ORIGINAL PAPER

ORDINARY GENERATING FUNCTIONS OF BINARY PRODUCTS OF (p,q)-MODIFIED PELL NUMBERS AND k-NUMBERS AT POSITIVE AND NEGATIVE INDICES

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Abstract. *In this paper, we introduce a operator in order to derive some new symmetric properties of (p,q)-modified Pell numbers and we give some new generating functions of the products of (p,q)-modified Pell numbers with k-Fibonacci and k-Lucas numbers, k-Pell and k-Pell Lucas numbers, k-Jacobsthal and k-Jacobsthal Lucas numbers at positive and negative indices. By making use of the operator defined in this paper, we give some new generating functions of the products of (p,q)-modified Pell numbers with k-balancing and k-Lucas-balancing numbers.*

Keywords: *symmetric functions, generating functions; (p,q)-modified Pell numbers; k-Fibonacci numbers; k-Jacobsthal numbers; k-balancing numbers; k-Pell numbers.*

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1. INTRODUCTION

The authors in [1] defined and studied the (*p*,*q*)-modified Pell numbers. They gave generating function, Binet's formula and some sums formulas of these numbers. Now, let we recall some properties of these numbers.

The (*p*,*q*)-modified Pell numbers $\{MP_{p,q,n}\}_{n\geq 0}$ satisfies the recurrence relation:

$$\begin{cases} MP_{p,q,n} = 2pMP_{p,q,n-1} + qMP_{p,q,n-2}, \text{ for } n \geq 2 \\ MP_{p,q,0} = 1,\ MP_{p,q,1} = p \end{cases}.$$

If $p = q = 1$ we get the sequence of **modified Pell numbers** $\{q_n\}_{n\geq 0}$.

The Binet's formula for (*p*,*q*)-modified Pell numbers is given by

$$MP_{p,q,n} = p\left(\frac{x_1^n + x_2^n}{x_1 + x_2}\right),$$

where $x_1 = p + \sqrt{p^2 + q}$ and $x_2 = p - \sqrt{p^2 + q}$ are roots of the characteristic equation $x^2 - 2px - q = 0.$ We note that

$$x_1 + x_2 = 2p,\ x_1 x_2 = -q \text{ and } x_1 - x_2 = 2\sqrt{p^2 + q}.$$

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Furthermore, the generating function of (*p*,*q*)-modified Pell numbers is given by

$$\sum_{n=0}^{+\infty} MP_{p,q,n} z^n = \frac{1-pz}{1-2pz-qz^2}.$$

The next proposition gives the complete homogeneous symmetric functions of (*p*,*q*)-modified Pell numbers.

Proposition 1.1. [2] *For* $n \in \mathbb{N}$, *we have*

$$MP_{p,q,n} = h_n\left(a_1,[-a_2]\right) - ph_{n-1}\left(a_1,[-a_2]\right), \text{ with } \begin{cases} a_1 = p+\sqrt{p^2+q} \\ a_2 = p-\sqrt{p^2+q} \end{cases}.$$

The *k*-balancing numbers $\{B_{k,n}\}_{n\geq 0}$ and *k*-Lucas-balancing numbers $\{C_{k,n}\}_{n\geq 0}$ are given either by the generating functions:

$$\sum_{n=0}^{+\infty} B_{k,n} z^n = \frac{z}{1-6kz+z^2},$$

$$\sum_{n=0}^{+\infty} C_{k,n} z^n = \frac{1-3kz}{1-6kz+z^2},$$

respectively, or by the recurrence relations:

$$\begin{cases} B_{k,n+1} = 6kB_{k,n} - B_{k,n-1}, \text{ for } k,\ n \geq 1 \\ B_{k,0} = 0,\ B_{k,1} = 1 \end{cases},$$

$$\begin{cases} C_{k,n+1} = 6kC_{k,n} - C_{k,n-1}, \text{ for } k,\ n \geq 1 \\ C_{k,0} = 1,\ C_{k,1} = 3k \end{cases},$$

respectively, for more information, please see the papers [3, 4] .

In 1965, Horadam [5, 6] defined a second-order linear recurrence sequence $\{W_n\left(a,b;p,q\right)\}_{n\geq 0}$ or briefly $\{W_n\}_{n\geq 0}$ by the recurrence relation:

$$W_n = pW_{n-1} + qW_{n-2}, \quad \text{for } n \geq 2,$$

with the initial conditions $W_0 = a$ and $W_1 = b$. The special cases of the numbers $\{W_n\}_{n\geq 0}$ are listed as follows:

1. For $a=q=1$ and $b=p=k$ we get the *k*-Fibonacci numbers $F_{k,n}$ (see [7]).
2. For $a=2, b=p=k$ and $q=1$ it yields *k*-Lucas numbers $L_{k,n}$ (see [8]).
3. For $a=0, b=1, p=k$ and $q=2$ it reduces to the *k*-Jacobsthal numbers $J_{k,n}$ (see [9]).
4. For $a=q=2$ and $b=p=k$ we get the *k*-Jacobsthal Lucas numbers $j_{k,n}$ (see [10]).
5. For $a=0, b=1, p=2$ and $q=k$ it yields the *k*-Pell numbers $P_{k,n}$ (see [11]).
6. For $a=b=p=2$ and $q=k$ it reduces to the *k*-Pell Lucas numbers $Q_{k,n}$ (see [12]).

