LOCALLY AFFINE FUNCTIONS

CORNELIU UDREA

University of Pitesti, Faculty of Mathematics and Computer Science, 110040, Pitesti, Romania

Abstract. *The Monge-Ampère equation and related boundary value problem are a source example for the non-linear potential theory, and for convex analysis ([1-5, 8]). Moreover the concept of the locally convex function is introduced in the study of the solutions of this equations ([1, 3, 5, 9]). In this paper an attempt is made to define a generalization of this type of functions (i.e. it is defined the quasi-locally convex functions), some properties of the quasi-locally convex functions are studied, and an important example of quasi-locally affine function is presented.*

Keywords: *locally affine, epigraph, function.*

1. INTRODUCTION

In this section we recall some notions, notations, and results of the convex analysis from [5], [6], [8] or [9], which will be used in the next sections. Here S is a nonempty subset of $\mathbb{R}^k$, f is a numerical function on S and F is a non-empty subset of $\mathbb{R}^k \times \mathbb{R}$.

Definition 1.1

(i). The set $\{(x,\xi) \in S \times \mathbb{R} : f(x) \le \xi\}$ is called the *epigraph* of f, and is denoted by $\operatorname{epi} f$. (Obviously $\operatorname{epi} f \subset \mathbb{R}^k \times \mathbb{R}$).

(ii). The function f is called *convex function* (respectively *lower semicontinuous function*) if $\operatorname{epi} f$ is a a convex (respectively closed) subset of $\mathbb{R}^{k+1}$ (respectively of $S \times \mathbb{R}$).

(iii). We define $\varphi_F : \operatorname{pr}_{\mathbb{R}^k} F \to [-\infty, \infty)$ by the following formula:

$$\varphi_F(x) := \inf\{\xi \in \mathbb{R} : (x,\xi) \in F\}, \text{ for all } x \in \operatorname{pr}_{\mathbb{R}^k} F.$$

Remark 1.2

(i). A real function g on S is convex (respectively lower semi-continuous) if and only if S is a convex subset of $\mathbb{R}^k$, and for all $x, y \in S$ and $t \in (0,1)$, $f((1-t)x + ty) \le (1-t) f(x) + tf(y)$ (respectively for all $x \in S$, $f(x) = \liminf_{S \ni y \to x} f(y)$) ([6]).

(ii). If F is a convex subset of $\mathbb{R}^k \times \mathbb{R}$, then $C := \operatorname{pr}_{\mathbb{R}^k} F$ is convex and $\varphi_F : C \to \overline{\mathbb{R}}$ is convex.

(iii). Suppose that for all $x \in \operatorname{pr}_{\mathbb{R}^k} F$ there exists $\alpha_x \in \mathbb{R}$ such that $\{\mu \in \mathbb{R} : (x,\mu) \in F\} \subset [\alpha_x, \infty)$. Then φ_F is a real function.

(iv). If F is a closed subset of $\mathbb{R}^k \times \mathbb{R}$, and $F \subset \mathbb{R}^k \times [\alpha_0, \infty)$, where $\alpha_0 \in \mathbb{R}$, then the function φ_F is a real, lower semicontinuous function.

Definition 1.3

(i). Denote by $\mathrm{co}f$ the function $\varphi_{\mathrm{co}(\mathrm{epi}f)}$ i.e. the greatest numerical con-vex function g on $\mathbb{R}^k$ such that $g|_S \le f$. The function $\mathrm{co}f$ is called the convex hull of f.

(ii). Similar $\varphi_{\overline{\mathrm{epi}f}} := \mathrm{sci}f$, and $\mathrm{sci}f$ is the greatest numerical lower semicontinuous function g on $\mathbb{R}^k$ such that $g|_S \le f$. The map $\mathrm{sci}f$ is called *the lower semicontinuous hull* of f.

(iii). We define $\mathrm{cl}f := \mathrm{sci}(\mathrm{co}f)$. This function is called *the lower semicontinuous convex hull off*. If f is convex, then $\mathrm{cl}f$ is said to be *the closed hull of* f.

Remark 1.4

(i). From now on we shall use the following notations.

$J := \{1,2,\dots,k+2\}$, and $\Delta := \{t \in [0,1]^{k+2} : t = (t_1,t_2,\dots,t_{k+2})$, and $\sum_{j=1}^{k+2} t_j = 1\}$.

