ORIGINAL PAPER

DISPERSIVE SOLITARY WAVE SOLUTIONS OF COUPLING BOITI-LEON-PEMPINELLI SYSTEM USING TWO DIFFERENT METHODS

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Abstract. *In this paper, new exact traveling wave solutions for the coupling Boiti-Leon-Pempinelli system are obtained by using two important different methods. The first is the modified extended tanh function methods which depend on the balance rule and the second is the Ricatti-Bernoulli Sub-ODE method which doesn't depend on the balance rule. The solitary waves solutions can be derived from the exact wave solutions by give the parameters a special value. The consistent and inconsistent of the obtained solutions are studied not only between these two methods but also with that relisted by the other methods.*

Keywords: *modified extended tanh-function method; Ricatti-Bernoulli Sub-ODE method; coupling Boiti-Leon-Pempinelli system; traveling wave solutions.*

1. INTRODUCTION

The study of various complex phenomenon of physics and engineering using the nonlinear evolution equations (NLEEs) are still important. In fluid dynamics the analytical solutions for the nonlinear evolution equations of shallow water waves and the equations related to it (the Korteweg-de Vries (KdV) equation, modified KdV equation, Boussinesq equation, Green-Naghdi equation, Gardner's equation, Whitham-Broer-Kaup equation, Jaulent-Miodek (JM) equations and coupling Boiti-Leon-Pempinelli system) are not usually available. To discuss these types of equations which still an open area in the theory of solitons several methods are applied successively to understanding these complex phenomena in mathematical physics through many authors [1-30]. As example of such methods the modified simple equation-method, the first integral method, the (G′ /G)-expansion method ,the modified (G′ /G, 1/G)-expansion method, extended Jacobi elliptic function method, the $\exp(-\phi(\zeta))$-expansion method, the modified extended $\exp(-\phi(\zeta))-$expansion method, the Riccati-Bernoulli Sub-ODE method, exp-function method, the tanh-function method, the extended tanh-function method, the new extended direct algebraic method and so on.

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The objective of this article is using these two different methods mentioned above to find the exact solutions of the coupling Boiti-Leon-Pempinelli system [31-39] in terms of some parameters. If these parameters take definite values the solitary wave solutions can be derived from it.

2. MODIFIED EXTENDED TANH-FUCTION METHOD

According to the constructed method the solution is

$$u(\zeta) = a_0 + \sum_{i=1}^{m}\left(a_i\phi_i + \frac{b_i}{\phi_i}\right), \tag{1}$$

where the positive integer m can be determined by balancing the highest order derivative term and the nonlinear term such that either $a_m \neq 0$ or $b_m \neq 0$ while a_i, b_i are constants to be determined. The function ϕ mention in this equation must be satisfies the Riccati equation $\phi' = b + \phi^2$, and admits three forms of solutions according to the value of b namely:

(1) If $b < 0$, then

$$\varphi = -\sqrt{-b}\tanh\left(\sqrt{-b}\zeta\right), \quad \text{or} \quad \varphi = -\sqrt{-b}\coth\left(\sqrt{-b}\zeta\right). \tag{2}$$

(2) If $b > 0$, then

$$\varphi = \sqrt{b}\tan\left(\sqrt{b}\zeta\right), \quad \text{or} \quad \varphi = -\sqrt{b}\cot\left(\sqrt{b}\zeta\right).$$

(3) If $b = 0$, then

$$\varphi = -\frac{1}{\zeta}$$

Equating the coefficient of different power of ϕ^i $(i = 0, \pm1, \pm2, \pm3, ...)$ to zero after substituting in the given problem we get a system of algebraic equations, which can be solved by Maple or any other computer program to get the values of the required constants.

2.1 APPLICATION

Here, we use the modified extended tanh-function method described [22] to find the exact traveling wave solutions and then the solitary wave solutions for the coupling Boiti-Leon-Pempinelli system [31]

$$\begin{aligned} u_{ty} &= (u^2 - u_x)_{xy} + 2v_{xxx}, \\ v_t &= v_{xx} + 2uv_x. \end{aligned} \tag{3}$$

Let $u(x,y,t) = u(\zeta)$, $v(x,y,t) = v(\zeta)$ .and $\zeta = x + y - \lambda t,$ substitute at Eq. (9) we get

$$\begin{aligned} -\lambda u'' &= \left(u^2\right)'' - u''' + 2v''', \\ -\lambda v' &= v'' + 2uv', \end{aligned} \tag{4}$$