The Binet's formula for $\{W_n\}_{n\geq 0}$ is given by

$$W_n = \frac{Ax_1^n - Bx_2^n}{x_1 - x_2},$$

with $A = b - ax_2$ and $B = b - ax_1$.

The special cases of the Binet's formula for $\{W_n\}_{n\geq 0}$ are listed in the Table 1.

Table 1. Binet's formulas of k-numbers.

a	b	p	q	Roots (x_1 and x_2)	Binet's formula (W_n)
1	k	k	1	$x_1 = \frac{k+\sqrt{k^2+4}}{2}, x_2 = \frac{k-\sqrt{k^2+4}}{2}$	$F_{k,n} = \frac{x_1^{n+1}-x_2^{n+1}}{x_1-x_2}$
2	k	k	1	$x_1 = \frac{k+\sqrt{k^2+4}}{2}, x_2 = \frac{k-\sqrt{k^2+4}}{2}$	$L_{k,n} = x_1^n + x_2^n$
0	1	k	2	$x_1 = \frac{k+\sqrt{k^2+8}}{2}, x_2 = \frac{k-\sqrt{k^2+8}}{2}$	$J_{k,n} = \frac{x_1^n-x_2^n}{x_1-x_2}$
2	k	k	2	$x_1 = \frac{k+\sqrt{k^2+8}}{2}, x_2 = \frac{k-\sqrt{k^2+8}}{2}$	$j_{k,n} = x_1^n + x_2^n$
0	1	2	k	$x_1 = 1+\sqrt{1+k}, x_2 = 1-\sqrt{1+k}$	$P_{k,n} = \frac{x_1^n-x_2^n}{x_1-x_2}$
2	2	2	k	$x_1 = 1+\sqrt{1+k}, x_2 = 1-\sqrt{1+k}$	$Q_{k,n} = x_1^n + x_2^n$

The next proposition gives the relations between k-numbers at positive and negative indices.

Proposition 1.2. [13, 14] *For $n \geq 0$ and $k \geq 1$, the following equalities are holds:*

$$F_{k,-n} = (-1)^{n+1} F_{k,n}. \tag{1.1}$$

$$L_{k,-n} = (-1)^n L_{k,n}. \tag{1.2}$$

$$P_{k,-n} = (-1)^{n+1} P_{k,n}. \tag{1.3}$$

$$Q_{k,-n} = (-1)^n Q_{k,n}. \tag{1.4}$$

$$J_{k,-n} = \frac{(-1)^{n+1}}{2^n} J_{k,n}. \tag{1.5}$$

$$j_{k,-n} = \frac{(-1)^n}{2^n} j_{k,n}. \tag{1.6}$$

In this contribution, we will define an operator denoted by $\delta_{a_1a_2}^k$ and the complete homogeneous symmetric function for which we can formulate, extends and proves results based on our previous ones [13, 15-18]. In order to determine new generating functions of the products of (p,q)-modified Pell numbers with k-Fibonacci and k-Lucas numbers, k-Pell and k-Pell Lucas numbers, k-Jacobsthal and k-Jacobsthal Lucas numbers at positive and negative indices. By making use of the operator defined in this paper, we give some new generating functions of the products of (p,q)-modified Pell numbers with k-balancing and k-Lucas-balancing numbers, we use analytical means and series manipulation methods. In the sequel, we derive new symmetric functions and some new properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in section 3.

2. PRELIMINARIES AND DEFINITIONS

In this section, we introduce a symmetric function and give some properties of this symmetric function.

Definition 2.1. [19] *Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables. Then, the k-th elementary symmetric function $e_k\left(a_1, a_2, ..., a_n\right)$ is defined by*

$$e_k^{(n)} = e_k\left(a_1, a_2, ..., a_n\right) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad \left(0 \leq k \leq n\right), \tag{2.1}$$

with $i_1, i_2, ..., i_n = 0$ or 1.

Definition 2.2. [19] *Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables. Then, the k-th complete homogeneous symmetric function $h_k\left(a_1, a_2, ..., a_n\right)$ is defined by*

$$h_k^{(n)} = h_k\left(a_1, a_2, ..., a_n\right) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad \left(k \geq 0\right), \tag{2.2}$$

with $i_1, i_2, ..., i_n \geq 0$.

Remark 2.1. *Set $e_0\left(a_1, a_2, ..., a_n\right) = 1$ and $h_0\left(a_1, a_2, ..., a_n\right) = 1$, by usual convention. For $k < 0$, we set $e_k\left(a_1, a_2, ..., a_n\right) = 0$ and $h_k\left(a_1, a_2, ..., a_n\right) = 0$.*

If $n = 2$, the k-th complete homogeneous symmetric function (2.2) gives us

$$h_k(a_1, a_2) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2}, \; k \in \mathbb{N}_0.$$

with

$$h_0(a_1, a_2) = 1,$$

$$h_1(a_1, a_2) = a_1 + a_2,$$

$$h_2(a_1, a_2) = a_1^2 + a_1 a_2 + a_2^2,$$

$$\vdots$$

Proposition 2.1. *Given an alphabet $A = \{a_1, a_2, ..., a_n\}$, we have*

$$\sum_{k=0}^{+\infty} e_k(a_1, a_2, ..., a_n) z^k = \prod_{a \in A} (1 + az).$$

Proposition 2.2. *Given an alphabet* $A=\{a_1,a_2,...,a_n\}$, *we have*

$$\sum_{k=0}^{+\infty} h_k(a_1,a_2,...,a_n)z^k = \frac{1}{\prod_{a\in A}(1-az)}.$$

There is a fundamental relation between the elementary symmetric functions and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^j e_j(a_1,a_2,...,a_n)h_{k-j}(a_1,a_2,...,a_n)=0,$$

which is valid for all $k>0$.