Obviously Δ is a convex compact subset of $\mathbb{R}^{k+2}$.

(ii). In view of the Carathéodory's theorem ([7]) we have that

$$\mathrm{co}(\mathrm{epi}f) = \left\{ \sum_{j\in J} t_j (x_j, \xi_j) : t \in \Delta \text{ and } \{(x_j,\xi_j)\}_{j\in J} \subset \mathrm{epi}f \right\}$$

and for all $x \in \mathrm{pr}_{\mathbb{R}^k}(\mathrm{co}(\mathrm{epi}f))$

$$(\mathrm{co}f)(x) = \inf\left\{ \sum_{j\in J} t_j \xi_j : t \in \Delta, \{(x_j,\xi_j)\}_{j\in J} \subset \mathrm{epi}f \text{ and } x = \sum_{j\in J} t_j x_j \right\}.$$

(iii). According to the previous remark if $a \in S$ is an extremal point of the set $\mathrm{co}S$, then $(\mathrm{co}f)(a) = f(a)$.

Proposition 1.5

Let F be compact, and convex. The function φ_F is a real convex, and lower semicontinuous function on the compact convex set $C = \mathrm{pr}_{\mathbb{R}^k} F$ ([8, 9]).

Proof. Obviously φ_F is a real convex function on C, and since

$$\begin{aligned}\mathrm{epi}\varphi_F &= \{(x, \varphi_F(x) + y) : x \in C \text{ and } y \in [0,\infty)\} \\ &= \{(x, \xi + y) : (x,\xi) \in F, \text{and } y \in [0,\infty)\} = F + \{0_k\} \times [0,\infty)\end{aligned}$$

the set $\mathrm{epi}\varphi_F$ is closed and φ_F is lower semicontinuous on C. ■

Example 1.6

We consider C a non-empty, compact, and convex subset of $\mathbb{R}^k$, f a real continuous function on C, $G_f := \{(x, f(x)) : x \in C\}$ (the graph of f) and $K_f := \mathrm{co}G_f$. Obviously K_f is a convex, compact subset of $\mathbb{R}^k \times \mathbb{R}$. We define $\varphi_f := \varphi_{K_f}$ ([8, 9]).

Theorem 1.7

The function φ_f *(defined in Example 1.6) has the following properties* ([8, 9]).

(i). *It is a real, convex, and lower semicontinuous function on* C.

(ii). *For all* $a \in C$, *there exist* $t \in \Delta$ *and* $\{x_j\}_{j\in J} \subset C$ *such that* $a = \sum_{j\in J} t_j x_j$ *and* $\varphi_f(a) = \sum_{j\in J} t_j f(x_j)$.

(iii).*We have the relations* $\varphi_f = \mathrm{co}f = \mathrm{cl}f$, *and* $\varphi_f|_{exC} = f|_{exC}$ *(where* exC *denotes the extremal points of* C*).*

Proof.

(i). We use Proposition 1.5.

(ii). Let us consider $\mathrm{pr}_{\mathbb{R}} : K_f \to \mathbb{R}$, $\mathrm{pr}_{\mathbb{R}}(x,\xi) = \xi$ for all $(x,\xi) \in K_f$. According to the definition for all $a \in C$, $\varphi_f(a) = \inf\{\mathrm{pr}_{\mathbb{R}}(a,\xi) : \xi \in \mathbb{R} \text{ such that } (a,\xi) \in K_f\}$.

Since K_f is a compact set, and $\mathrm{pr}_{\mathbb{R}}$ is a continuous function there exists $\xi \in \mathbb{R}$ with the properties $(a,\xi) \in K_f$, and $\varphi_f(a) = \xi$.

Therefore there exist $t = (t_1, t_2, \ldots, t_{k+2}) \in \Delta$, and $\{x_j\}_{j\in J} \subset C$ such that

$$a = \sum_{j\in J} t_j x_j,\ \xi = \sum_{j\in J} t_j f(x_j) \Rightarrow \varphi_f(a) = \xi = \sum_{j\in J} t_j f(x_j).$$

(iii). It is a consequence of Remark 1.4.(iii). ■

Remark 1.8

For all C non-empty, convex subset of $\mathbb{R}^k$ we define the relative interior of the set C as the interior which results when C is regarded as a subset of its affine hull. This set is denoted by $\mathrm{int}_r C$. The set $\overline{C} \setminus \mathrm{int}_r C$ is called the relative boundary of C and it is denoted by $\partial_r C$.