By integrating the first Eq. (4) twice with respect to ζ , and neglecting the constants of integration, we have

$$v' = \frac{1}{2}u' - \frac{u^2 + \lambda u}{2}, \tag{5}$$

That gives

$$v = \frac{1}{2}u - \frac{1}{2}\int (u^2 + \lambda u)d\zeta. \tag{6}$$

Inserting Eq. (6) into second Eq. (4) yields

$$u'' - 2u^3 - 3\lambda u^2 - \lambda^2 u = 0. \tag{7}$$

Balancing the nonlinear term with the highest order derivative term, we find $m+2 = 3m \Rightarrow m = 1$.Consequently, according to the constructed method the solution is

$$u(\zeta) = a_0 + a_1\phi(\zeta) + \frac{b_1}{\phi(\zeta)}, \tag{8}$$

Substitute, about u'' ,u^3,u^2 and u at Eq. (8), and equating different power of $\phi(\zeta)$ to zero, we obtain algebraic system of equation

$$\begin{aligned} & a_1(1 - a_1^2) = 0, \\ & a_1^2(2a_0 + \lambda) = 0, \\ & 2a_1 b - 6a_0^2 a_1 - 6a_1^2 b_1 - 6ca_0 a_1 - \lambda^2 a_1 = 0, \\ & b_1(b^2 - b_1^2) = 0, \\ & b_1^2(2a_0 + \lambda) = 0, \\ & 2bb_1 - 6a_0^2 b_1 - 6a_1 b_1^2 - 6\lambda a_0 b_1 - \lambda^2 b_1 = 0, \\ & 2a_0^3 + 12a_0 a_1 b_1 + 3\lambda a_0^2 + 6\lambda a_1 b_1 + \lambda^2 a_0 = 0, \end{aligned} \tag{9}$$

Solving this system of algebraic equations by Maple, we get the following results:

$$
\begin{aligned}
&(1)\ a_1=-1,\ b_1=0,\ b=-a_0^2,\ \lambda=-2a_0,\\
&(2)\ a_1=-1,\ b_1=-\frac{a_0^2}{2},\ b=\frac{a_0^2}{2},\ \lambda=-2a_0,\\
&(3)\ a_1=-1,\ b_1=-\frac{a_0^2}{4},\ b=-\frac{a_0^2}{4},\ \lambda=-2a_0,\\
&(4)\ a_1=0,\ b_1=-a_0^2,\ b=-a_0^2,\ \lambda=-2a_0,\\
&(5)\ a_1=0,\ b_1=a_0^2,\ b=-a_0^2,\ \lambda=-2a_0,\\
&(6)\ a_1=1,\ b_1=0,\ b=-a_0^2,\ \lambda=-2a_0,\\
&(7)\ a_1=1,\ b_1=\frac{a_0^2}{4},\ b=-\frac{a_0^2}{4},\ \lambda=-2a_0,\\
&(8)\ a_1=1,\ b_1=\frac{a_0^2}{2},\ b=\frac{a_0^2}{2},\ c=-2a_0.
\end{aligned}
\tag{10}
$$

These obtained results will generate 16 solutions of the given equation according to the different cases of b at the investigated method. Now, we will choose only two of these solution say, (2) and (3).

Thus, the solution of case (2) above $a_0=-\frac{\lambda}{2}$, $a_1=-1$, $b_1=-\frac{\lambda^2}{8}$, $\mathrm{b}=\frac{\lambda^2}{8}$. So that the exact solution of Eq. (7) according to the (8) is

$$
u(\zeta)=-\frac{1}{2}\lambda+\frac{\sqrt{-2\lambda^2}}{4}\tanh\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)+\frac{\lambda^2}{2\sqrt{-2\lambda^2}\tanh\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)},
$$

$$
v(\zeta)=-\frac{1}{4}\lambda+\frac{\sqrt{-2\lambda^2}}{8}\tanh\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)-\frac{\lambda^2\tanh\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)}{4\sqrt{-2\lambda^2}}+\frac{\lambda^2 Ln\left(\tanh\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)-1\right)}{4\sqrt{-2\lambda^2}}-\frac{\lambda^2 Ln\left(\tanh\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)+1\right)}{4\sqrt{-2\lambda^2}}+\frac{\lambda^2\xi}{4}+K.
\tag{11}
$$

$$
u(\zeta)=-\frac{1}{2}\lambda+\frac{\sqrt{-2\lambda^2}}{4}\coth\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)+\frac{\lambda^2}{2\sqrt{-2\lambda^2}\coth\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)},
\tag{12}
$$

$$v(\zeta) = -\frac{1}{4}\lambda + \frac{\sqrt{-2\lambda^2}}{8}\coth\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right) - \frac{\lambda^2\coth\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)}{4\sqrt{-2\lambda^2}} + \frac{\lambda^2 Ln\left(\coth\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)-1\right)}{4\sqrt{-2\lambda^2}} - \frac{\lambda^2 Ln\left(\coth\left(\frac{\sqrt{-2\lambda^2}}{4}\zeta\right)+1\right)}{4\sqrt{-2\lambda^2}} + \frac{\lambda^2\xi}{4} + K,$$

where K is constant.