Definition 2.3. [20] *Let* f *be any function on* $\mathbb{R}^n$, *then we consider the divided difference operator as the following form*

$$\partial_{a_i a_{i+1}}(f)=\frac{f(a_1,\cdots,a_i,a_{i+1},\cdots,a_n)-f(a_1,\cdots,a_{i-1},a_{i+1},a_i,a_{i+2},\cdots,a_n)}{a_i-a_{i+1}}.$$

Definition 2.4. [15] *The symmetrizing operator* $\delta_{a_1a_2}^k$ *is defined by*

$$\delta_{a_1a_2}^k f(a_1)=\frac{a_1^k f(a_1)-a_2^k f(a_2)}{a_1-a_2}, \text{ for all } k\in\mathbb{N}_0.$$

3. GENERATING FUNCTIONS OF THE PRODUCTS OF (*p, q*)-MODIFIED PELL NUMBERS WITH *k*-NUMBERS AT POSITIVE AND NEGATIVE INDICES

The following propositions is one of the key tools of the proof of our main result. It has been proved in [17, 21]

Proposition 3.1. *Given two alphabets* $E=\{e_1,e_2\}$ *and* $A=\{a_1,a_2\}$, *then*

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2)h_n(e_1,e_2)z^n=\frac{1-a_1a_2e_1e_2z^2}{\left(\sum\limits_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2)e_2^n z^n\right)}. \tag{3.1}$$

Based on the relationship (3.1) we get

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2)h_{n-1}(e_1,e_2)z^n=\frac{z-a_1a_2e_1e_2z^3}{\left(\sum\limits_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2)e_2^n z^n\right)}. \tag{3.2}$$

Proposition 3.2. *Given two alphabets* $E=\{e_1,e_2\}$ *and* $A=\{a_1,a_2\}$, *then*

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2)h_n(e_1,e_2)z^n=\frac{(e_1+e_2)z-e_1e_2(a_1+a_2)z^2}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2)e_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2)e_2^n z^n\right)}. \tag{3.3}$$

In this part, we now derive the new generating functions of the products of (*p*,*q*)-modified Pell numbers with *k*-Fibonacci numbers, *k*-Lucas numbers, *k*-Pell numbers, *k*-Pell Lucas numbers, *k*-Jacobsthal numbers and *k*-Jacobsthal Lucas numbers at positive and negative indices, and we give the new generating functions of the products of (*p*,*q*)-modified Pell numbers with *k*-balancing and *k*-Lucas-balancing numbers.

For the case $A=\{a_1,-a_2\}$ and $E=\{e_1,-e_2\}$ with replacing a_2 by $(-a_2)$ and e_2 by $(-e_2)$ in (3.1), (3.2) and (3.3), we have

$$\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n=\frac{1-a_1a_2e_1e_2z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}. \tag{3.4}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n=\frac{z-a_1a_2e_1e_2z^3}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}. \tag{3.5}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n=\frac{(e_1-e_2)z+e_1e_2(a_1-a_2)z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}. \tag{3.6}$$

This case consists of four related parts. **Firstly**, the substitutions

$$\begin{cases} a_1-a_2=2p \\ a_1a_2=q \end{cases} \text{ and } \begin{cases} e_1-e_2=k \\ e_1e_2=1 \end{cases},$$

in (3.4), (3.5) and (3.6), we obtain

$$\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n=\frac{1-qz^2}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}, \tag{3.7}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n=\frac{z-qz^3}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}, \tag{3.8}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n=\frac{kz+2pz^2}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}, \tag{3.9}$$

respectively, and we have the following theorems.

Theorem 3.1. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (*p*,*q*)*-modified Pell numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} F_{k,n} z^n = \frac{1-kpz-\left(2p^2+q\right)z^2}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}. \quad (3.10)$$

Proof: Recall that, we have $F_{k,n} = h_n(e_1,[-e_2])$ (see [22]). We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} F_{k,n} z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2]) - p h_{n-1}(a_1,[-a_2])\right) h_n(e_1,[-e_2]) z^n \\ &= \sum_{n=0}^{+\infty} h_n(a_1,[-a_2]) h_n(e_1,[-e_2]) z^n - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2]) h_n(e_1,[-e_2]) z^n.\end{aligned}$$

Using the relationships (3.7) and (3.9), we obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} F_{k,n} z^n &= \frac{1-qz^2}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4} \\ &\quad - \frac{p\left(kz+2pz^2\right)}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4} \\ &= \frac{1-kpz-\left(2p^2+q\right)z^2}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}.\end{aligned}$$

So, the proof is completed.

