# NEW BOUNDS FOR A CONVERGENCE BY DETEMPLE

CRISTINEL MORTICI

*Valahia University of Targoviste, Faculty of Science and Arts, 130082, Targoviste, Romania*

***Abstract.*** *The aim of this paper is to improve results of DeTemple [A quicker convergence to Euler's constant Amer. Math. Monthly 100 (1993) 468—470] about the defining sequence of the Euler-Mascheroni constant.*

***Keywords:*** *Euler-Mascheroni constant, inequalities, approximations.*

***2000 Mathematics subject classification:*** *11Y60, 40A05.*

## 1. INTRODUCTION

It is of general knowledge that the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

is convergent to a limit denoted $\gamma = 0.577215...$ now known as Euler-Mascheroni constant.

Many authors have given bounds for $D_n - \gamma$, for example the following increasingly better

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)} \quad (n \geq 2) \quad [13]$$

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (n \geq 1) \quad [12,\ 16]$$

$$\frac{1-\gamma}{n} \leq D_n - \gamma < \frac{1}{2n} \quad (n \geq 1) \quad [2]$$

$$\frac{1}{2n+\frac{2}{5}} < D_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad (n \geq 1) \quad [14,\ 15]$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} < D_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad (n \geq 1) \quad [1,\ 14,\ 15].$$

Quicker approximations to the Euler-Mascheroni constant were established in the recent past, see, e.g., [1-16]. We mention here the sequence

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right) \; [3]$$

satisfying for every integer $n \geq 1$,

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}$$

## 2. RESULTS

We discuss now the faster convergence $R_n$ and we prove the following result which is much better than the estimate from the previous section of this paper.

**Theorem 2.1.** For every integer $n \geq 1$ it holds

$$\frac{1}{24\left(n + \frac{1}{2} + \frac{7}{80n}\right)^2} < R_n - \gamma < \frac{1}{24\left(n + \frac{1}{2}\right)^2}$$

*Proof:* The sequences

$$a_n = R_n - \gamma - \frac{1}{24\left(n + \frac{1}{2}\right)^2},$$

$$b_n = R_n - \gamma - \frac{1}{24\left(n + \frac{1}{2} + \frac{7}{80n}\right)^2}$$

converge to zero and we prove that $a_n$ is strictly increasing, while $b_n$ is strictly decreasing.

In this sense, we have $a_{n+1} - a_n = f(n)$, $b_{n+1} - b_n = g(n)$, where

$$f(n) = \frac{1}{n+1} - \ln\left(n + \frac{3}{2}\right) + \ln\left(n + \frac{1}{2}\right) - \frac{1}{24\left(n + \frac{3}{2}\right)^2} + \frac{1}{24\left(n + \frac{1}{2}\right)^2}$$

and

$$g(n) = \frac{1}{n+1} - \ln\left(n + \frac{3}{2}\right) + \ln\left(n + \frac{1}{2}\right) - \frac{1}{24\left(n + \frac{3}{2} + \frac{7}{80(n+1)}\right)^2} + \frac{1}{24\left(n + \frac{1}{2} + \frac{7}{80n}\right)^2}.$$

We have

$$f'(n) = -\frac{28n^2+56n+25}{3(n+1)^2(2n+1)^3(2n+3)^3} < 0$$

and

$$g'(n) = \frac{P(n)}{3(2n+1)(2n+3)(n+1)^2(80n^2+40n+7)^3(80n^2+200n+127)^3} > 0$$

where

$$P(n) = 2227973307 + 117922026320n + 1150657635680n^2 + 5047861062400n^3 +$$

$$+12302961177600n^4 + 1804973056000n^5 + 1632099225600n^6 +$$

$$+8903843840000n^7 + 2686689280000n^8 + 344064000000n^9.$$

Now $f$ is strictly decreasing, $g$ is strictly increasing on $[1, \infty)$, with $f(\infty) = (\infty) = 0$, so $f(x)>0$ and $g(x)<0$ for every $x \in [1,\infty)$.

Finally, $a_n$ is strictly increasing, and $b_n$ is strictly decreasing and the conclusion follows.

## REFERENCES

[1] Alzer, H., *Abh. Math. Sem. Univ. Hamburg*, **68**, 363-372, 1998.
[2] Anderson, G. D., Barnard, R. W., Richards, K. C., Vamanamurthy, M. K., Vuorinen, M., *Trans. Amer. Math. Soc.*, **347**, 1713-1723, 1995.
[3] DeTemple, D. W., *Amer. Math. Monthly*, **100**, 468-470, 1993.
[4] Mortici, C., *Appl. Math. Lett.*, **23**, 97-100, 2010.
[5] Mortici, C., *Math. Comput. Modelling*, **51**(9-10), 1154-1159, 2010.
[6] Mortici, C., *Comput. Math. Appl.*, in press, 2010.
[7] Mortici, C., *Comput. Math. Appl.*, **59**(8), 2610-2614, 2010.
[8] Mortici, C., *Appl. Math. Comput.*, **215**(9), 3443-3448, 2010.
[9] Mortici, C., *Anal. Appl. (Singap.)*, 8(1), 99-107, 2010.
[10] Mortici, C., *Carpathian J. Math.*, 26(1), 2010.
[11] Negoi, T., *A zarter convergence to the constant of Euler*, Gazeta Matematică, seria A, 15, 111-113, 1997.
[12] Rippon, P. J., *Amer. Math. Monthly*, **93**, 476-478, 1986.

[13] Tims, S. R., Tyrrell, J.A., *Math. Gaz.*, **55**, 65-67, 1971.
[14] Tóth, L., *Amer. Math. Monthly*, **98**, 264, 1991.
[15] Tóth, L., *Amer. Math. Monthly*, **99**, 684-685, 1992.
[16] Young, R. M., *Math. Gaz.*, **75**, 187-190, 1991.

Manuscript received: 14.07.2010
Accepted paper: 26.08.2010
Published online: 04.10.2010