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CONTINUATION METHOD FOR BOUNDARY VALUE PROBLEMS WITH UNIFORM ELLIPTICAL OPERATORS

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Abstract*. In this note we study further a method that linking problem for operators of Laplace with Diriclet problem corresponding to a linear operator Diriclet elliptical. So we are able to prove existence and uniqueness stability problem using the existence, uniqueness and Diriclet stability problem.*

Keywords*: uniform elliptical operators, Hilbert space, bundary value.*

1. INTRODUCTION

Let X be a real Hilbert space with scalar product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. Consider a linear operator $B : D(B) \subseteq X \to X$ with domain $D(B)$ an infinite dimensional subspace which is *symmetrical*:

$$(Bu, v) = (u, Bv), \quad \forall u, v \in D(B),$$

and *strictly monotone (positive)*, i.e. there is $c > 0$ such that:

$$(Bu, u) \geq c\|u\|^2, \quad \forall u \in D(B).$$

Introducing the $D(B)$ *energy scalar product*:

$$(u, v)_E := (Bu, v), \quad \forall u, v \in D(B),$$

and *energy norm* $\|u\|_E := \sqrt{(u,u)_E}, \quad \forall u \in D(B)$.

We denote by E the linear subspace $D(B) \subseteq X$ supplement in relation to the energy norm, which we call *energy space* of B. It contains all the elements of $u \in X$, limited by Cauchy sequences $\{u_n\} \subset D(B)$ in norm energy. Extending by continuity to the whole space E energy scalar product, i.e. for $u_n \to u$ and $v_n \to v$ we take

$$(u, v)_E := \lim (u_n, v_n)_E.$$

Energy space becomes a Hilbert space E, which contains $D(B)$ as a dense subspace and $E \hookrightarrow X$ is continuous embedding because we can write:

$$\|u\| \leq c^{-\frac{1}{2}}\|u\|_E, \quad \forall u \in E.$$

The application of duality $J : E \to E^*$, defined by:

$$\langle Ju, v \rangle := (u, v)_E, \quad \forall u, v \in E,$$

is a *homeomorfism isometric*, i.e.:

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$$\|Ju\| = \|u\|_E, \quad \forall u \in E,$$

(see [2, p.112)]) and an extension of B (i.e., $Ju = Bu, \quad \forall u \in D(B)$).

Friederichs extension of the operator $A : D(A) \subseteq X \to X$ is defined:

$$Au := Ju, \quad \forall u \in D(A),$$

where $D(A) := \{u \in E;\ Ju \in X\}$. We see that:

$$u \in D(A) \Leftrightarrow \exists g \in X \text{ so} \langle Ju, v \rangle = (g, v)_E, \forall v \in E,$$

as $D(B) \subseteq E \subseteq X \equiv X^* \subseteq E^*$ (see[5, p. 280]). Note that this extension is maximal monotone extension of B in X, because $D(A)$ is dense in X and A is a closed operator, autoadjunct, bijective and strictly monotone (see [2, p.48]). Also, inverse operator $A^{-1} : X \mapsto X$ is linear, continuous, autoadjunct and compact whenever embedding $E \hookrightarrow X$ is compact.

So in this case, we can apply *Fredholm's theory* and we establish the following variant of *theorem of existence*.

Theorem 1.1. *If embedding $E \hookrightarrow X$ is compact, then there is unique generalized solution, $u \in X$, the equation:*

$$(Au, v) = (f, v), \quad \forall v \in X. \tag{1}$$

If $u \in D(B)$, then u is the solution equation $Bu = f$.

Example 1.1. *Let $X := L^2(\Omega)$ and $E := \{u \in H^1(\Omega); u/_\Gamma = 0\}$ where $\Gamma \subseteq \partial\Omega$ is part of the boundary as $means(\Gamma) > 0$. Both spaces are Hilbert spaces with scalar product:*

$$(u, v) := \int_\Omega u(x)v(x)dx; \qquad (u, v)_E := \int_\Omega \nabla u(x) \cdot \nabla v(x)dx.$$

Also, the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact when $\Omega \in \Re^N$, $N \geq 2$, is a regular field, (see [1]), and we can apply theorem because the energy space of the operator $H^1(\Omega)$ is linear, symmetric, strictly monotonous $B := -\Delta$ with $D(B) := \{u \in C^2(\Omega) \cap C(\overline{\Omega}); u/_\Gamma = 0\}$.
In this case equation (1) takes the form:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = \int_\Omega f(x)v(x)dx, \qquad \forall v \in E.$$