Theorem 3.2. *The new generating function of the product of* (*p*,*q*)*-modified Pell numbers and k-Fibonacci numbers* $F_{k,-n}$ *is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} F_{k,-n} z^n = \frac{-1-kpz+\left(2p^2+q\right)z^2}{1+2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2+2kpqz^3+q^2z^4}, \text{ for all } n \in \mathbb{N}. \quad (3.11)$$

Proof: We use the change of variable $z = -z$ in (3.10) and according to relationship (1.1), we get

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} F_{k,-n} z^n &= \sum_{n=0}^{+\infty} MP_{p,q,n} \left(-1\right)^{n+1} F_{k,n} z^n \\ &= \frac{-1-kpz+\left(2p^2+q\right)z^2}{1+2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2+2kpqz^3+q^2z^4}.\end{aligned}$$

Thus, this completes the proof.

Theorem 3.3. *For $n \in \mathbb{N}$, the new generating function of the product of* (p,q)*-modified Pell numbers with k-Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} L_{k,n} z^n = \frac{2-3kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-kpqz^3}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}. \tag{3.12}$$

Proof: By [22], we have $L_{k,n}=2h_n(e_1,[-e_2])-kh_{n-1}(e_1,[-e_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} L_{k,n} z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2])-ph_{n-1}(a_1,[-a_2])\right)\left(2h_n(e_1,[-e_2])-kh_{n-1}(e_1,[-e_2])\right)z^n \\ &= 2\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n - k\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &\quad -2p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n + kp\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n.\end{aligned}$$

Using the relationships (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} L_{k,n} z^n &= \frac{2\left(1-qz^2\right)}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4} \\ &\quad -\frac{k\left(2pz+kqz^2\right)}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4} \\ &\quad -\frac{2p\left(kz+2pz^2\right)}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4} \\ &\quad +\frac{kp\left(z-qz^3\right)}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4} \\ &= \frac{2-3kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-kpqz^3}{1-2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2-2kpqz^3+q^2z^4}.\end{aligned}$$

This completes the proof.

Theorem 3.4. *The new generating function of the product of* (p,q)*-modified Pell numbers and k-Lucas numbers* $L_{k,-n}$ *is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} L_{k,-n} z^n = \frac{2+3kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2+kpqz^3}{1+2kpz-\left(4p^2+q\left(k^2+2\right)\right)z^2+2kpqz^3+q^2z^4}, \text{ for all } n \in \mathbb{N}. \tag{3.13}$$

Proof: We use the change of variable $z = -z$ in (3.12) and according to relationship (1.2), we get

$$\sum_{n=0}^{+\infty} MP_{p,q,n} L_{k,-n} z^n = \sum_{n=0}^{+\infty} MP_{p,q,n} (-1)^n L_{k,n} z^n$$
$$= \frac{2+3kpz - \left(4p^2 + q\left(k^2+2\right)\right)z^2 + kpqz^3}{1+2kpz - \left(4p^2+q\left(k^2+2\right)\right)z^2 + 2kpqz^3 + q^2z^4}.$$

So, the proof is completed.

- Put $k = 1$ in the relationships (3.10), (3.11), (3.12) and (3.13), we obtain the following results.

Corollary 3.1. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} F_n z^n = \frac{1-pz-(2p^2+q)z^2}{1-2pz-(4p^2+3q)z^2-2pqz^3+q^2z^4}.$$

Corollary 3.2. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Fibonacci numbers at negative indice is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} F_{-n} z^n = \frac{-1-pz+(2p^2+q)z^2}{1+2pz-(4p^2+3q)z^2+2pqz^3+q^2z^4}.$$

Corollary 3.3. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} L_n z^n = \frac{2-3pz-(4p^2+3q)z^2-pqz^3}{1-2pz-(4p^2+3q)z^2-2pqz^3+q^2z^4}.$$

Corollary 3.4. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} L_{-n} z^n = \frac{2+3pz-(4p^2+3q)z^2+pqz^3}{1+2pz-(4p^2+3q)z^2+2pqz^3+q^2z^4}.$$

- By putting $p = q = 1$ in the relationships (3.10), (3.11), (3.12) and (3.13) and the corollaries 3.1, 3.2, 3.3 and 3.4, we obtain the following new generating functions, calculation results are indicated in Table 2.

Secondly, the substitutions

$$\begin{cases} a_1 - a_2 = 2p \\ a_1 a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2 \\ e_1 e_2 = k \end{cases},$$

in (3.4), (3.5) and (3.6), we give

$$\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n = \frac{1-kqz^2}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}, \tag{3.14}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n = \frac{z-kqz^3}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}, \tag{3.15}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n = \frac{2z+2kpz^2}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}, \tag{3.16}$$

respectively, thus we have the following theorems.

Table 2. A new generating functions for products of some sequences.

Coefficient of z^n	Generating function
$q_nF_{k,n}$	$\frac{1-kz-3z^2}{1-2kz-\left(k^2+6\right)z^2-2kz^3+z^4}$
$q_nF_{k,-n}$	$\frac{-1-kz+3z^2}{1+2kz-\left(k^2+6\right)z^2+2kz^3+z^4}$
$q_nL_{k,n}$	$\frac{2-3kz-\left(k^2+6\right)z^2-kz^3}{1-2kz-\left(k^2+6\right)z^2-2kz^3+z^4}$
$q_nL_{k,-n}$	$\frac{2+3kz-\left(k^2+6\right)z^2+kz^3}{1+2kz-\left(k^2+6\right)z^2+2kz^3+z^4}$
q_nF_n	$\frac{1-z-3z^2}{1-2z-7z^2-2z^3+z^4}$
q_nF_{-n}	$\frac{-1-z+3z^2}{1+2z-7z^2+2z^3+z^4}$
q_nL_n	$\frac{2-3z-7z^2-z^3}{1-2z-7z^2-2z^3+z^4}$
q_nL_{-n}	$\frac{2+3z-7z^2+z^3}{1+2z-7z^2+2z^3+z^4}$

