ORIGINAL PAPER

A NUMERICAL TREATMENT OF GENERALIZED HUXLEY EQUATION

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Abstract. *In this paper, numerical solutions of generalized Huxley are obtained by using a new scheme: Crank-Nicolson logarithmic finite difference method (CN-LFDM). The efficiency of the presented method is illustrated by a numerical example for different cases of parameters which confirm that obtained results are in good agreement with the exact solutions and numerical solutions obtained by some other methods in literature. The method is analyzed by von-Neumann stability analysis method and it is displayed that the method is unconditionally stable.*

Keywords: *Generalized Huxley equation; Crank Nicolson logarithmic finite difference method; von Neumann Satbility analysis.*

1. INTRODUCTION

Nonlinear partial differential equations are often used to model most of the problems in various fields such as physics, chemistry, biology, mathematics, and engineering. One of these nonlinear partial differential equations is generalized Huxley equation.

The generalized Huxley equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right), \qquad a<x<b, \quad t>0 \tag{1}$$

with initial condition

$$u(x,0)=f(x), \qquad a<x<b$$

and boundary conditions

$$u(a,t)=g_1(t), \quad u(b,t)=g_2(t), \quad t>0$$

describe the propagation of a nerve impulse in nerve fibers and the movement of the wall in liquid crystals. Where $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions, $\delta,\beta\geq 0$ and $\gamma\in(0.1)$ are given parameters.

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Various numerical methods have been used to solve the equation (1) numerically by many researchers. . Variational iteration method (VIM) has been used to obtain the numerical solutions of the equation by Batiha et. al. [1]. Hashemi et. al. [2] used the homotopy perturbation method (HPM) to obtain the numerical solutions of the equation. Hashim et al. [3] applied the Adomian decomposition method to solve the equation numerically. Hemida and Mohamed [4] used the homotopy analysis method (HAM) for obtaining the numerical solutions of the equation. Inan [5, 6] used the explicit exponential finite difference method and implicit exponential finite difference method (I-EFDM) to solve the equation.

In this study, we present the Crank-Nicolson logarithmic finite difference method to obtain the numerical solutions of the generalized Huxley equation. Logarithmic finite difference methods have been used to solve various equations in literature. Celikten et. al. [7] used the explicit logarithmic finite difference schemes to solve the Burgers equation. Modified Burgers equation has solved by Celikten [8] using the explicit logarithmic finite difference schemes.

Celikten [9] obtained the numerical solutions of Burgers equation by using implicit and fully implicit logarithmic finite difference methods. Celikten and Surek [10] used the explicit logarithmic finite difference method to solve the generalized Burgers-Fisher equation numerically. El-Azab et al. [11] obtained numerical solutions of the Korteweg de Vries Burger (KdVB) equation using the open logarithmic finite difference method. Ismail and Al-Basyoni [12] used the implicit logarithmic finite difference method to solve the Troesch problem numerically. Srivastava et al. [13] used the closed logarithmic finite difference method to solve two-dimensional Burgers equation systems. The one-dimensional coupled Burgers equation was solved by Srivastava et al. [14] using the implicit logarithmic finite difference method. Aljaboori [15] used the Crank-Nicolson logarithmic finite difference method to solve the combined Burgers equation numerically.

2. MATERIALS AND METHODS

2.1. CRANK NICOLSON LOGARITHMIC FINITE DIFFERENCE METHOD

We demonstrate the finite difference approximation of $u(x,t)$ at the node point (x_i, t_n) by u_i^n in which $x_i = ih(i = 0,1,\ldots,N)$, $t_n = t_0 + nk(n = 0,1,2,\ldots)$, $h = \dfrac{b-a}{N}$ is the node size in x direction and k is the time step.

We reorganize Equation (1) to acquire

$$\frac{\partial u}{\partial t} = \beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right)+\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

Multiplying equation (2) by e^u, we acquire the following equation:

$$\frac{\partial e^u}{\partial t} = e^u\left(\beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right)+\frac{\partial^2 u}{\partial x^2}\right) \tag{3}$$

using the finite difference approximations for derivatives in Equation (3) the following Crank Nicolson logarithmic finite difference scheme is acquired

CN-EFDM $$u_i^{n+1} = u_i^n + \ln\left\{1 + k\left[\beta u_i^n\left(1-\left(u_i^n\right)^\delta\right)\left(\left(u_i^n\right)^\delta - \gamma\right) + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2h^2}\right]\right\} \quad (4)$$

where $1 \le i \le N$ -1.