If $u \in D(B)$, then Green's formula we see that u is the classical solution of the problem to the limit:

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Gamma, \\ \dfrac{\partial u(x)}{\partial n} = 0, & x \in \partial\Omega \setminus \Gamma. \end{cases}$$

2. POINCARÉ'S CONTINUATION METHOD

Let X be a Banach space, Y a normed space and $L_0, L_1 \in L(X, Y)$ bounded operators. Homotopia define:

$$L_t := (1-t)L_0 + tL_1$$

and suppose

$$\|L_t x\|_Y \geq c\|x\|_X, \qquad \forall x \in X, t \in [0,1].$$

We show that L_1 *applied X on Y if and only if* L_0 *has this property.*

Indeed, if L_s is surjective for $s \in [0, 1]$, then the equation $L_t x = y$ can be equivalent rewritten as

$$x = L_s^{-1} y + (t-s) L_s^{-1}(L_0 - L_1)x := Tx.$$

Tx application is a *c*-contraction on *X* if

$$|s-t| < \delta := \frac{c}{\|L_0\| + \|L_1\|}$$

So L_t is surjective for all $t \in [0, 1]$ for which $|s-t| < \delta$. Dividing [0, 1] in subintervale long $< \delta$, we see that L_t is surjective for all $t \in [0,1]$ provided that he is surjective for $s = 0$. In particular for *t*=1 (see [3, p. 57]).

We consider the limit problem:

$$\begin{cases} Lu = f & in\ \Omega \\ u/_{\Gamma} = 0 & on\ \Gamma \\ \dfrac{\partial u}{\partial n} := \displaystyle\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} n_j = g & on\ \partial\Omega \setminus \Gamma, \end{cases} \tag{2}$$

where *L* is an elliptic operator divergential:

$$(Lu)(x) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left[a_{ij}(x) \frac{\partial u}{\partial x_i} \right] + b(x)u(x).$$

Assume for simplicity that are met classical conditions:

$$\begin{cases} a_{ij} \in C^1(\overline{\Omega}), b \in C(\overline{\Omega}), \displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\xi_i \xi_j \geq k|\xi|^2, k > 0 \\ \displaystyle\sum_{i,j=1}^{N} |a_{ij}(x)|^2 \leq M^2, \frac{|b(x)|}{k} \leq m^2, \forall \xi \in \Re^N, x \in \Omega, \end{cases} \tag{3}$$

and formulate the problem to the limit:

$$\begin{cases} Lu = f & on\ \Omega, \\ u/_{\Gamma} = 0 & on\ \Gamma, \\ \dfrac{\partial u}{\partial n} = g & on\ \partial\Omega \setminus \Gamma. \end{cases} \tag{4}$$

As before we take $X := L^2(\Omega)$ and $E := \{u \in H^1(\Omega); u/_{\Gamma} = 0\}$. The weak solution of problem (4) understand an element $u \in E$ for which:

$$a(u,v) = (f,v), \quad \forall v \in E,$$

where $a(u,v)$ is the Diriclet form associated problem (4):

$$a(u,v) := \int_{\Omega} \left[\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + b(x)u(x)v(x) \right] dx.$$

We note that *E* is the energy space of the operator *L*, with $D(L) := \{u \in C^2(\Omega) \cap C(\overline{\Omega}); u/_{\Gamma} = 0\}$, and Friederichs extension of *L* is defined by:

$$(Au, v) := a(u, v) \quad \forall v \in E.$$

So states the theorem can:

Theorem 2.1. *If embedding $E \hookrightarrow X$ is compact, then there is unique generalized solution, $u \in X$, of equation (1). If $u \in D(L)$ then u is classical solution of the problem (4).*

Proof. Apply the continue method with $L_0 := J \equiv -\Delta$ and $L_1 := A$. Then we have

$$(L_t u, u) := (1-t)(Ju, u) + t(Au, u) \geq (1-t)\|u\|^2 + tk\|u\|^2.$$

Because $(L_t u, u) \leq \|L_t u\| \cdot \|u\|$ by Schwarz's inequality, we deduce that $\|L_t u\| \geq c\|u\|$ with $c := \min(1, k)$. (q.e.d.)

3. THE PROBLEM OF ELASTIC EQUILIBRIUM

We apply the results of previous study of the mixed equilibrium in linear elasticity.