Theorem 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-modified pell numbers with k-Pell numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}P_{k,n}z^n = \frac{pz+2qz^2+kpqz^3}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}. \tag{3.17}$$

Proof: By referred to [22], we have

$$P_{k,n} = h_{n-1}(e_1,[-e_2]).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n}P_{k,n}z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2])-ph_{n-1}(a_1,[-a_2])\right)h_{n-1}(e_1,[-e_2])z^n \\ &= \sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &= \frac{2pz+2qz^2}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}\end{aligned}$$

$$-\frac{p\left(z-kqz^3\right)}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}$$

$$=\frac{pz+2qz^2+kpqz^3}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}.$$

This completes the proof.

Theorem 3.6. *The new generating function of the product of* (p,q)*-modified Pell numbers and k-Pell numbers* $P_{k,-n}$ *is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}P_{k,-n}z^n=\frac{pz-2qz^2+kpqz^3}{1+4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2+4kpqz^3+k^2q^2z^4},\text{ for all } n\in\mathbb{N}. \quad (3.18)$$

Proof: We use the change of variable $z=-z$ in (3.17) and according to relationship (1.3), we get

$$\sum_{n=0}^{+\infty} MP_{p,q,n}P_{k,-n}z^n=\sum_{n=0}^{+\infty} MP_{p,q,n}\left(-1\right)^{n+1}P_{k,n}z^n$$

$$=\frac{pz-2qz^2+kpqz^3}{1+4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2+4kpqz^3+k^2q^2z^4}.$$

So, the proof is completed.

Theorem 3.7. *For* $n\in\mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with k-Pell Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}Q_{k,n}z^n=\frac{2-6pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-2kpqz^3}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}. \quad (3.19)$$

Proof: Recall that, we have [22].

$$Q_{k,n}=2h_n(e_1,[-e_2])-2h_{n-1}(e_1,[-e_2]).$$

We see that

$$\sum_{n=0}^{+\infty} MP_{p,q,n}Q_{k,n}z^n=\sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2])-ph_{n-1}(a_1,[-a_2])\right)\left(2h_n(e_1,[-e_2])-2h_{n-1}(e_1,[-e_2])\right)z^n$$

$$=2\sum_{n=0}^{+\infty}h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n-2\sum_{n=0}^{+\infty}h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n$$

$$-2p\sum_{n=0}^{+\infty}h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n+2p\sum_{n=0}^{+\infty}h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n,$$

by using the relationships (3.14), (3.15) and (3.16), we get

$$\sum_{n=0}^{+\infty} MP_{p,q,n}Q_{k,n}z^n = \frac{2\left(1-kqz^2\right)}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}$$
$$-\frac{2\left(2pz+2qz^2\right)}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}$$
$$-\frac{2p\left(2z+2kpz^2\right)}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}$$
$$+\frac{2p\left(z-kqz^3\right)}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}$$
$$=\frac{2-6pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-2kpqz^3}{1-4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2-4kpqz^3+k^2q^2z^4}.$$

This completes the proof.

Theorem 3.8. *The new generating function of the product of* (*p*,*q*)*-modified Pell numbers and k-Pell Lucas numbers* $Q_{k,-n}$ *is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}Q_{k,-n}z^n = \frac{2+6pz-\left(4kp^2+2q\left(k+2\right)\right)z^2+2kpqz^3}{1+4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2+4kpqz^3+k^2q^2z^4},\ \text{for all } n\in\mathbb{N}. \quad (3.20)$$

Proof: We use the change of variable $z=-z$ in (3.19) and according to relationship (1.4), we get

$$\sum_{n=0}^{+\infty} MP_{p,q,n}Q_{k,-n}z^n = \sum_{n=0}^{+\infty} MP_{p,q,n}\left(-1\right)^n Q_{k,n}z^n$$
$$=\frac{2+6pz-\left(4kp^2+2q\left(k+2\right)\right)z^2+2kpqz^3}{1+4pz-\left(4kp^2+2q\left(k+2\right)\right)z^2+4kpqz^3+k^2q^2z^4}.$$

Thus, this completes the proof.

- By setting $k=1$ in the relationships (3.17), (3.18), (3.19) and (3.20) we obtain the following corollaries.

Corollary 3.5. *For* $n\in\mathbb{N}$, *the new generating function of the product of* (*p*,*q*)*-modified Pell numbers with Pell numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}P_n z^n = \frac{pz+2qz^2+pqz^3}{1-4pz-(4p^2+6q)z^2-4pqz^3+q^2z^4}.$$

Corollary 3.6. *For* $n\in\mathbb{N}$, *the new generating function of the product of* (*p*,*q*)*-modified Pell numbers with Pell numbers at negative indice is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} P_{-n} z^n = \frac{pz - 2qz^2 + pqz^3}{1+4pz - (4p^2+6q)z^2 + 4pqz^3 + q^2z^4}.$$

Corollary 3.7. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Pell Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} Q_n z^n = \frac{2-6pz - \left(4p^2+6q\right)z^2 - 2pqz^3}{1-4pz - (4p^2+6q)z^2 - 4pqz^3 + q^2z^4}.$$

Corollary 3.8. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Pell Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} Q_{-n} z^n = \frac{2+6pz - \left(4p^2+6q\right)z^2 + 2pqz^3}{1+4pz - (4p^2+6q)z^2 + 4pqz^3 + q^2z^4}.$$

- Put $p = q = 1$ in the relationships (3.17), (3.18), (3.19) and (3.20) and the corollaries 3.5, 3.6, 3.7 and 3.8, we obtain the following new generating functions, calculation results are indicated in Table 3.