Equation (4) is a system of nonlinear difference equations. We assume this nonlinear system of equations in the form

$$G(V) = 0 \quad (5)$$

where $G = \left[g_1, g_2, \ldots, g_{N-1}\right]^T$ and $V = \left[u_1^{n+1}, u_2^{n+1}, \ldots, u_{N-1}^{n+1}\right]^T$. Newton's iterative method is used to linearize the nonlinear Equation (5) results in the following iteration:

1) Set $V^{(0)}$, an initial guess.

2) For $m = 0,1,2,\ldots$ until convergence do:

Solve $J\left(V^{(m)}\right)\delta^{(m)} = -G\left(V^{(m)}\right)$;

Set $V^{(m+1)} = V^{(m)} + \delta^{(m)}$ where $J(V^{(m)})$ is the Jacobian matrix which is appraised analytically. The solution at the previous time-step is taken as the initial estimate. The Newton's iteration at each time-step is stopped when $\left\|G(V^{(m)})\right\| \le 10^{-5}$.

2.2. STABILITY ANALYSIS

To investigate the stability of scheme, we will use the von Neumann stability analysis in which the growth factor of a typical Fourier mode is defined as follows:

$$u_i^n = \varepsilon^n e^{I\phi ih}, \ I = \sqrt{-1}. \quad (6)$$

von Neumann stability analysis is used to analyze the stability of finite difference scheme applied to linear partial differential equations. So we will investigate the stability of linear form of the scheme. The nonlinear term of the scheme (4) have been linearized by replacing the quantity $\left(u_i^n\right)^\delta$ by local constant $\tilde{U}$. Hence the numerical scheme (4), convert into

$$u_i^{n+1} = u_i^n + \ln\left\{1 + k\left[\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U} - \gamma\right) + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2h^2}\right]\right\}. \quad (7)$$

Since the scheme (7) is logarithmic, the examination will be improved by expanding the logarithmic term of the scheme into a Taylor's series. Hilal et al. [16] applied the same procedure to calculate the local truncation error of exponential finite difference schemes and examine their stability. If the logarithmic term of the scheme expands to Taylor series and use the first term of the expansion the scheme can be written as:

$$u_i^{n+1} = u_i^n + k\beta u_i^n \left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) + \frac{k}{2h^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]. \tag{8}$$

By substituting the (6) equality into the (8) linear form of the scheme, we get the growth factor as follows:

$$\varepsilon = \frac{1 + k\beta\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) - \dfrac{k}{h^2}\sin^2\dfrac{\phi h}{2}}{1 + \dfrac{k}{h^2}\sin^2\dfrac{\phi h}{2}}.$$

Stability condition in von-Neumann method is $|\varepsilon| \le 1$ and this condition is satisfied since $\beta \ge 0$ and $\gamma \in (0.1)$. Therefore CN-LFDM generalized Huxley equation is unconditionally stable.

3. NUMERICAL RESULTS AND DISCUSSION

Crank Nicolson logarithmic finite difference method is used to acquire the numerical solutions of the generalized Huxley equation. To demonstrate the correctness of results L_2 and L_∞ error norms:

$$L_2 = \left\|U - u_N\right\|_2 = \sqrt{h\sum_{j=0}^{N}\left|U_j - \left(u_N\right)_j\right|^2},$$

$$L_\infty = \left\|U - u_N\right\|_\infty = \max_j\left|U_j - \left(u_N\right)_j\right|$$

are used, in which U and u indicate the exact and computed numerical solutions, respectively. In all numerical computations we took as $h = 0.01$ and $k = 0.000001$.

3.1. NUMERICAL EXAMPLE OF GENERALIZED HUXLEY EQUATION

Consider the generalized Huxley equation of the form Equation (1) in domain $0 \le x \le 1,\ t > 0$ with initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)\right]^{\frac{1}{\delta}}$$

and boundary conditions

$$u(0,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left\{\frac{\left(1+\delta-\gamma\right)\rho}{2\left(1+\delta\right)}\right\}t\right\}\right]^{\frac{1}{\delta}},$$

$$u(1,t)=\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(1+\left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{\frac{1}{\delta}}.$$

The exact solution of this problem is [17]:

$$u(x,t)=\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(x+\left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{\frac{1}{\delta}}$$

where $\rho=\sqrt{4\beta(1+\delta)}$ and $\sigma=\delta\rho/4(1+\delta)$.