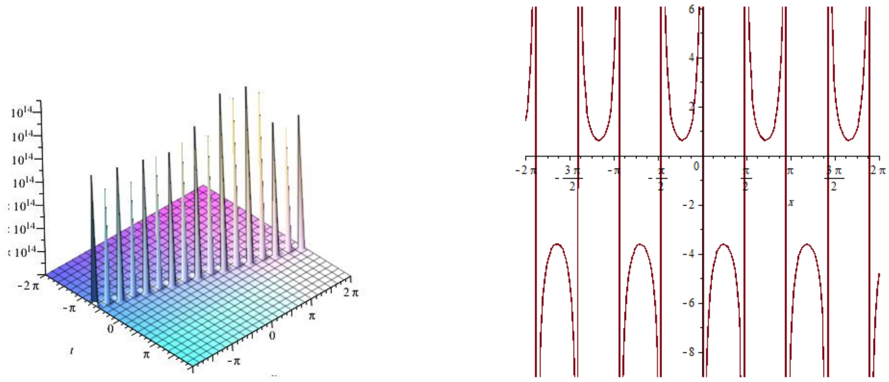

Figure 1. Solitary wave solutions for (u) of Eq. (11) by considering the values $\lambda = 2,\ t = 0.001, -2 < x < 2, -2 < y < 2$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

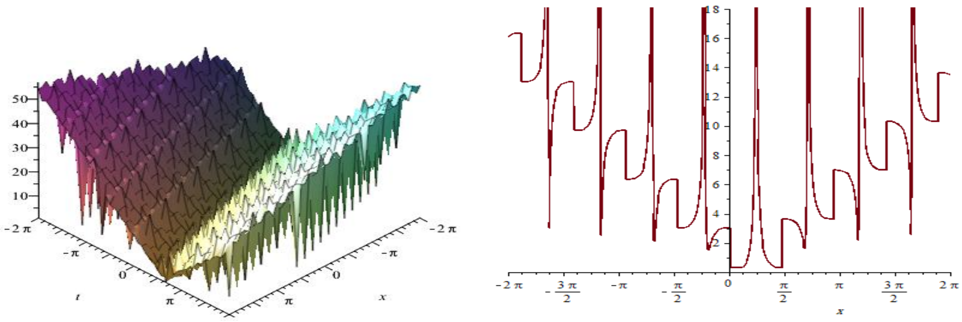

Figure 2. Solitary wave solutions for (v) of Eq. (11) by considering the values $\lambda = 2,\ t = 0.001, -2 < x < 2, -2 < y < 2$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

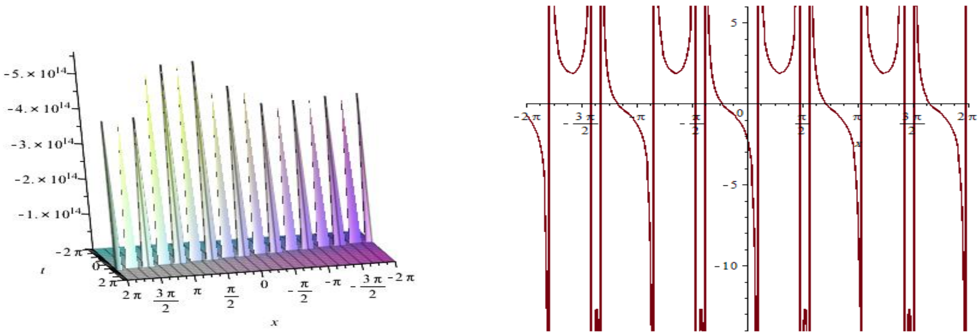

Figure 3. Solitary wave solutions of (u) Eq. (12) by considering the values $\lambda = 2,\ t = 0.001, -2 < x < 2, -2 < y < 2$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

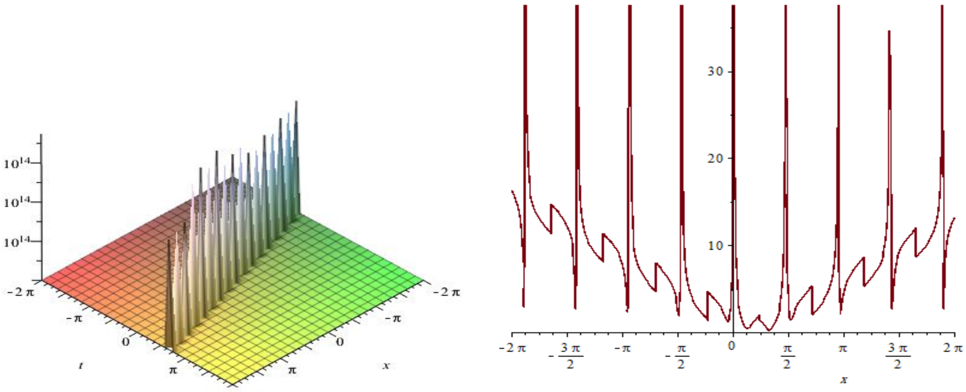

Figure 4. Solitary wave solutions for (v) of Eq. (12) by considering the values $\lambda = 2,\ t = 0.001, -2 < x < 2, -2 < y < 2$ **for the 3D plots and** $y = 0.001, -2 < t < 2,$ **for the 2D plots.**

