ORIGINAL PAPER

OPERATION CALCULUS OF GENERALIZED FRACTIONAL COMPLEX MELLIN TRANSFORM

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Abstract. *This paper generalizes the fractional complex Mellin Transform to the spaces of generalized functions and obtained many operation formulae for the transform. Some properties of the kernel are discussed and inverse of the transform is also obtained. The values of the transform for common functions are presented in tabular form. Lastly the applications of Fractional Mellin transform are given.*

Keywords: *Mellin transform, testing function space, scale transform, fractional Mellin transform.*

1. INTRODUCTION

Fourier analysis is one of the most frequently used tools in signal processing and is frequently used in many other scientific disciplines. In the mathematics literature a generalization of the Fourier transform known as the fractional Fourier transform was proposed by Namias [6] and Mc Bride developed it in [5]. As it is potentially useful for signal processing and many applications, the fractional Fourier transform has been independently reinvented by a number of researchers. L.B.Almeida [3] had introduced the fractional Fourier transform as an angular transform. He discussed the many operational formulae and presented the new results including the fractional Fourier transform. Also presents the simple relationships of the fractional Fourier transform with several time-frequency representations that supports the interpretation of it as a rotation operator. Akay [1] had discussed many Unitary and Hermitian operators on fractional Fourier transform.

O.Akay [2] had defined the one dimensional fractional Mellin transform with angular parameter ϕ of $f(x)$ denoted by $M^{\phi} f(x)$ which performs a linear operation given by the integral transform,

$$\left[M^{\varphi} f(x)\right](s) = \sqrt{1 - i\cot\varphi}\int_0^{\infty} f(x).e^{i\pi s^2 \cot\varphi}.e^{i\pi(\ln x)^2\cot\varphi - i2\pi(\ln x)s.\csc\varphi}\frac{dx}{\sqrt{x}}, \text{ for } \phi \neq n\pi$$

$$= e^{s/2} f(e^{s}), \qquad \text{for } \phi = 2n\pi$$

$$= e^{-s/2} f(e^{-s}), \qquad \text{for } \phi = (2n+1)\pi \tag{1.1}$$

The above fractional Mellin transform is the generalization of the conventional complex Mellin transform

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$$\text{i.e. } \int_0^\infty f(x).x^{ic-\frac{1}{2}}dx, \tag{1.2}$$

which is nothing but Mellin transform with complex argument $-ic+\frac{1}{2}$. For the parameter $\varphi=\frac{\pi}{2}$ the fractional Mellin transform reduces to the above complex Mellin transform. The fractional Mellin transform can be interpreted as a rotation on the scale – log modulation – phase plane. Sezbon [7] had given another version of fractional Mellin transform and used it for optical navigation.

This paper is organized as follows. In section II the fractional Mellin transform with parameter ϕ given in (1.1) is extended in the sense of generalized function as per [9]. In section III some useful properties of kernel along with inversion formula for the fractional Mellin transform was proved. Section IV lists some operation transform formulae for the generalized fractional Mellin transform. In section V we give fractional Mellin transform of some simple functions. Applications of this transform are discussed in section VI.

2. GENERALIZED FRACTIONAL MELLIN TRANSFORM

For defining fractional Mellin transform in the generalized sense, first we define,

2.1. THE TESTING FUNCTION SPACE E

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ or E if for each compact set $K \subset S_a$

where $S_a = \{x \in R^n, |x| \le a, a > 0\}, \quad k \in N^n$,

$$\gamma_{E,k}(\psi) = \sup_{x\in K}\left|D^k\psi(x)\right| < \infty$$

Clearly E is complete and so a Frechet space.

Moreover we say that f is a fractional Mellin transformable if it is a member of E' (the dual space of E).

2.2. THE GENERALIZED FRACTIONAL MELLIN TRANSFORM ON E'

It is easily seen that for each $s \in R^n$ and $0 \le \varphi \le \frac{\pi}{2}$, the function $K_\phi(x,s)$ belongs to E as a function of x.

