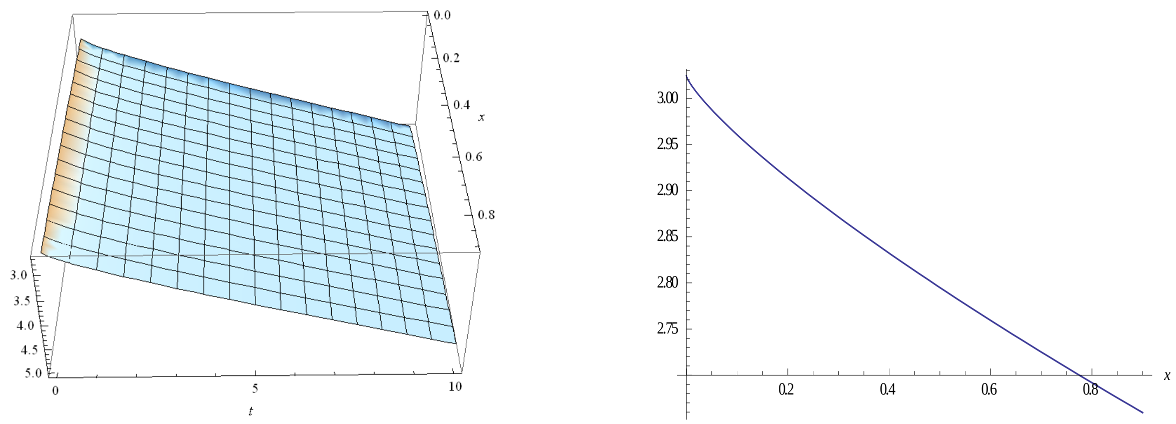


Figure 8. the plot of Eq.(33) in 2D and 3D with values: $l_1 = 2, l_2 = 1, \mu = 3,$
$\gamma = 0.8, a = 0.3, b = 0.001, c = -0.3, \; d = 0.2,$ **0 < x < 1, 0 < t < 10,**

5. CONCLUSION

In this work, the $(G'/G)-$expansion method has been used effectively to realize some new exact travelling wave solution the 3D-FWBBM equation in two and three dimensions (Figs. 1-6) some of which are equivalent with that obtained by [20] and the others are new. Furthermore, a comparison between one of these exact solutions with the numerical solution obtained by the variational iteration method (Figs. 7-8) has been listed. It is clear that there exist agreement between the exact and the numerical solutions which are positive forward future studies for the suggested equation.

REFERENCES

[1] Wazwaz, A.M., *Open Engineering*, **7**, 169, 2017.
[2] Bekir, A., Uygun, F., *Arab Journal of Mathematical Sciences*, **18**, 73, 2012.
[3] Zahran E H. M., Khater, M.M.A., *American Journal of computational Mathematics*, **4**, 455, 2014.
[4] Shehata M. S. M., *International Journal of Computer Applications*, **109**, 1, 2015.
[5] Wazwaz A.M., *Chaos Solitons Fractals*, **12**, 1549, 2001.
[6] Shehata, M. S., *International Journal of Physical Sciences*, **11**, 80, 2016.
[7] Yang, X. F., Deng, Z. C., Wei, Y., *Advances in Difference Equations*, **1**, 1, 2015.
[8] Khater, M. M., Lu, D., & Zahran, E. H., *Appl. Math. Inf. Sci*, **11**, 1, 2017.
[9] Zayed, E. M., & SA, H. I., *International Journal of Computer Applications*, **67**, 6, 2013.
[10] Khater, M. M., Zahran, E. H. & Shehata, M. S., *Journal of Egyptian Mathematical Society*, **25**, 8, 2017.
[11] Shehata, M. S., *American Journal of Computational Mathematics*, **5**, 468, 2015.
[12] Zahran, E. H., *Journal of Computational and Theoretical Nanoscience*, **12**, 5716, 2015.
[13] Biswas A, Ekici M, Sonmezoglu A, Alqahtani, R.T., *Optik*, **165**, 102, 2018.
[14] Eslami, M., Rezazadeh, H., *Calcolo*, **53**, 475, 2016.
[15] Zhou, Q., Ekici, M., Sonmezoglu, A., Mirzazadeh, M., Eslami, M., *Nonlinear Dynamics*, **84**, 1883, 2016.
[16] Khalid, K.A, Nuruddeen RI., *Int J Phys Res*, **5**, 109, 2017.
[17] Nuruddeen, R.I., *Ocean J Eng Sci*, **3**, 1, 2018.
[18] Khalid, K.A., Nuruddeen, R.I., Raslan, K.R., *Opt Quant Electron*, **50**, 61, 2018.
[19] Benjamin T, Bona J, Mahony J., *Philos Trans R Soc London Ser A*, **272**, 47, 1972.
[20] Seadawy, A. R., Khalid, K.A., Nuruddeen, R.I., *Results in Physics*, **12**, 2234, 2019.
[21] Wazwaz, A.M., *Applied Mathematics and Computation*, **212**, 120, 2009.
[22] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., *J Comput Appl Math*, **265**, 65, 2014.