Table 3. A new generating functions for products of some sequences.

Coefficient of z^n	Generating function
$q_n P_{k,n}$	$\frac{z+2z^2+kz^3}{1-4z-(6k+4)z^2-4kz^3+k^2z^4}$
$q_n P_{k,-n}$	$\frac{z-2z^2+kz^3}{1+4z-(6k+4)z^2+4kz^3+k^2z^4}$
$q_n Q_{k,n}$	$\frac{2-6z-(6k+4)z^2-2kz^3}{1-4z-(6k+4)z^2-4kz^3+k^2z^4}$
$q_n Q_{k,-n}$	$\frac{2+6z-(6k+4)z^2+2kz^3}{1+4z-(6k+4)z^2+4kz^3+k^2z^4}$
$q_n P_n$	$\frac{z+2z^2+z^3}{1-4z-10z^2-4z^3+z^4}$
$q_n P_{-n}$	$\frac{z-2z^2+z^3}{1+4z-10z^2+4z^3+z^4}$
$q_n Q_n$	$\frac{2-6z-10z^2-2z^3}{1-4z-10z^2-4z^3+z^4}$
$q_n Q_{-n}$	$\frac{2+6z-10z^2+2z^3}{1+4z-10z^2+4z^3+z^4}$

Thirdly, the substitutions

$$\begin{cases} a_1 - a_2 = 2p \\ a_1a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = k \\ e_1e_2 = 2 \end{cases},$$

in (3.4), (3.5) and (3.6) we give

$$\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n = \frac{1-2qz^2}{1-2kpz - \left(8p^2+q\left(k^2+4\right)\right)z^2 - 4kpqz^3 + 4q^2z^4}, \tag{3.21}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n = \frac{z-2qz^3}{1-2kpz - \left(8p^2+q\left(k^2+4\right)\right)z^2 - 4kpqz^3 + 4q^2z^4}, \tag{3.22}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n = \frac{kz+4pz^2}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4}, \tag{3.23}$$

respectively, and we have the following theorems.

Theorem 3.9. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}J_{k,n}z^n = \frac{pz+kqz^2+2pqz^3}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4}. \tag{3.24}$$

Proof: By [22], we have $J_{k,n} = h_{n-1}(e_1,[-e_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n}J_{k,n}z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2])-ph_{n-1}(a_1,[-a_2])\right)h_{n-1}(e_1,[-e_2])z^n \\ &= \sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &= \frac{2pz+kqz^2}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4} \\ &\quad -\frac{p\left(z-2qz^3\right)}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4},\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} MP_{p,q,n}J_{k,n}z^n = \frac{pz+kqz^2+2pqz^3}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4}.$$

So, the proof is completed.

Theorem 3.10. *The new generating function of the product of* (p,q)*-modified Pell numbers and k-Jacobsthal numbers* $J_{k,-n}$ *is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}J_{k,-n}z^n = \frac{2pz-kqz^2+pqz^3}{4+4kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2+2kpqz^3+q^2z^4}, \text{ for all } n \in \mathbb{N}. \tag{3.25}$$

Proof: We use the change of variable $z = \frac{-z}{2}$ in (3.24) and according to relationship (1.5), we get

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} J_{k,-n} z^n &= \sum_{n=0}^{+\infty} MP_{p,q,n} \frac{(-1)^{n+1}}{2^n} J_{k,n} z^n \\ &= -\frac{p\left(\frac{-z}{2}\right)+kq\left(\frac{-z}{2}\right)^2+2pq\left(\frac{-z}{2}\right)^3}{1-2kp\left(\frac{-z}{2}\right)-\left(8p^2+q\left(k^2+4\right)\right)\left(\frac{-z}{2}\right)^2-4kpq\left(\frac{-z}{2}\right)^3+4q^2\left(\frac{-z}{2}\right)^4} \\ &= \frac{2pz-kqz^2+pqz^3}{4+4kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2+2kpqz^3+q^2z^4}.\end{aligned}$$

This completes the proof.

Theorem 3.11. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with k-Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} j_{k,n} z^n = \frac{2-3kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-2kpqz^3}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4}. \tag{3.26}$$

Proof: We know that

$$j_{k,n} = 2h_n(e_1,[-e_2]) - kh_{n-1}(e_1,[-e_2]), \ \left(\text{see } [18]\right).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} j_{k,n} z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)\left(2h_n(e_1,[-e_2]) - kh_{n-1}(e_1,[-e_2])\right)z^n \\ &= 2\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n - k\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &\quad -2p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n + kp\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &= \frac{2\left(1-2qz^2\right)}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4} \\ &\quad -\frac{k\left(2pz+kqz^2\right)}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4} \\ &\quad -\frac{2p\left(kz+4pz^2\right)}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4} \\ &\quad +\frac{kp\left(z-2qz^3\right)}{1-2kpz-\left(8p^2+q\left(k^2+4\right)\right)z^2-4kpqz^3+4q^2z^4},\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} MP_{p,q,n}\, j_{k,n} z^{n} = \frac{2-3kpz-\left(8p^{2}+q\left(k^{2}+4\right)\right)z^{2}-2kpqz^{3}}{1-2kpz-\left(8p^{2}+q\left(k^{2}+4\right)\right)z^{2}-4kpqz^{3}+4q^{2}z^{4}}.$$

Thus, this completes the proof.