Also for the solution (3) we have,

$$a_0 = -\frac{\lambda}{2}\ ,\ a_1 = -1,\ b_1 = -\frac{\lambda^2}{16}\ , \mathrm{b} = -\frac{\lambda^2}{16}$$

So that the exact solution of Eq. (7) according to the (8) is

$$u(\zeta) = -\frac{1}{2}\lambda + \frac{\sqrt{\lambda^2}}{4}\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right) + \frac{\lambda^2}{2\sqrt{\lambda^2}\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)},$$

$$v(\zeta) = -\frac{1}{4}\lambda + \frac{\sqrt{\lambda^2}}{8}\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right) + \frac{\lambda^2}{4\sqrt{\lambda^2}\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)} + \frac{\lambda^2\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)}{8\sqrt{\lambda^2}} - \frac{\lambda^2 Ln\left(\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)+1\right)}{2\sqrt{\lambda^2}} + \frac{\lambda^2 Ln\left(\tanh\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)-1\right)}{2\sqrt{\lambda^2}} + \frac{\lambda^2\xi}{4} + K. \qquad (13)$$

$$u(\zeta) = -\frac{1}{2}\lambda + \frac{\sqrt{\lambda^2}}{4}\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right) + \frac{\lambda^2}{4\sqrt{\lambda^2}\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)},. \qquad (14)$$

$$v(\zeta)=-\frac{1}{4}\lambda+\frac{\sqrt{\lambda^2}}{8}\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)+\frac{\lambda^2}{4\sqrt{\lambda^2}\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)}+\frac{\lambda^2\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)}{8\sqrt{\lambda^2}}-$$

$$\frac{\lambda^2 Ln\left(\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)-1\right)}{2\sqrt{\lambda^2}}-\frac{\lambda^2 Ln\left(\coth\left(\frac{\sqrt{\lambda^2}}{4}\zeta\right)+1\right)}{2\sqrt{\lambda^2}}+\frac{\lambda^2\xi}{4}+K.$$

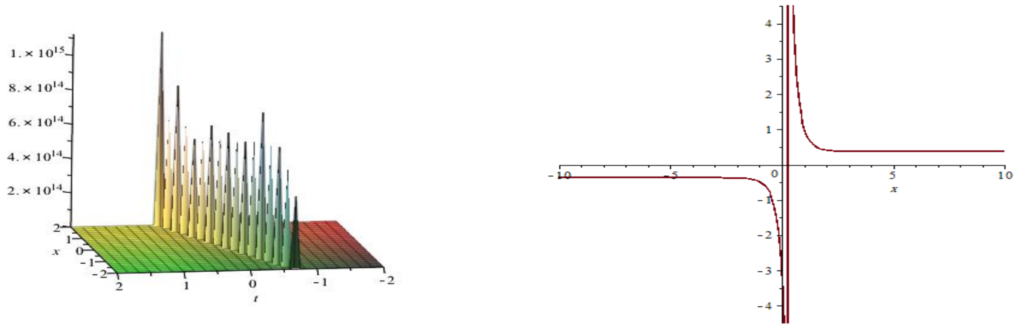

Figure 5. Solitary wave solutions for (u) of Eq. (13) by considering the values $\lambda=2,\ t=0.001, -2<x<2, -2<y<2$ for the 3D plots and $y=0.001, -2<t<2$ for the 2D plots.

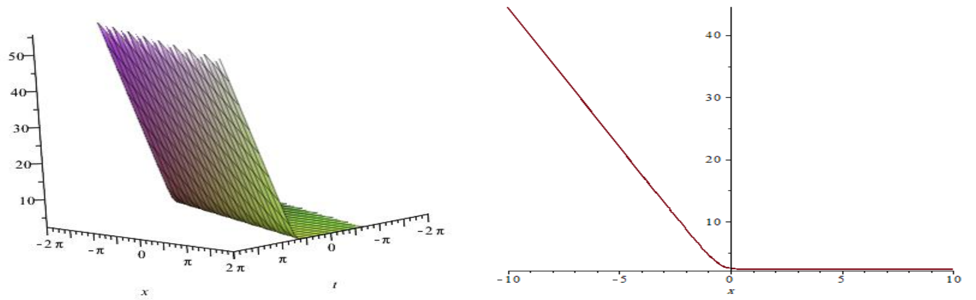

Figure 6. Solitary wave solutions for (v) of Eq. (13) by considering the values $\lambda=2,\ t=0.001, -2<x<2, -2<y<2$ for the 3D plots and $y=0.001, -2<t<2$ for the 2D plots.