2. LOCALLY CONVEX FUNCTIONS

We generalize the notion of the locally convex function (defined in [1] and [3]) and we present some properties of these functions.

In this section F is a non-empty subset of $\mathbb{R}^k$, and f is a real function on a subset A of $\mathbb{R}^k$ where $A \supset F$.

Definition 2.1

(i). The function f is called *quasi-locally convex function* on F if for all $x \in F$ there exists V a neighbourhood of x such that $V \cap F$ is a convex set, and the function $f|_{V\cap F}$ is convex.

(ii). If the functions f and $-f$ are quasi-locally convex functions on F, then f is called *quasi-locally affine function on* F.

Remark 2.2

(i). The following assertions are equivalent. (a). The function f is quasi-locally convex on F. (b). There exists $\mathcal{R}$ an open covering of F such that $R \cap F$ is a convex set, and $f|_{R\cap F}$ is a convex function, for all $R \in \mathcal{R}$.

(ii). If F is an open subset of $\mathbb{R}^k$, then the following assertions are equivalent. (a). The function f is quasi-locally convex. (b). There exists $\mathcal{R}$ an open covering of F such that R is convex subset of F, and $f|_R$ is a convex function, for all $R \in \mathcal{R}$ (i.e. according to [1] and [3] f is *locally convex function on* F).

(iii). Since every quasi-locally affine function on F is locally continuous on F we have that all quasi-locally affine function on F is continuous on F.

(iv). Let s be a real, convex function on the line segment $[a,b]$ (respectively (a,b)). According to [4], [6], and [9] we have the following assertions.

(a). If c is a point from (a,b) such that $s(c) = \max s([a,b])$ (respectively $s(c) = \max s((a,b))$) then $s(x) = s(c)$ for all $x \in [a,b]$ (respectively for all $x \in (a,b)$).
(b). The function s is upper semicontinuous on $[a,b]$.

Lemma 2.3

If f is a real quasi-locally convex function on the set F, and a,b are two points of F such that $[a,b] \subset F$, then f is an upper semicontinuous function on $[a,b]$.

Proof.

In view of the general convex analysis f is continuous at all point $c \in (a,b) = \operatorname{int}_r([a,b])$. If V is an open convex subset of $\mathbb{R}^k$ such that $V \cap F$ is a convex set, $f|_{V\cap F}$ is a convex function, and either $b \in V$, and $a \notin V$ or $b \notin V$, and $a \in V$, then either $V \cap F \cap [a,b] = V \cap [a,b] = (b_1, b]$, or $V \cap F \cap [a,b] = V \cap [a,b] = [a, a_1)$.
According to the previous remark $f|_{(b_1,b]}$ (respectively $f|_{[a,a_1)}$) is upper semicontinuous at the point b (respectively at the point a). ■

Theorem 2.4

Given f a real quasi-locally convex function on the set F, and C a nonempty convex subset of F it follows that the function $f|_C$ is convex.

Proof. (Similar to [3]). We want to prove that $f((1-t)a + tb) \le (1-t)f(a) + tf(b)$ for all $a,b \in C$, and $t \in (0,1)$. So that we consider a,b different points from C.

Suppose that $f(a) = f(b)$, and define $M := \sup f([a,b])$, and $S := \{c \in [a,b] : f(c) = M\} = \{f = M\} \cap [a,b]$. It is obvious that $M \in \mathbb{R}$ and S is non-empty. Let T be the set $\{t \in [0,1] : (1-t)a + tb \in S\}$, $t_0 := \sup T$, and $\varphi : [0,1] \to \mathbb{R}$, $\varphi(t) := f((1-t)a + tb)$.

Assume that $t_0 \in (0,1)$, hence $c_0 := (1-t_0)a + t_0 b \in (a,b)$, $t_0 \in T$ and $c_0 \in S$. Take V an open convex subset of $\mathbb{R}^k$ such that $c_0 \in V$, $a,b \notin V$, $V \cap F$ is convex, and $f|_{V\cap F}$ is a convex function. Hence $V \cap F \cap [a,b] = V \cap [a,b] = (a_1, b_1)$ where $c_0 \in (a_1, b_1) \subset (a,b)$, and $b_1 \notin S$. Therefore $f(a_1) \le M$, and $f(b_1) < M$.