Theorem 3.12. *The new generating function of the product of* (p,q)*-modified Pell numbers and k-Jacobsthal Lucas numbers* $j_{k,-n}$ *is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}\, j_{k,-n} z^{n} = \frac{8+6kpz-\left(8p^{2}+q\left(k^{2}+4\right)\right)z^{2}+kpqz^{3}}{4+4kpz-\left(8p^{2}+q\left(k^{2}+4\right)\right)z^{2}+2kpqz^{3}+q^{2}z^{4}}, \text{ for all } n \in \mathbb{N}. \quad (3.27)$$

Proof: We use the change of variable $z = \frac{-z}{2}$ in (3.26) and according to relationship (1.6), we get

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n}\, j_{k,-n} z^{n} &= \sum_{n=0}^{+\infty} MP_{p,q,n} \frac{(-1)^{n}}{2^{n}} j_{k,n} z^{n} \\ &= \frac{2-3kp\left(\frac{-z}{2}\right)-\left(8p^{2}+q\left(k^{2}+4\right)\right)\left(\frac{-z}{2}\right)^{2}-2kpq\left(\frac{-z}{2}\right)^{3}}{1-2kp\left(\frac{-z}{2}\right)-\left(8p^{2}+q\left(k^{2}+4\right)\right)\left(\frac{-z}{2}\right)^{2}-4kpq\left(\frac{-z}{2}\right)^{3}+4q^{2}\left(\frac{-z}{2}\right)^{4}} \\ &= \frac{8+6kpz-\left(8p^{2}+q\left(k^{2}+4\right)\right)z^{2}+kpqz^{3}}{4+4kpz-\left(8p^{2}+q\left(k^{2}+4\right)\right)z^{2}+2kpqz^{3}+q^{2}z^{4}}.\end{aligned}$$

This completes the proof.

- By putting $k = 1$ in the relationships (3.24), (3.25), (3.26) and (3.27), we obtain the following corollaries.

Corollary 3.9. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} J_{n} z^{n} = \frac{pz+qz^{2}+2pqz^{3}}{1-2pz-\left(8p^{2}+5q\right)z^{2}-4pqz^{3}+4q^{2}z^{4}}.$$

Corollary 3.10. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Jacobsthal numbers at negative indice is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} J_{-n} z^{n} = \frac{2pz-qz^{2}+pqz^{3}}{4+4pz-\left(8p^{2}+5q\right)z^{2}+2pqz^{3}+q^{2}z^{4}}.$$

Corollary 3.11. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}\, j_{n} z^{n} = \frac{2-3pz-\left(8p^{2}+5q\right)z^{2}-2pqz^{3}}{1-2pz-\left(8p^{2}+5q\right)z^{2}-4pqz^{3}+4q^{2}z^{4}}.$$

Corollary 3.12. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Jacobsthal Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}\, j_{-n} z^n = \frac{8+6pz-\left(8p^2+5q\right)z^2+pqz^3}{4+4pz-\left(8p^2+5q\right)z^2+2pqz^3+q^2z^4}.$$

- Put $p=q=1$ in the relationships (3.24), (3.25), (3.26) and (3.27) and the corollaries 3.9, 3.10, 3.11 and 3.12, we obtain the following new generating functions, calculation results are indicated in Table 4.

Table 4. A new generating functions for products of some sequences.

Coefficient of z^n	Generating function
$q_n J_{k,n}$	$\frac{z+kz^2+2z^3}{1-2kz-\left(k^2+12\right)z^2-4kz^3+4z^4}$
$q_n J_{k,-n}$	$\frac{2z-kz^2+z^3}{4+4kz-\left(k^2+12\right)z^2+2kz^3+z^4}$
$q_n j_{k,n}$	$\frac{2-3kz-\left(k^2+12\right)z^2-2kz^3}{1-2kz-\left(k^2+12\right)z^2-4kz^3+4z^4}$
$q_n j_{k,-n}$	$\frac{8+6kz-\left(k^2+12\right)z^2+kz^3}{4+4kz-\left(k^2+12\right)z^2+2kz^3+z^4}$
$q_n J_n$	$\frac{z+z^2+2z^3}{1-2z-13z^2-4z^3+4z^4}$
$q_n J_{-n}$	$\frac{2z-z^2+z^3}{4+4z-13z^2+2z^3+z^4}$
$q_n j_n$	$\frac{2-3z-13z^2-2z^3}{1-2z-13z^2-4z^3+4z^4}$
$q_n j_{-n}$	$\frac{8+6z-13z^2+z^3}{4+4z-13z^2+2z^3+z^4}$

Fourthly, the substitutions

$$\begin{cases} a_1-a_2=2p \\ a_1a_2=q \end{cases} \text{and} \begin{cases} e_1-e_2=6k \\ e_1e_2=-1 \end{cases},$$

in (3.4), (3.5) and (3.6), we obtain

$$\sum_{n=0}^{+\infty} h_n\left(a_1,[-a_2]\right)h_n\left(e_1,[-e_2]\right)z^n = \frac{1+qz^2}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}, \quad (3.28)$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n = \frac{z+qz^3}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}, \quad (3.29)$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n = \frac{6kz-2pz^2}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}, \quad (3.30)$$

respectively, and we have the following theorems.