Hence the fractional Mellin transform of $f \in E'$ can be defined by

$$\left[M^\varphi f(x)\right](s) = M_\varphi(s) = \left\langle f(x),\ K_\varphi(x,s)\right\rangle, \tag{2.1}$$

where

$$K_\phi(x,s) = \sqrt{1-i\cot\phi}\ e^{i\pi s^2\cot\phi}.e^{i\pi(\ln x)^2\cot\phi - i2\pi(\ln x)s.\csc\phi}\ \frac{1}{\sqrt{x}}, \tag{2.2}$$

then the right hand side of (2.1) has a meaning as the application of $f \in E'$ to $K_\phi(x,s) \in E$.

3. PROPERTIES OF KERNEL

The kernel $K_\phi(x,s)$ given in 2.2 satisfies the following properties.

1. The kernel $K_\phi(x,s)$ is continuous even at multiple of π.
2. $K_{-\phi}(x,s)=K_\phi^*(x,s),$ where '∗' denotes the conjugation.
3. For $\phi=\dfrac{\pi}{2}$ the kernel coincides with the kernel of the complex Mellin transform given in (1.2)
4. $K_\phi(x,s)=e^{-\frac{S}{2}\sec\phi(1+2i\pi\, s.\sin\phi)}K_\phi\left(\dfrac{x}{e^{s.\sec\phi}},0\right),$

Properties (1), (2) and (3) are trivial, hence not proved. The fourth property can be proved as follows.

From (2.2) we have

Proof :

$$K_\phi\left(\frac{x}{e^{s.\sec\phi}},0\right)=\sqrt{1-i\cot\phi}\; e^{i\pi\left[\ln\left(\frac{x}{e^{s.\sec\phi}}\right)\right]^2\cot\phi}.\frac{\sqrt{e^{s.\sec\phi}}}{\sqrt{x}}$$

$$=\sqrt{1-i\cot\phi}\; e^{i\pi[\ln x-s.\sec\phi]^2\cot\phi}.\frac{\sqrt{e^{s.\sec\phi}}}{\sqrt{x}}$$

$$=K_\phi(x,s).e^{i\pi s^2\tan\phi}.e^{\frac{s}{2}\sec\phi}$$

Therefore $K_\varphi(x,s)=e^{-\frac{S}{2}\sec\varphi(1+2i\pi s.\sin\varphi)}K_\varphi\left(\dfrac{x}{e^{s.\sec\varphi}},\ 0\right).$

Inversion formula for generalized fractional Mellin transform is proved, using the method as in [4]

Theorem: If (1.1) gives the fractional complex Mellin transform then

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}M_\phi(s)\,\overline{K_\phi(x,s)}\,ds\,,$$

where

$$\overline{K_\phi(x,s)}=\frac{\pi}{\sin\phi}(i\sin\phi)^{1/2}\exp\left(\frac{-i\phi}{2}\right)\exp\left\{-\frac{i\pi}{\sin\phi}\left[\left(s^2+(\ln x)^2\right)\cos\phi-2\ln x.s\ \right]\right\}\frac{1}{\sqrt{x}}$$

Proof: The one dimensional fractional complex Mellin transform is given by

$$\left[M^\phi f(x)\right](s)=M_\phi(s)=\int_0^\infty K_\phi(x,s)\,f(x)\,dx, \tag{4.1}$$

where the kernel

$$K_\phi(x,s)=\sqrt{1-i\cot\phi}\;e^{i\pi s^2\cot\phi}.\,e^{i\pi(\ln x)^2\cot\phi-i2\pi\ln x\,s.\csc\phi}\,\frac{1}{\sqrt{x}}$$

$$=C_{1\phi}\exp\left\{iC_{2\phi}\left[\left(s^2+(\ln x)^2\right)\cos\phi-2.\ln x.s\ \right]\right\}\frac{1}{\sqrt{x}}$$

where

$C_{1\phi}=(i\sin\phi)^{-\frac{1}{2}}e^{i\frac{\phi}{2}}$ *and* $C_{2\phi}=\dfrac{\pi}{\sin\phi}.$