If $c_0 = (1-t)a_1 + tb_1$, $t \in (0,1)$, then

$$M = f(c_0) \le (1-t)f(a_1) + tf(b_1) < M \Rightarrow t_0 = 1, c_0 = b \in S.$$

Therefore for all $t \in (0,1)$

$$f((1-t)a + tb) \le M = (1-t)M + tM = (1-t)f(a) + tf(b).$$

Suppose now that $f(a) \ne f(b)$, and let u be a linear functional on $\mathbb{R}^k$ such that $f(a) - u(a) = f(b) - u(b)$. Obviously $f - u$ is a quasi-locally convex function, $(f-u)(a) = (f-u)(b)$, and in view of the previous step it follows that

$$(f-u)((1-t)a + tb) \le (1-t)(f-u)(a) + t(f-u)(b)$$

for all $t \in (0,1)$. Since u is a linear functional we have that

$$f((1-t)a + tb) \le (1-t)f(a) + tf(b), \forall t \in (0,1). \blacksquare$$

Corollary 2.5

Let F be such that for all $x \in F$ there exists V a neighborhood of x such that $V \cap F$ is convex, and let f be a real function on F. The following assertions are equivalent.

(i). *The function f is quasi-locally convex on F.*

(ii). *For all C convex subset of F, $f|_C$ is a convex function.*

Proof. It is obvious. ■

Corollary 2.6

If F is a convex set, and f is a quasi-locally convex function on F, then f is a convex function on F.

Proof. It is obvious. ■

Proposition 2.7

Take $\{C_i\}_{i\in\{1,2\}}$ convex subsets of $\mathbb{R}^k$ such that $C_1 \subset C_2$, and C_1 is a closed set. Let u_1 (respectively u_2) be a real convex function on C_1 (respectively C_2) with the following properties $u_2|_{C_1} \le u_1$, and $u_2|_{\partial_r C_1} = u_1|_{\partial_r C_1}$.

Then the function $u : C_2 \to \mathbb{R}$, $u = \begin{cases} u_1 & \text{on } C_1 \\ u_2 & \text{on } C_2 \setminus C_1 \end{cases}$ is convex on C_2.

Proof. We want to prove that $u|_{[a,b]}$ is a convex function for all $a, b \in C_2$. If $[a,b] \subset C_1 \setminus C_2$, then $u|_{[a,b]} = u_2|_{[a,b]}$, and $u|_{[a,b]}$ is a convex function. Otherwise $[a,b] \cap C_1 \ne \emptyset$. Take $\varepsilon \in (0,\infty)$, define $u_\varepsilon : C_2 \to \mathbb{R}$, $u_\varepsilon = \begin{cases} \sup(u_1 - \varepsilon, u_2) & \text{on } C_1 \\ u_2 & \text{on } C_2 \setminus C_1 \end{cases}$ and remark that by the hypothesis u_ε is quasi-locally convex function on $[a,b]$. Hence u_ε is a convex function on $[a,b]$. Since $u|_{[a,b]} = \lim_{\varepsilon \downarrow 0} u_\varepsilon|_{[a,b]}$ we have that $u|_{[a,b]}$ is a convex function.■

3. LOCALLY AFFINE FUNCTIONS

We prove some results about continuity of convex functions, and we present an example of a locally affine function.

In this section U is a non-empty, open, bounded, and strictly convex subset of $\mathbb{R}^k$ (the strictly convex property means that for all $a,b \in \partial U$, $a \neq b$, we have $(a,b) \subset U$).

Lemma 3.1

For all $a \in \partial U$, and $r \in (0,\infty)$ there exists $d_r \in (0,\infty)$ such that $\overline{B}\left(\frac{1}{2}(a+x), d_r\right) \subset U$ for all $x \in \overline{U} \setminus B(a,r)$ ([1]).

Proof. Obviously $\frac{1}{2}(a+x) \in (a,x) \subset U$ for all $x \in \overline{U} \setminus \{a\}$, and $K = \frac{1}{2}(a + (\overline{U} \setminus B(a,r)))$ is a compact set which is contained in U, hence $d := dist(K, \partial U)$ is strictly positive. Define $d_r := \frac{d}{2}$ and remark that for all $x \in \overline{U} \setminus B(a,r)$, $dist\left(\frac{1}{2}(a+x), \partial U\right) \geq d > d_r$, therefore $\overline{B}\left(\frac{1}{2}(a+x), d_r\right) \subset B\left(\frac{1}{2}(a+x, d)\right) \subset U$.