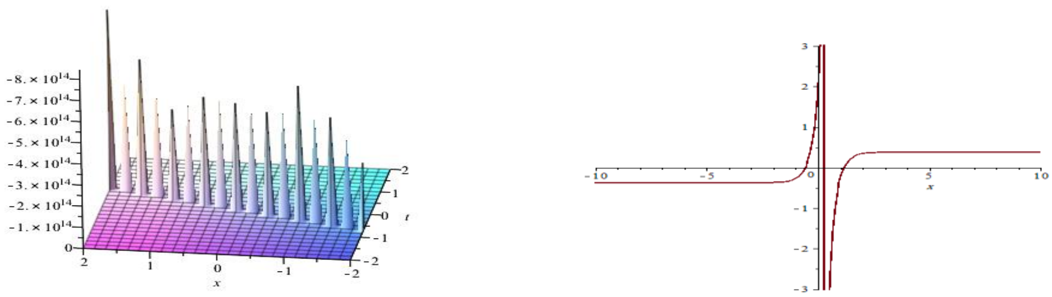

Figure 7. Solitary wave solutions for (u) of Eq. (14) by considering the values $\lambda=2,\ t=0.001, -2<x<2, -2<y<2$ for the 3D plots and $y=0.001, -2<t<2$ for the 2D plots.

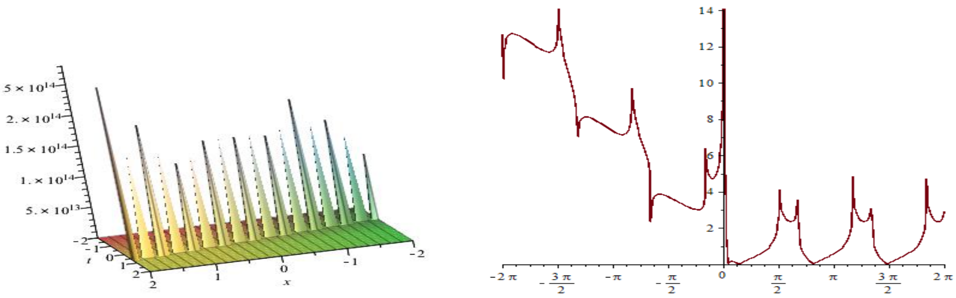

Figure 8. Solitary wave solutions for (v) of Eq. (14) by considering the values $\lambda = 2,\ t = 0.001, -2 < x < 2, -2 < y < 2$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

3. THE RICCATI-BERNOULLI SUB-ODE METHOD

According to the Riccati-Bernoulli Sub-ODE method [13, 14] the suggested solution is

$$u' = au^{2-m} + bu + cu^m, \tag{15}$$

where a,b,c and m are constants to be determined later. It is important to knots that when $ac \neq 0$ and $m = 0$, Eq.(15) is a Riccati equation. When $a \neq 0,\ c = 0,$ and $m \neq 1$, Eq.(15) is a Bernoulli equation.

Differentiate (15) once we get,

$$u'' = ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^m + (2ac+b^2)u. \tag{16}$$

Substituting the derivatives of u into Eq. (15) yields an algebraic equation of u, by consider the symmetry of the right-hand item of Eq. (15) and setting equivalence for the highest power exponents of u we can determine m. Comparing the coefficients of u^i yields a set of algebraic equations for a,b,c and λ which solving to get $a,\ b,\ c,\ \lambda$.

According to the obtained values of these constants and use the transformation $\zeta = x + y - \lambda t$ the Riccati-Bernoulli Sub-ODEE equation admits the following solutions:

(1) When $m = 1$, the solution of Eq. (15) is

$$u(\zeta) = C_1\, e^{(a+b+c)\zeta}. \tag{17}$$

(2) When $m \neq 1,\ b = 0$ and $c = 0$, the solution of Eq. (15) is

$$u(\zeta) = \left(a(m-1)(\zeta + C_1)\right)^{1/(1-m)}. \tag{18}$$

(3) When $m \neq 1,\ b \neq 0$ and $c = 0$, the solution of Eq. (15) is

$$u(\zeta)=\left(-\frac{a}{b}+C_1 e^{b(m-1)\zeta}\right)^{1/(m-1)}. \tag{19}$$

(4) When $m\neq 1,\ a\neq 0$ and $b^2-4ac<0$, the solution of Eq. (15) is

$$u(\zeta)=\left(\frac{-b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left(\frac{(1-m)\sqrt{4ac-b^2}}{2}(\zeta+C_1)\right)\right)^{\frac{1}{(1-m)}}, \tag{20}$$

and

$$u(\zeta)=\left(\frac{-b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\cot\left(\frac{(1-m)\sqrt{4ac-b^2}}{2}(\zeta+C_1)\right)\right)^{\frac{1}{(1-m)}}. \tag{21}$$

