# THE EQUIVALENCE BETWEEN SOME DIFFERENTIAL EQUATIONS

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***Abstract:*** *The classical method for solving Riccati equations uses a change which leads to a first order linear equation. We give here a new method of the variation of constants.*

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## 1. Introduction

One method for solving linear equations of the form

$$y' = b(x)y + c(x), \text{ with } b, c : I \subseteq \mathbf{R} \to \mathbf{R} \qquad (1.1)$$

is the Lagrange method of the variation of the constants. First, the corresponding equation with separable variables

$$y' = b(x)y$$

has solutions of the form $y = ce^{B(x)}$, where $B \in \int b$ and $c \in \mathbf{R}$. Hence the general solution of the linear equation (1.1) is $y = c(x)e^{B(x)}$ where $c(x)$ is deduced by substituting in (1.1).

Using this idea, we will give a simpler method for solving, Riccati equations of the form

$$y' = a(x)y^2 + b(x)y + c(x), \text{ with } a, b, c : I \subseteq \mathbf{R} \to \mathbf{R} \text{ continuous} \quad (1.2)$$

than the classical method which we remember here. If $y_0$ is a particular solution of the equation (1.2), then

$$u = y - y_0 \qquad (1.3)$$

satisfies the Bernoulli equation

$$u' = (2a(x)y_0 + b(x))u + a(x)u^2, \qquad \text{with } r = 2$$

On the natural way, denote further

$$u = z^{-1} \qquad (1.4)$$

to obtain the linear equation

$$z' = -(2a(x)y_0 + b(x))z - a(x) \qquad (1.5)$$

Now by substitute z in, we derive

$$\frac{1}{z} = y - y_0, \text{ so } \quad y = y_0 + \frac{1}{z}$$

In fact, this is the classical method for solving the Riccati equation (1.2). If $y_0$ is a particular solution, then the substitution $y = y_0 + \frac{1}{z}$ leads to a linear equation of the first order.

## 2. The Result

Next we give a method which transforms directly the Riccati equation into a equation with separable variables. As we mentioned, the solution of that linear equation(1.5) is of the form

$$z = c(x)e^{-v(x)}, \text{ where } v \in \int (2a(x)y_0 + b(x))dx.$$

Now by substitute z in (1.3)-(1.4), we derive

$$u(x)e^{v(x)} = y - y_0, \ or \ \ y = y_0 + u(x)e^{v(x)},$$

with the renotation $u(x) = \frac{1}{c(x)}$ We can state

**Theorem 2.1** *Assume that the Riccati equation (1.2) has a particular solution $y_0$ Then the general solution of the Riccati equation(1.2) is of the form*

$$y = y_0 + u(x)e^{v(x)},$$

*where $v(x) \in \int (2a(x)y_0 + b(x))dx$ and u(x) can be deduced by substituting in (1.2). More precisely, u(x) is the general solution of the equation with separable variables*

$$u'(x) = a(x)e^{v(x)}u^2(x) \qquad (2.1)$$

It follows that

$$\frac{1}{u(x)} = -\int a(x)e^{v(x)}dx,$$

so we can stat the following result which is the direct formula of the general solution of the Riccati equation(1.2):

**Theorem 2.2** *Assume that the Riccati equation (1.2) has a particular solution $y_0$. Then the general solution of the Riccati equation (1.2) is of the form*

$$y = y_0 - \frac{e^{v(x)}}{w(x)}, \qquad (2.2)$$

*where v(x) is a primitive of the function $2a(x)y_0 + b(x)$ and w(x) is any primitive of the function $a(x)e^{v(x)}$ .*

Now we can remark that the solving of a Riccati equation is reduced to a direct substitution in the formula (2.2), so we do not need calculations for each example of Riccati equations; we purely can replace in the formula (2.2) in the equivalent form

$$y = y_0 - \frac{e^{\int_{x_0}^{x}(2a(s)y_0(s)+b(s))ds}}{c + \int_{x_0}^{x} a(s)e^{\int_{x_0}^{x}(2a(t)y_0(t)+b(t)dt} ds}, \ c \in R \qquad (2.3)$$

## 3. An example

Let us consider the equation

$$y' = \alpha y^2 + \frac{\beta}{x}y + \frac{\gamma}{x^2}, \qquad x > 0 \qquad (3.1)$$

where $\alpha, \beta, \gamma \in R$ are such that

$$(\beta + 1)^2 = 4\alpha\gamma \neq 0$$

That type of equation has the particular solution $y_0 = \frac{m}{x}$, where m ∈ R is the unique solution of the quadric equation

$$\alpha m^2 + (\beta + 1)m + \gamma = 0, \quad \text{so} \qquad m = -\frac{\beta + 1}{2\alpha}$$

Note that

$$2m\alpha + \beta = -1$$

Using the classical way with the notation $y = \frac{m}{x} + \frac{1}{z}$, the unknown function z satisfies the linear equation

$$z' = \frac{1}{x} \cdot z - \alpha$$

After two steps, using the Lagrange's method, we obtain $z = c - \alpha \ln x$. After replace also $m = -\frac{\beta + 1}{2\alpha}$, the general solution of the equation (3.1) is

$$y = -\frac{\beta + 1}{2\alpha x} + \frac{1}{c - \alpha \ln x}, \; c \in R \qquad (3.2)$$

Now, by using the formula (2.3) we obtain the solution directly, with $x_0=1$

$$y = \frac{m}{x} - \frac{e^{\int_1^x (\frac{2m}{s} + \frac{\beta}{s}) ds}}{c + \int_1^x \alpha e^{\int_1^s (\frac{2m}{t} + \frac{\beta}{t}) dt} ds} = \frac{m}{x} - \frac{1}{x(c + \alpha \ln x)},$$

which is the family (3.2)

**References**

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