Lemma 3.2

For all $a \in \partial U$, and $r \in (0,\infty)$ there exists u an affine function on $\mathbb{R}^k$ such that $u(a) = 1$, $u \leq 1$ on $\overline{U}$, and $u < 0$ on $\overline{U} \setminus B(a,r)$ ([1]).

Proof. According to the first separation theorem we have v an affine function on $\mathbb{R}^k$ with the following properties $v(a) = 0$, and $v|_U < 0$. In view of the previous lemma let d_r be a positive number such that $\overline{B}\left(\frac{1}{2}(a+x), d_r\right) \subset U$ for all $x \in \overline{U} \setminus B(a,r)$.

We fix a point $b \in B(a, d_r) \setminus U$ such that $\alpha := v(b) > 0$, and we remark that for all $x \in \overline{U} \setminus B(a,r)$, $\frac{1}{2}(x+b) \in B\left(\frac{1}{2}(x+b), d_r\right) \subset U$, $v(\frac{1}{2}(x+b)) < 0$, and $v(x) < -v(b) = -\alpha$.

If we define $u := \frac{1}{\alpha}(v + \alpha)$, then u is affine function, $u(a) = 1$, $u(x) = \frac{1}{\alpha}(v(x) + v(b)) = \frac{1}{\alpha} v(x+b) < 0$, for all $x \in \overline{U} \setminus B(a,r)$, and $u(x) = \frac{1}{\alpha}(v(x) + \alpha) \leq 1$ for all $x \in \overline{U}$. ■

Theorem 3.3

If f is a real, lower semicontinuous and convex function on $\overline{U}$ such that $f|_{\partial U}$ is a continuous function, then f is continuous on $\overline{U}$.

Proof. Let a be a boundary point of U, $\varepsilon \in (0,\infty)$, and $r_\varepsilon \in (0,\infty)$ such that $|f(x)-f(a)| \le \varepsilon$ for all $x \in \overline{B}(a,r_\varepsilon) \cap \partial U$. Take u the affine function of Lemma 3.2 for r_ε, a, and U.

Then $\overline{U} \cap \{u>0\} \subset B(a,r_\varepsilon)$, and $W := \{u>0\} \cap B(a,r_\varepsilon)$ is an open convex neighbourhood of a. If $x \in W \cap U$ (hence $u(x) \subset (0,1)$), and d is a line contained in the hyperplane $\{u = u(x)\}$, and passing trough the point x, then $d \cap (\partial U) = \{x_1, x_2\}$ where $u(x_1) = u(x_2) = u(x) > 0$, $x_1, x_2 \in B(0,r_\varepsilon)$, and $x = (1-t)x_1 + x_2$ where $t \in (0,1)$.

Since f is a convex function we have

$$f(x) \le (1-t)f(x_1) + tf(x_2) \le (1-t)(f(a)+\varepsilon) + t(f(a)+\varepsilon) = f(a)+\varepsilon .$$

Therefore $f(x) \le f(a)+\varepsilon$ for all $x \in W \cap U$, and $\lim\limits_{U \ni x \to a} \sup f(x) \le f(a)+\varepsilon$. Hence $\lim\limits_{U \ni x \to a} \sup f(x) \le f(a)$ i.e. f is upper semicontinuous in every point $a \in \partial U$, so that f is continuous in any boundary point of U. ■

Corollary 3.4

(i). *Let f be a real, and continuous function on $\overline{U}$. The function* $\mathrm{co}f$ *is continuous on $\overline{U}$.*

(ii). *Take f_1, and f_2 real, convex, and continuous on $\overline{U}$. We have that $f_1 \wedge f_2 := \mathrm{co}(\inf(f_1, f_2))$ is continuous on $\overline{U}$.*

Proof. (i). According to the Theorem 1.7, $\mathrm{co}f = \mathrm{cl}f$ is a lower semicontinuous, and convex function on $\overline{U}$ such that $(\mathrm{co}f)|_{\partial U} = f|_{\partial U}$, hence $(\mathrm{co}f)|_{\partial U}$ is continuous. We apply Theorem 3.3 to the function $\mathrm{co}f$.