(5) When $m\neq 1,\ a\neq 0$ and $b^2-4ac>0$, the solution of Eq. (15) is

$$u(\zeta)=\left(\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\coth\left(\frac{(1-m)\sqrt{b^2-4ac}}{2}(\zeta+C_1)\right)\right)^{\frac{1}{(1-m)}}, \tag{22}$$

and

$$u(\zeta)=\left(\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\tanh\left(\frac{(1-m)\sqrt{b^2-4ac}}{2}(\zeta+C_1)\right)\right)^{\frac{1}{(1-m)}}. \tag{23}$$

(6) When $m\neq 1,\ a\neq 0$ and $b^2-4ac=0$ the solution of Eq. (15) is

$$u(\zeta)=\left(\frac{1}{a(m-1)(\zeta+C_1)}-\frac{b}{2a}\right)^{1/(1-\mathrm{m})}, \tag{24}$$

where C_1 is an arbitrary constant.

3.1. APPLICATION

Now, we apply this method for solving the coupling Boiti-Leon-Pempinelli system mention above:

$$u''-2u^3-3\lambda u^2-\lambda^2 u=0. \tag{25}$$

According to Riccati-Bernoulli Sub-ODE method,

$$u' = au^{2-m} + bu + cu^{m}, \tag{26}$$

Differentiate once we get,

$$u'' = ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^{m} + (2\,\mathrm{ac} + b^2)u. \tag{27}$$

Substitute about u'' at Eq. (25), by a suitable choose of m and equating the coefficients of different power of u to zero, we get this system of algebraic equations,

$$\begin{aligned} &a^2 - 1 = 0, \\ &ab - \lambda = 0, \\ &2ac + b^2 - \lambda^2 = 0, \\ &bc = 0. \end{aligned} \tag{28}$$

According to the obtained solutions of this system and the cases of the constructed method we will take only case (4) and case (5).

(4) When $a > 0$, ($a = 1, c = 0, \lambda = b$) the solution is

$$\begin{aligned} u(\zeta) &= \frac{1}{-\frac{1}{\lambda} + C_1 e^{-\lambda\zeta}}, \\ v(\zeta) &= K. \end{aligned} \tag{29}$$

$-2 < y < 2,$ and $y = 0.001, -2 < t < 2$ for the 2D graphics

(5) When $a > 0$, ($a = 1, b = -\lambda, c = 0$) the solution is

$$\begin{aligned} u(\zeta) &= -\frac{\lambda}{2} + \frac{\lambda}{2}\coth\left(\frac{\lambda}{2}(\zeta + C_1)\right), \\ v(\zeta) &= -\frac{\lambda}{4} + \frac{\lambda}{2}\coth\left(\frac{\lambda}{2}(\zeta + C_1)\right) - \frac{\lambda}{4}Ln\left(\coth\left(\frac{\lambda}{2}(\zeta + C_1)\right) + 1\right) + \\ &\qquad \frac{\lambda}{4}Ln\left(-1 + \coth\left(\frac{\lambda}{2}(\zeta + C_1)\right)\right) + \frac{\lambda^2}{4}\xi + K. \end{aligned} \tag{30}$$

$$\begin{aligned} u(\zeta) &= \frac{-\lambda}{2} + \frac{\lambda}{2}\tanh\left(\frac{\lambda}{2}(\zeta + C_1)\right), \\ v(\zeta) &= -\frac{\lambda}{4} + \frac{\lambda}{2}\tanh\left(\frac{\lambda}{2}(\zeta + C_1)\right) - \frac{\lambda}{4}Ln\left(\tanh\left(\frac{\lambda}{2}(\zeta + C_1)\right) + 1\right) + \\ &\qquad \frac{\lambda}{4}Ln\left(-1 + \tanh\left(\frac{\lambda}{2}(\zeta + C_1)\right)\right) + \frac{\lambda^2}{4}\xi + K. \end{aligned} \tag{31}$$

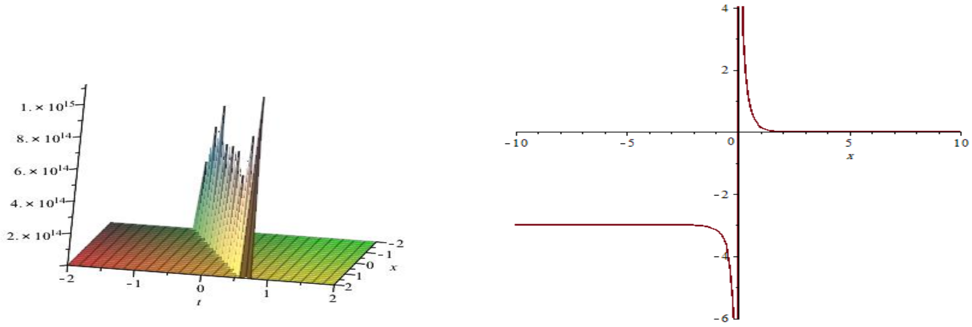

Figure 9. Solitary wave solutions for (u) of Eq. (30) by considering the values $C_1 = 1, \lambda = 2,\ t = 0.001, -2 < x < 2,\ -2 < y < 2,$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