Therefore from (1)

$$\exp\left(-iC_{2\phi}s^2\cos\phi\right)M_\phi(s)=\int_0^\infty g(x)x^{-2C_{2\phi}si-\frac{1}{2}}\,dx,$$
$$=M\left[g(x)\right](2C_{2\phi}si),$$

where

$$g(x)=f(x).C_{1\phi}\exp(iC_{2\phi}(\ln x)^2\cos\phi), \tag{4.2}$$

and $M\left[g(x)\right]$ is the complex Mellin transform of *g(x)* with argument $\eta=2C_{2\phi}si$ (say)

Thus

$$\exp\left(-iC_{2\phi}s^2\cos\phi\right)M_\phi\left(\frac{\eta}{2C_{2\phi}i}\right)=M\left[g(x)\right]\eta,$$

Invoking Mellin inversion we can write,

$$g(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}G(\eta)x^{\eta-\frac{1}{2}}d\eta,$$

where

$$G(\eta)=\exp\left(iC_{2\phi}\cos\phi\frac{\eta^2}{4C^2_{2\phi}}\right)M_\phi\left(\frac{\eta}{2C_{2\phi}i}\right).$$

Putting the value of *g(x)* from (2) and on simplifying we get inversion formula ,

$$f(x)=\frac{1}{\pi}\int_{-i\infty}^{i\infty}M_\phi(s)\,\overline{K_\phi(x,s)}\,ds,$$

where

$$\overline{K_\phi(x,s)}=\frac{\pi}{\sin\phi}(i\sin\phi)^{\frac{1}{2}}\exp\left(\frac{-i\phi}{2}\right)\exp\left\{-\frac{i\pi}{\sin\phi}\left[\left(s^2+(\ln x)^2\right)\cos\phi-2\ln x.s\right]\right\}\frac{1}{\sqrt{x}}.$$

4. PROPERTIES OF FRACTIONAL COMPLEX MELLIN TRANSFORM

In Table 1 we list a number of useful properties of the fractional Mellin transform which are extensions of the corresponding properties of the complex Mellin transform.

Table 1. Properties of the fractional Mellin transform

Sr.No.	Function	Fractional Mellin transform with angle ϕ
1.	*g(cx)*	$\frac{1}{\sqrt{c}}e^{i\pi\ln c.\sin\phi(2s+\ln c.\cos\phi)}\left[M^\phi g(x)\right]\left(s+lnc.\cos\phi\right)$
2.	*x.g(x)*	$e^{\cos\phi\left(s+\frac{i}{4\pi}\sin\phi\right)}\left[M^\phi g(x)\right]\left(s+\frac{i}{2\pi}sin\phi\right)$
3.	*lnx.g(x)*	$s.\cos\phi\left[M^\phi g(x)\right](s)+\frac{i}{2\pi}\sin\phi\left[M'^\phi g(x)\right](s)$
4.	$g(x)e^{iclnx}$	$e^{ic\cos\phi\left(s-\frac{c}{4\pi}\sin\phi\right)}\left[M^\phi g(x)\right]\left(s-\frac{c}{2\pi}sin\phi\right)$

5.	*g'(x)*	$M^{\phi}\left\{\frac{g(x)}{x}\left[\frac{1}{2}-2i\pi.\csc\phi(\ln x.\cos\phi-s)\right]\right\}(s)$ and $\left(\frac{1}{2}+2\pi is.\sin\phi\right)\left[M^{\phi}\frac{g(x)}{x}\right](s)+\cos\phi\left[M'^{\phi}\frac{g(x)}{x}\right](s)$
6.	*g(x+τ)*	$\left[M^{\phi}g(x).e^{i\pi\ln\left(1-\frac{\tau}{x}\right)\cot\phi\left[\ln(x^{2}-\tau x)-2s.\sec\phi\right]}\frac{1}{\sqrt{1-\frac{\tau}{x}}}\right](s)$

where M'^{ϕ} denotes the derivative of M^{ϕ} with respect to *s*.