The numerical solutions of Generalized Huxley Equation obtained by CN-LFDM are compared with the exact solutions and numerical solutions obtained by some other methods [1-5] in literature in Tables 1-3. The comparisons for the case $\delta=1$, $\beta=1$ and $\gamma=0.001$ are shown in Table 1, while the comparisons for the case $\delta=2$, $\beta=1$ and $\gamma=0.001$ are shown in Table 2 and for the case $\delta=3$, $\beta=1$ and $\gamma=0.001$ are shown in Table 3.

Table 1. Exact and numerical solutions for the case *δ* = 1, *β* = 1 and *γ* = 0.001.

x	*t*	Exact	CN-LFDM	VIM [1], HPM [2], ADM [3]	HAM [4]	I-EFDM [5]
0.1	0.05	5.000302E-4	5.000195E-4	5.000052E-4	5.000100E-4	5.000125 E-4
	0.1	5.000427E-4	5.000250E-4	4.999927E-4	5.000030E-4	5.000102 E-4
	1	5.002676E-4	5.002363E-4	4.997678E-4	4.998680E-4	5.000064 E-4
0.5	0.05	5.001009E-4	5.000677E-4	5.000759E-4	5.000810E-4	5.000768 E-4
	0.1	5.001134E-4	5.000517E-4	5.000634E-4	5.000730E-4	5.000692 E-4
	1	5.003383E-4	5.002316E-4	4.998385E-4	4.999380E-4	5.000572 E-4
0.9	0.05	5.001716E-4	5.000949E-4	5.001466E-4	5.001520E-4	5.001540 E-4
	0.1	5.001841E-4	5.000963E-4	5.001341E-4	5.001440E-4	5.001516 E-4
	1	5.004090E-4	5.003070E-4	4.999092E-4	5.000090E-4	5.001479 E-4

Table 2. Exact and numerical solutions for the case *δ* = 1, *β* = 1 and *γ* = 0.001.

x	*t*	Exact	CN-LFDM	VIM [1]	HPM [2], ADM [3]	HAM [4]	I-EFDM [5]
0.1	0.05	2.236188E-2	2.236141E-2	2.236077E-2	2.236077E-2	2.236100E-2	2.236110E-2
	0.1	2.236244E-2	2.236167E-2	2.236021E-2	2.236021E-2	2.236070E-2	2.236099E-2
	1	2.237250E-2	2.237117E-2	2.235015E-2	2.235015E-2	2.223546E-2	2.236082E-2
0.5	0.05	2.236447E-2	2.236306E-2	2.236335E-2	2.236335E-2	2.236360E-2	2.236339E-2
	0.1	2.236503E-2	2.236246E-2	2.236279E-2	2.236279E-2	2.236320E-2	2.236305E-2
	1	2.237508E-2	2.237067E-2	2.235273E-2	2.235273E-2	2.235720E-2	2.236251E-2

x	t	Exact	CN-LFDM	VIM [1]	HPM [2], ADM [3]	HAM [4]	I-EFDM [5]
0.9	0.05	2.236705E-2	2.236416E-2	2.236593E-2	2.236593E-2	2.236620E-2	2.236114E-2
	0.1	2.236761E-2	2.236428E-2	2.236537E-2	2.236537E-2	2.236580E-2	2.236615E-2
	1	2.237766E-2	2.237375E-2	2.235532E-2	2.235531E-2	2.235980E-2	2.236599E-2

Table 3. Exact and numerical solutions for the case δ = 3, β = 1 and γ = 0.001.