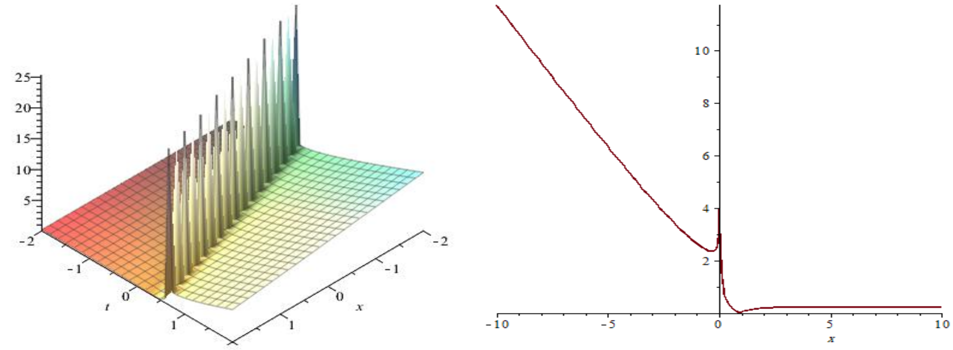

Figure 10. Solitary wave solutions for (v) of Eq. (30) by considering the values $C_1 = 1, \lambda = 2,\ t = 0.001, -2 < x < 2,\ -2 < y < 2,$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

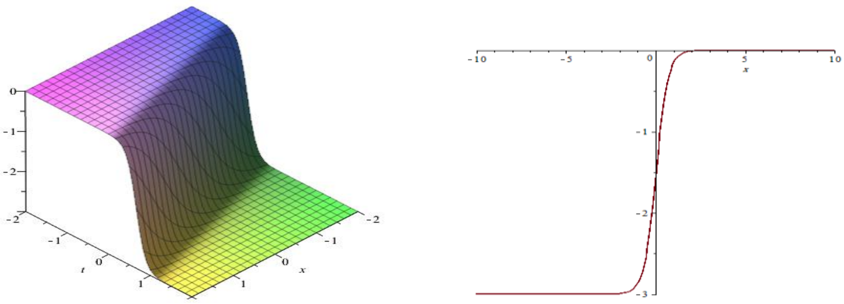

Figure 11. Solitary wave solutions for (u) of Eq. (31) by considering the values $C_1 = 1, \lambda = 2,\ t = 0.001, -2 < x < 2,\ -2 < y < 2,$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

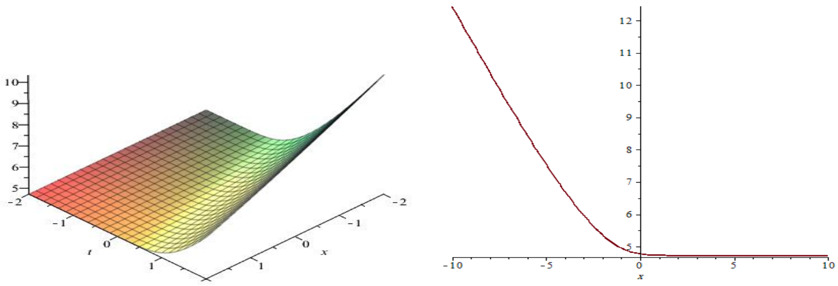

Figure 12. Solitary wave solutions for (v) of Eq. (31) by considering the values $C_1 = 1, \lambda = 2,\ t = 0.001, -2 < x < 2,\ -2 < y < 2,$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

Case 5. When $a<0$, ($a=-1, b=\lambda, c=0$) the solution is

$$
\begin{aligned}
&u(\zeta)=-\frac{\lambda}{2}-\frac{\lambda}{2}\coth\left(\frac{\lambda}{2}(\zeta+C_1)\right),\\
&v(\zeta)=-\frac{\lambda}{4}+\frac{\lambda}{4}Ln\left(\coth\left(\frac{\lambda}{2}(\zeta+C_1)\right)-1\right)-\frac{\lambda}{4}Ln\left(\coth\left(\frac{\lambda}{2}(\zeta+C_1)\right)+1\right)+\frac{\lambda^2}{4}\xi+K.
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&u(\zeta)=-\frac{\lambda}{2}-\frac{\lambda}{2}\tanh\left(\frac{\lambda}{2}(\zeta+C_1)\right),\\
&v(\zeta)=-\frac{\lambda}{4}+\frac{\lambda}{4}Ln\left(\tanh\left(\frac{\lambda}{2}(\zeta+C_1)\right)-1\right)-\frac{\lambda}{4}Ln\left(\tanh\left(\frac{\lambda}{2}(\zeta+C_1)\right)+1\right)+\frac{\lambda^2}{4}\xi+K.
\end{aligned}
\tag{33}
$$