(ii). We use (i) for the function $\inf(f_1, f_2)$. ■

Theorem 3.5

Let D be a non-empty, open, bounded, convex subset of $\mathbb{R}^k$, f a real continuous function on $\overline{D}$, and $F := \{\mathrm{co}f < f\}$. Then $\mathrm{co}f$ *is a locally affine function on the set $F \cap D$.*

Proof. Denote by s the function $\mathrm{co}f : \overline{D} \to \mathbb{R}$ and remark that in view of Theorem 1.7 the function s is convex, lower semicontinuous on $\overline{D}$, and continuous on D. Moreover $F \cap D = \{x \in D : s(x) < f(x)\}$ is an open subset of D. If $a \in F \cap D$ then $s(a) < f(a)$ and there exists $r_0 \in (0,\infty)$ such that $B(a,r_0) \subset F \cap D$, and $s(x) < f(y)$ for all $x, y \in B(a,r_0)$.

Take b and c from $B(a,r_0)$ ($b \ne c$), and define for all $t \in [0,1]$ $h : [b,c] \to \mathbb{R}$, $h((1-t)b+tc) := (1-t)s(b) + ts(c)$. Obviously h is an affine function on $[b,c]$, and for all $t \in [0,1]$ we have that

$$h((1-t)b+tc) = (1-t)s(b)+ts(c) < (1-t)f((1-t)b+tc) + tf(1-t)b+tc) = f((1-t)b+tc)$$

i.e. $h(x) < f(x)$ for all $x \in [b,c]$. Moreover $h(b) = s(b)$, $h(c) = s(c)$, and $s|_{[b,c]} \le h$.

We apply Proposition 2.7 and we have that the function $u:\overline{D}\to\mathbb{R}$, $u=\begin{cases} h & \text{on } [b,c] \\ u_2 & \text{on } \overline{D}\setminus[b,c] \end{cases}$ is convex on $\overline{D}$, and $u\le f$ on $\overline{D}$. According to the definition $u\le \mathrm{co}f = s$, hence for all $t\in[0,1]$

$s((1-t)b+tc)\le(1-t)s(b)+ts(c) = h((1-t)b+tc)=u((1-t)b+tc)\le s((1-t)b+tc)$

i.e. $s((1-t)b+tc)=(1-t)s(b)+ts(c)$ for all $t\in[0,1]$, and $b,c\in B(a,r_0)$. ■

Corollary 3.6

If f is a real continuous function on $\overline{U}$, then $\mathrm{co}f$ is a locally affine function on the set $\{\mathrm{co}f < f\}$

Proof. By the Theorems 1.7 and 3.3 $\mathrm{co}f$ is a continuous, convex function on $\overline{U}$ such that $(\mathrm{co}f)|_{\partial U} = f$. Therefore $\{\mathrm{co}f < f\}\subset U$, and we apply the previous theorem. ■

REFERENCES

[1] Bertin, E.M.J., *Fonctions convexes et theorie du potentiel*, INDAG. MATH., Univ. Utrecht, Utrecht, 385-409, 1979.
[2] Bertin, E.M.J., *Axiomatic Convex Potential Theory*, Preprint 607, Univ. Utrecht, Utrecht, 1990.
[3] Bertin, E.M.J., *L'équation de Monge-Ampère dans un espace de Banach*, in Classical and Modern Potential Theory and Applications, 61-75, 1994.
[4] van Beusekan, P., *Monotone Dirichlet Forms and Resolvents*, Prep. no. 731, Univ. Utrecht, Utrecht, 1992.
[5] van Gool, F., *Topics in Non Linear Potential Theory*. Graduate thesis, Univ. Utrecht, Utrecht, 1992. Proc. of I.C.P.T.91, Kluwer Ac. Publ, 61-89, 1994.
[6] Rockafellar, T., *Convex Analysis*, Princeton University Press, Princeton, 1970.
[7] Rudin, W., *Functional Analysis.*, McGraw-Hill Book Company, 1973.
[8] Udrea, C., *Rev. Roum. de Math. Pures et Appl.*, **XL**, 691-712, 1995.
[9] Udrea, C., *The Perron-Wiener-Brelot Method and the Monge-Ampère Equation*, Proc. of XVI-th R. Nevanlinna Coll., Walter de Gruyters & Co, 279-287, 1996.

Manuscript received: 28.04.2010
Accepted paper: 10.09.2010
Published online: 04.10.2010