x	t	Exact	CN-LFDM	VIM [1]	HPM [2], ADM [3]	HAM [4]	I-EFDM [5]
0.1	0.05	7.937402E-2	7.937235E-2	7.937005E-2	7.937005E-2	7.937080E-2	7.937122E-2
	0.1	7.937601E-2	7.937331E-2	7.936807E-2	7.936807E-2	7.936970E-2	7.937084E-2
	1	7.941169E-2	7.940713E-2	7.933236E-2	7.933234E-2	7.934820E-2	7.937025E-2
0.5	0.05	7.938196E-2	7.937716E-2	7.937799E-2	7.937799E-2	7.937880E-2	7.937814E-2
	0.1	7.938394E-2	7.937523E-2	7.937601E-2	7.937601E-2	7.937760E-2	7.937692E-2
	1	7.941962E-2	7.940476E-2	7.934031E-2	7.934029E-2	7.935620E-2	7.937501E-2
0.9	0.05	7.938989E-2	7.938080E-2	7.938592E-2	7.938592E-2	7.938670E-2	7.938709E-2
	0.1	7.939187E-2	7.938132E-2	7.938394E-2	7.938394E-2	7.938550E-2	7.938671E-2
	1	7.942755E-2	7.941506E-2	7.934825E-2	7.934823E-2	7.936410E-2	7.938612E-2

As can be seen from the tables, numerical solutions obtained by the presented method are quite compatible with exact solutions and numerical solutions obtained by some other methods in the literature.

In addition, the numerical solutions obtained by the method presented at time $t=1$ are better than the numerical solutions obtained by some other methods in the literature. L_2 and L_∞ error norms for the case $\delta=1$, $\gamma=0.01$ and different values of β are given in Table 4. L_2 and L_∞ error norms for the case $\delta=1, \beta=1$ and different values of γ are given in Table 5. Table 6 presents L_2 and L_∞ error norms for the case $\beta=1$, $\gamma=0.001$ and different values of δ. As it can be seen from the tables, the L_2 and L_∞ error norms acquired by the CN-LFDM are quite small in all cases.

Table 4. L_2 and L_∞ error norms for the case δ=1 and γ=0.01.

t	L_2			L_∞		
	β = 1	β = 10	β = 100	β = 1	β = 10	β = 100
0.01	2.349602E-6	9.186968E-6	5.433684E-5	8.393999E-6	2.690926E-5	8.870622E-5
0.1	6.226516E-6	3.792347E-5	3.120904E-4	8.864600E-6	4.742339E-5	4.068564E-4
1	9.105172E-6	5.884348E-5	4.180346E-4	1.142221E-5	7.678065E-5	5.646421E-4
10	9.082588E-6	4.666173E-5	1.141340E-7	1.139386E-5	6.091816E-5	1.550568E-7

Table 5. L_2 and L_∞ error norms for the case $\delta = 1$ and $\beta = 1$.

t	L_2			L_∞		
	$\gamma = 0.01$	$\gamma = 0.001$	$\gamma = 0.0001$	$\gamma = 0.01$	$\gamma = 0.001$	$\gamma = 0.0001$
0.01	2.349602E-6	2.350512E-8	2.350589E-10	8.393999E-6	8.394278E-8	1.183651E-10
0.1	6.226516E-6	6.237805E-8	6.238358E-10	8.864600E-6	8.867026E-8	1.250322E-10
1	9.105172E-6	9.124808E-8	9.126230E-10	1.142221E-5	1.144705E-7	1.614362E-10
10	9.082588E-6	9.125080E-8	9.126260E-10	1.139386E-5	1..447403E-7	1.614370E-10

Table 6. L_2 and L_∞ error norms for the case $\beta = 1$ and $\gamma = 0.001$.

t	L_2			L_∞		
	$\delta = 1$	$\delta = 2$	$\delta = 4$	$\delta = 1$	$\delta = 2$	$\delta = 4$
0.01	2.350512E-8	8.751590E-7	4.684094E-6	8.394278E-8	3.069117E-6	1.593123E-5
0.1	6.237805E-8	2.485496E-6	1.462652E-5	8.867026E-8	3.329747E-6	1.867144E-5
1	9.124808E-8	3.684982E-6	2.201687E-5	1.144705E-7	4.637855E-6	2.793710E-5
10	9.125080E-8	3.668338E-6	2.171734E-5	1..447403E-7	4.616923E-6	2.755716E-5

4. CONCLUSION

In this study, Crank-Nicolson logarithmic finite difference method is used to obtain the numerical solutions of the generalized Huxley equation. The comparison of the numerical solutions obtained by presented method with the exact solutions and the numerical solutions obtained by previous studies in the literature is given by tables. It is clear from the tables that the numerical solutions obtained by CN-LFDM are in good agreement with the exact solutions and better than numerical solutions obtained by some other methods in literature. The presented method is an efficient technique for finding numerical solutions for various kinds of nonlinear problems.

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