5. TRANSFORMS OF SOME COMMON FUNCTIONS

Table 2 gives the fractional Mellin transforms of a number of common functions.

Table 2. Transform of some common functions.

Sr.No.	Function	Fractional Mellin transform with angle ϕ
1.	$\delta(x-a)$	$\sqrt{1-i\cot\phi}\,.\,e^{i\pi\left[\left(s^{2}+(\ln a)^{2}\right)\cot\phi-2\ln a\, s.\csc\phi\right]}\frac{1}{\sqrt{a}}$, if ϕ is not a multiple of π.
2.	*1*	$\sqrt{1+i\tan\phi}\,.\,e^{i\tan\phi\left(\frac{1}{16\pi}-\pi s^{2}\right)+\frac{s}{2}\sec\phi}$, if $\phi-\frac{\pi}{2}$ is not a multiple of π.
3.	x^{n}	$\sqrt{1+i\tan\phi}\,.\,e^{i\tan\phi\left(\frac{1}{4\pi}\left(n+\frac{1}{2}\right)^{2}-\pi s^{2}\right)+\left(n+\frac{1}{2}\right)s.\sec\phi}$, if $\phi-\frac{\pi}{2}$ is not a multiple of π.
4.	x^{2ic}	$\sqrt{1+i\tan\phi}\,.\,e^{-i\tan\phi\left(\pi s^{2}+\frac{1}{\pi}\left(c^{2}-\frac{1}{16}\right)-\frac{ic}{2\pi}\right)+s.\left(2ic+\frac{1}{2}\right).\sec\phi}$, if $\phi-\frac{\pi}{2}$ is not a multiple of π.
5.	$e^{\ln x}$	$\sqrt{1+i\tan\phi}\,.\,e^{i\tan\phi\left(\frac{9}{16\pi}-s^{2}\pi\right)+\frac{3}{2}.s.\sec\phi}$, if $\phi-\frac{\pi}{2}$ is not a multiple of π.
6.	$e^{ic\ln x}$	$\sqrt{1+i\tan\phi}.e^{-i\tan\phi\left(\pi s^{2}+\frac{1}{4\pi}\left(c^{2}-\frac{1}{4}\right)-\frac{ic}{4\pi}\right)+s\left(ic+\frac{1}{2}\right).\sec\phi}$,

		if $\phi - \frac{\pi}{2}$ is not a multiple of π.
7.	$e^{ic(\ln x)^2}$	$\sqrt{\frac{i\pi + \pi\cot\phi}{c + \pi\cot\phi}} \, . e^{i\pi s^2 \cot\phi - \frac{i\left(\pi s . \csc\phi + \frac{i}{4}\right)^2}{c + \pi\cot\phi}}$, if ϕ is not a multiple of π.
8.	$e^{-(\ln x)^2}$	$\sqrt{\frac{i\pi + \pi\cot\phi}{i + \pi\cot\phi}} \, . e^{i\pi s^2 \cot\phi - \frac{i\left(2\pi s . \csc\phi + \frac{i}{2}\right)^2}{4(\pi\cot\phi + i)}}$, if ϕ is not a multiple of π.

6. APPLICATION OF FRACTIONAL COMPLEX MELLIN TRANSFORM

Scale transform is a powerful mathematical tool for processing images (for detecting) that are arbitrarily scaled. Hence it is used in the class of linear stretch invariant systems. Xiaohong Hu [8] developed Mellin transform technique of probability modeling for accurate solution of problems in some industrial statistic.

