ON A CONVERGENCE BY DETEMPLE

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Abstract. *The aim of this paper is to discuss the sequence defined by DeTemple in [6].*

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INTRODUCTION

The Euler-Mascheroni constant γ=0.577215664… is defined as the limit of:

$$D_n = H_n - \ln n \tag{1.1}$$

where H_n denotes the *n*th harmonic number, defined for $n \in \mathbb{N}$ by $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Several bounds for $D_n - \gamma$ have been given in the literature [2, 3, 5, 7-9]. The convergence of the sequence Dn to γ is very slow. Some quicker approximations to the Euler-Mascheroni constant were established and we mention here the following sequence introduced by DeTemple [6]: $R_n = H_n - \ln\left(n + \frac{1}{2}\right)$,

for which

$$\frac{1}{24(n+1)^2} < Rn - \gamma < \frac{1}{24n^2} \tag{1.2}$$

First we use the asymptotic series of the digamma function ψ in terms of Bernoulli numbers

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}} = \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + ... \tag{1.3}$$

(see, e.g., [1]) to deduce the standard asymptotic series of DeTemple's sequence

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right) \sim \gamma + \frac{1}{24n^2} - \frac{1}{24n^3} + \frac{23}{960n^4} - \frac{1}{160n^5} - \frac{11}{8064n^6} + ... \tag{1.4}$$

Recently, Chen [4] obtained the following sharp form of the inequality (1.2):

$$\frac{1}{24(n+a)^2} \le Rn - \gamma < \frac{1}{24(n+b)^2}, \quad n \ge 1 \tag{1.5}$$

with the best possible constants $a = \frac{1}{\sqrt{24\left[-\gamma + 1 - \ln\frac{3}{2}\right]}} - 1 = 0.55106...$ and $b = \frac{1}{2}$

We propose the following series in negative powers of $n - 1/2$

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right) \sim \gamma + \frac{1}{24\left(n + \frac{1}{2}\right)^2} - \frac{\frac{7}{960}}{\left(n + \frac{1}{2}\right)^4} + \frac{\frac{31}{8064}}{\left(n + \frac{1}{2}\right)^6} - \frac{\frac{127}{30720}}{\left(n + \frac{1}{2}\right)^8} + ... \tag{1.6}$$

and we expect to be much faster than (1.4). Moreover we find the following:

Theorem 1. For every $n \in \mathbb{N}$, we have

$$\frac{1}{24\left(n+\frac{1}{2}\right)^2}-\frac{\frac{7}{960}}{\left(n+\frac{1}{2}\right)^4}+\frac{\frac{31}{8064}}{\left(n+\frac{1}{2}\right)^6}-\frac{\frac{127}{30720}}{\left(n+\frac{1}{2}\right)^8}<\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)<\frac{1}{24\left(n+\frac{1}{2}\right)^2}-\frac{\frac{7}{960}}{\left(n+\frac{1}{2}\right)^4}+\frac{\frac{31}{8064}}{\left(n+\frac{1}{2}\right)^6}.$$

The Results. We give the following

Theorem 2. The following standard asymptotic expansion holds as $n \to \infty$

$$\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)-\gamma \sim \frac{1}{2n}+\sum_{k=2}^{\infty}\left(\frac{(-1)^k}{2^k}-B_k\right)\frac{1}{kn^k}=\frac{1}{24n^2}-\frac{1}{24n^3}+\frac{1}{960n^4}-\ldots$$

Proof. We have $\psi(x+1)=H_n-\gamma$ and using (1.3), we get

$$\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)=\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)-\ln\left(1+\frac{1}{2n}\right)\sim\gamma+\frac{1}{2n}-\sum_{k=1}^{\infty}\frac{B_{2k}}{2kx^{2k}}-\ln\left(1+\frac{1}{2n}\right)$$

$$\sim\gamma+\frac{1}{2n}-\sum_{k=2}^{\infty}\frac{B_k}{kx^k}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k2^kn^k}\sim\gamma+\frac{1}{2n}+\sum_{k=2}^{\infty}\left(\frac{(-1)^k}{2^k}-B_k\right)\frac{1}{kn^k}.$$

Theorem 1 can be proved by defining the sequences

$$a_n=\frac{1}{24\left(n+\frac{1}{2}\right)^2}-\frac{\frac{7}{960}}{\left(n+\frac{1}{2}\right)^4}+\frac{\frac{31}{8064}}{\left(n+\frac{1}{2}\right)^6}-\frac{\frac{127}{30720}}{\left(n+\frac{1}{2}\right)^8}-\left(\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)\right)$$

$$b_n=\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)-\left(\frac{1}{24\left(n+\frac{1}{2}\right)^2}-\frac{\frac{7}{960}}{\left(n+\frac{1}{2}\right)^4}+\frac{\frac{31}{8064}}{\left(n+\frac{1}{2}\right)^6}\right)$$

and showing that they are strictly increasing to zero. As consequence, $a_n<0$ and $b_n<0$.

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