Elastic equilibrium of an object B, homogeneous and anisotropic, which occupies a bounded domain $\Omega \subset \Re^m, m \geq 2$, with border $\partial\Omega$ the smooth portions, described - small deformations assumption - the equations of Cauchy in Ω [4]:

$$\sum_{j=1}^{N} \sigma_{ij,j} + f_i = 0 \tag{5}$$

where $\sigma := (\sigma_{ij})_{3\times 3}$ is the Cauchy tensor power, and $f := (f_i)_{3\times 1}$ is the density of mass forces acting on unit volume of B. The field is homogeneous and anisotropic constitutive law satisfies:

$$\sigma_{ij} = \sum_{k,h=1}^{N} a_{ijkh}\varepsilon_{kh}, \tag{6}$$

where the elastic coefficients satisfy $a_{ijkh} = a_{jikh} = a_{khij}$, and $\varepsilon := (\varepsilon_{ij})_{3\times 3}$ is infinitesimal deformation tensor:

$$\varepsilon_{ij} := \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad u_{i,j} := \frac{\partial u_i}{\partial x_j},$$

dispacement vector associated $u := (u_i)_{3\times 1}$.

Equilibrium Problem in Linear Elasticity is similar problem to the limit (4). She asked to find the vector displacement $u := u(x)$ solution of the system:

$$\begin{cases} -div\ \sigma(\varepsilon(u)) = f, & in\ \Omega \\ u/_{\Gamma} = U & on\ \Omega \\ \sum_{j=1}^{N} \sigma_{ij} n_j /_{\partial\Omega\setminus\Gamma} = F & on\ \partial\Omega \setminus \Gamma. \end{cases}$$

Here the problem is: *f* is the density of mass forces, *U* is Γ border movement, and F is complementary border traction $\partial\Omega \setminus \Gamma$. By standard arguments - taking the form of translations of $y := u - U$ - get to the problem:

$$\begin{cases} -div\ \sigma(\varepsilon(y)) = f, & in\ \Omega \\ y/_{\Gamma} = 0 & on\ \Gamma \\ \sum_{j=1}^{N} \sigma_{ij} n_j /_{\partial\Omega\setminus\Gamma} = F & on\ \partial\Omega\setminus\Gamma. \end{cases}$$

Define:

$$D(B) := \left\{ y \in [C^2(\overline{\Omega})]^3;\ y/_{\Gamma} = 0 \right\}$$

Lamé's operator:

$$By := -div\ \sigma(\varepsilon(y)).$$

Then, taking the product in $X := [L^2(\Omega)]^3$ and applying Green's formula, we have:

$$(Bu, v) = -\int_{\Omega} div\ \sigma(\varepsilon(u)) \cdot v dx = \frac{1}{2}\int_{\Omega} a_{ijkh}\varepsilon_{ij}(v)\varepsilon_{kh}(u)dx =$$

$$= -\int_{\Omega} div\ \sigma(\varepsilon(v)) \cdot u dx = (u, Bv), \qquad \forall u, v \in D(B).$$

For $v := u$, the symmetry of deformation tensor, we have:

$$(Bu, u) = \frac{1}{2}\int_{\Omega} a_{ijkh}\varepsilon_{ij}(u)\varepsilon_{kh}(u)dx \geq \frac{k}{2}\int_{\Omega} \varepsilon_{ij}(u)\varepsilon_{ij}(u)dx = k\int_{\Omega} |\nabla u|^2 dx.$$

So we can define the energy space *E* by the complement of *D*(*B*) in norm $\|\cdot\|_E$ induced by scalar product:

$$(u, v)_E := \frac{1}{2}\int_{\Omega} a_{ijkh}\varepsilon_{ij}(u)\varepsilon_{kh}(v)dx.$$

This is a Hilbert space that contains elements of $V := \left\{ u \in [H^1(\Omega)]^3;\ u/_{\Gamma} = 0 \right\} \hookrightarrow X$ the Sobolev-Kondrashov theorem. So we have $E \hookrightarrow X \equiv X^* \hookrightarrow E^*$.

Scalar product defines the application of duality energy $J := E \mapsto E^*$, which introduce the Friederichs extension of Lamé operator:

$$Au := Ju, \qquad D(A) := \left\{ u \in E;\quad Ju \in X \right\}.$$

In conclusion, we can apply theorem, upon which states:

Theorem 3.1. If $f \in [L^2(\Omega)]^3$ then there is a variational solution (weak) unique. If, in addition, $u \in D(B)$, then this solution is strong solution (classical).

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