Fractional Mellin transform given by Akay adds one more parameter (angle) to scale transform and hence it is also used in pattern recognition problems, industrial statistic. Moreover fractional Mellin based correlators are used to obtain time to impact and controlling moments in the navigation task.

The generalized fractional Mellin transform we have introduced in this paper is the extension of fractional Mellin transform given by Akay and can be used in all above cases. The advantage of our generalized fractional Mellin transform is it can be used even when the signals (functions) are singular functions.

In the next section, we have given the application of generalized fractional Mellin transform for solving the particular partial differential equation.

An application of the fractional Mellin transform to differential equation :

Consider the differential equation –

$$P(xD_x)(u(x)) = f(x) \tag{6.1}$$

Where $f \in E'$ and $P(xD_x) = \sum_{|\beta| \le m} a_\beta x^\beta D_x^\beta$ is a linear differential operator of order m with constant coefficients a_β.

Suppose that the equation (1) posseses a solution u.

Applying the fractional mellin transform to (1)

$M^\phi[P(xD)u] = M^\phi(f)$

Therefore,

$$M^\phi[P(xD)u] = M^\phi\left[\left(\sum_{|\beta| \le m} a_\beta x^\beta D^\beta\right)u\right]$$

$$= \left[\sum a_\beta x^\beta \left(x^{-\beta} \sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right) \right] M^\phi(u)$$

$$= \left[\sum a_\beta \left(\sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right) \right] M^\phi(u)$$

Therefore,

$$\left[\sum a_\beta \left(\sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right) \right] M^\phi(u) = M^\phi(f) \qquad (6.2)$$

Under the assumption that the polynomial p is such that

$$\left| p\left[a_\beta \sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right] \right| > 0 \text{ for } s = (s_1, s_2 \ldots\ldots s_n) \in R^n \quad (6.3)$$

Equation (2) gives

$$M^\phi(u) = \left[p\left[a_\beta \sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right] \right]^{-1} M^\phi(f) \qquad (6.4)$$

Applying inversion of fractional mellin transform $[M^\phi]^{-1}$ to (4) we get

$$u = [M^\phi]^{-1} \left[M^\phi(f) \Big/ p\left[a_\beta \sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right] \right] \qquad (6.5)$$

Next we show that if $f \in E'$ and p satisfies (6.3), then equation (6.5) defines a generalized function which is a solution of equation (6.1)

Indeed since $f \in E'$ then for $0 < \phi \le \frac{\pi}{2}$, $M^\phi f \in E'$ and hence by assumption (6.3) and the definition that $\theta \in E$ and $f \in E'$, then the product 'θf' is defined by

$$\langle \theta f, \phi \rangle = \langle f, \theta\phi \rangle \qquad \forall \phi \in E$$

We have

$$M^\phi(f) \Big/ p\left[a_\beta \sum_{i=0}^{\beta} c_{\phi i} (\ln x \cos\phi - s)^i \right] \quad \in E' \text{ and so } u \in E'$$

To show that *u* satisfies (6.1) we apply M^ϕ to both sides of (6.5) and get (6.4). Since generalized function admit multiplication by polynomials, we hence obtain equality (6.2). Finally applying $[M^\phi]^{-1}$ to (6.2) we get (6.1).

7. CONCLUSIONS

We have introduced an extension of complex Mellin transform, that is designated fractional Mellin transform. This linear transform depends on a parameter ϕ and can be interpreted as a rotation by an angle ϕ in scale-log modulation phase plane. When $\phi = \frac{\pi}{2}$ the fractional Mellin transform coincides with the conventional complex Mellin transform. Inversion formula for this transform is also established. Number of properties of the fractional Mellin transform are given which coincides with corresponding properties for complex Mellin transform in special case and which can be used when we solve partial differential equations by using this transform. Fractional Mellin transform of some simple functions are also obtained. The applications of fractional Mellin transform in various fields are given. Also the application of generalized fractional Mellin transform for solving the particular partial differential equations is given.

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