Theorem 3.13. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with k-balancing numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} B_{k,n} z^n = \frac{pz + 6kqz^2 - pqz^3}{1-12kpz - 2\left(q\left(18k^2-1\right)-2p^2\right)z^2 + 12kpqz^3 + q^2z^4}. \quad (3.31)$$

Proof: Recall that, we have $B_{k,n} = h_{n-1}(e_1,[-e_2])$, (see [16]). We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} B_{k,n} z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)h_{n-1}(e_1,[-e_2])z^n \\ &= \sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &= \frac{2pz + 6kqz^2}{1-12kpz - 2\left(q\left(18k^2-1\right)-2p^2\right)z^2 + 12kpqz^3 + q^2z^4} \\ &\quad - \frac{p\left(z + qz^3\right)}{1-12kpz - 2\left(q\left(18k^2-1\right)-2p^2\right)z^2 + 12kpqz^3 + q^2z^4},\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} MP_{p,q,n} B_{k,n} z^n = \frac{pz + 6kqz^2 - pqz^3}{1-12kpz - 2\left(q\left(18k^2-1\right)-2p^2\right)z^2 + 12kpqz^3 + q^2z^4}.$$

So, the proof is completed.

Theorem 3.14. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with k-Lucas-balancing numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n} C_{k,n} z^n = \frac{1-9kpz + \left(2p^2 + q\left(1-18k^2\right)\right)z^2 + 3kpqz^3}{1-12kpz - 2\left(q\left(18k^2-1\right)-2p^2\right)z^2 + 12kpqz^3 + q^2z^4}. \quad (3.32)$$

Proof: We know that

$$C_{k,n} = h_n(e_1,[-e_2]) - 3kh_{n-1}(e_1,[-e_2]), \ \left(\text{see}[16]\right).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n} C_{k,n} z^n &= \sum_{n=0}^{+\infty}\left(h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)\left(h_n(e_1,[-e_2]) - 3kh_{n-1}(e_1,[-e_2])\right)z^n \\ &= \sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_n(e_1,[-e_2])z^n - 3k\sum_{n=0}^{+\infty} h_n(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n \\ &\quad - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_n(e_1,[-e_2])z^n + 3kp\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(e_1,[-e_2])z^n,\end{aligned}$$

by using the relationships (3.28), (3.29) and (3.30), we obtain the following result:

$$\begin{aligned}\sum_{n=0}^{+\infty} MP_{p,q,n}C_{k,n}z^n &= \frac{1+qz^2}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}\\ &-\frac{3k\left(2pz+6kqz^2\right)}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}\\ &-\frac{p\left(6kz-2pz^2\right)}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}\\ &+\frac{3kp\left(z+qz^3\right)}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}\\ &=\frac{1-9kpz+\left(2p^2+q\left(1-18k^2\right)\right)z^2+3kpqz^3}{1-12kpz-2\left(q\left(18k^2-1\right)-2p^2\right)z^2+12kpqz^3+q^2z^4}.\end{aligned}$$

Thus, this completes the proof.

- Put $k=1$ in the relationships (3.31) and (3.32) we obtain the following results.

Corollary 3.13. *For* $n\in\mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with balancing numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}B_n z^n = \frac{pz+6qz^2-pqz^3}{1-12pz-2(17q-2p^2)z^2+12pqz^3+q^2z^4}.$$

Corollary 3.14. *For* $n\in\mathbb{N}$, *the new generating function of the product of* (p,q)*-modified Pell numbers with Lucas-balancing numbers is given by*

$$\sum_{n=0}^{+\infty} MP_{p,q,n}C_n z^n = \frac{1-9pz+(2p^2-17q)z^2+3pqz^3}{1-12pz-2(17q-2p^2)z^2+12pqz^3+q^2z^4}.$$

- By taking $p=q=1$ in the relationships (3.31) and (3.32) and the corollary 3.13 and corollary 3.14, we obtain the following new generating functions, calculation results are indicated in Table 5.

Table 5. A new generating functions for products of some sequences.

Coefficient of z^n	Generating function
$q_nB_{k,n}$	$\frac{z+6kz^2-z^3}{1-12kz-6(6k^2-1)z^2+12kz^3+z^4}$
$q_nC_{k,n}$	$\frac{1-9kz+3(1-6k^2)z^2+3kz^3}{1-12kz-6(6k^2-1)z^2+12kz^3+z^4}$
q_nB_n	$\frac{z+6z^2-z^3}{1-12z-30z^2+12z^3+z^4}$
q_nC_n	$\frac{1-9z-15z^2+3z^3}{1-12z-30z^2+12z^3+z^4}$

4. CONCLUSIONS

In this paper, by making use of Eqs. (3.1) and (3.3), we have derived some new generating functions of the products of (p,q)-modified pell numbers with k-Fibonacci numbers, k-Lucas numbers, k-Jacobsthal numbers, k-Jacobsthal Lucas numbers, k-Pell numbers and k-Pell Lucas numbers at positive and negative indices, and the products of (p,q)-modified pell numbers with k-balancing and k-Lucas-balancing numbers. The derived theorems and corollaries are based on symmetric functions and the products of these numbers.

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