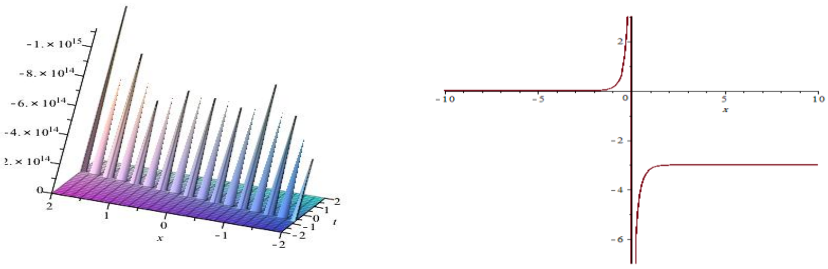

Figure 13. Solitary wave solutions for (u) of Eq. (32) by considering the values $C_1=1, \lambda=2,\ t=0.001, -2<x<2,\ -2<y<2,$ for the 3D plots and $y=0.001, -2<t<2$ for the 2D plots.

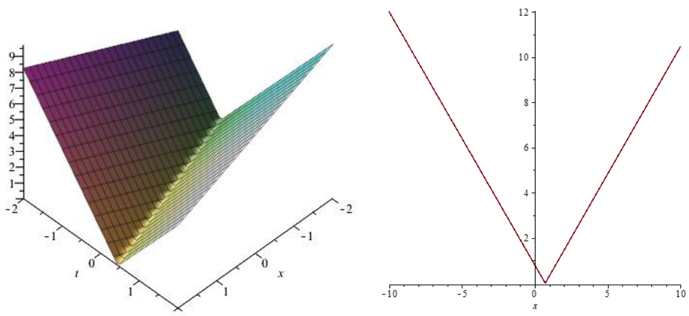

Figure 14. Solitary wave solutions for (v) of Eq. (32) by considering the values $C_1=1, \lambda=2,\ t=0.001, -2<x<2,\ -2<y<2,$ for the 3D plots and $y=0.001, -2<t<2$ for the 2D plots.

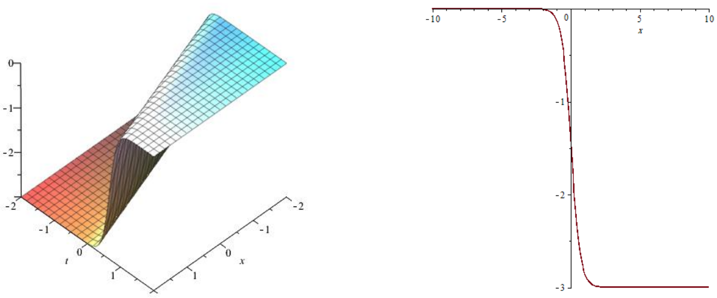

Figure 15. Solitary wave solutions for (u) of Eq. (33) by considering the values $C_1=1, \lambda=2,\ t=0.001, -2<x<2,\ -2<y<2,$ for the 3D plots and $y=0.001, -2<t<2$ for the 2D plots.

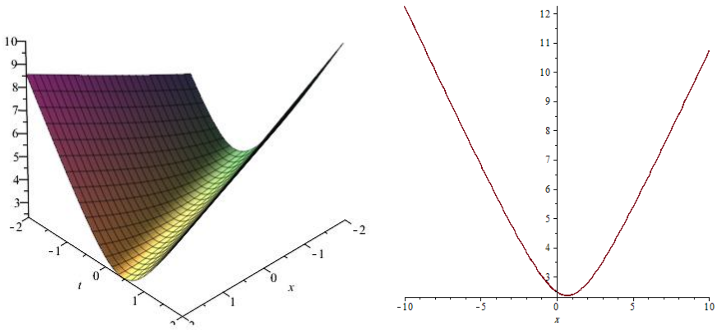

Figure 16. Solitary wave solutions for (v) of Eq. (33) by considering the values $C_1 = 1, \lambda = 2,\ t = 0.001, -2 < x < 2,\ -2 < y < 2,$ **for the 3D plots and** $y = 0.001, -2 < t < 2$ **for the 2D plots.**

4. CONCLUSIONS

The modified extended tanh-function method which depends on the balance rule has been successfully used to find many dispersive solitary traveling wave solutions of the Boiti-Leon-Pempinelli system. This large numbers of obtained solutions which not repeated as [31-39] are equivalent in some cases but increase in the other. Also, a new technique according the Riccati-Bernoulli Sub-ODE method which does not depend on the balance rule have been used to find the (kink , periodic, soliton and multi-soliton) solutions of the coupling Boiti-Leon-Pempinelli system and can be used for any other NLEEs in which the balance rule fails. Finally, let us compared between our obtained results in the present study with that obtained by other methods, we can be concluded that these two methods are reliable and effective and can be applied to many other NLEEs.

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