ORIGINAL PAPER

ON A GEOMETRIC INEQUALITY OF OPPENHEIM

JIAN LIU[1]

Manuscript received: 20.08.2011; Accepted paper: 14.11.2011;

Published online: 10.03.2012.

Abstract. *In this paper we give a simple proof of an Oppenheim's geometric inequality by using a new lemma, we also prove a refinement of the Oppenheim inequality. Some related conjectures which have been checked by the computer are put forward.*

Keywords: *triangle, interior point, Oppenheim inequality, Erd¨os-Mordell inequality, conjecture.*

Mathematics Subject Classification 2000: *51M16.*

1. INTRODUCTION

Let P be an arbitrary interior point of the triangle ABC. Denote by R_1, R_2, R_3 the distances from P to the vertices A, B, C, and r_1, r_2, r_3 the distances from P to the sidelines BC, CA, AB respectively. Then

$$R_2R_3 + R_3R_1 + R_1R_2 \geq 4\left(r_2r_3 + r_3r_1 + r_1r_2\right) \tag{1.1}$$

with equality if and only if ΔABC is equilateral and P is its center. This inequality was first published by J.M.Child [1]. In 1964, L.Carlitz [2] established the following stronger inequality:

$$R_2R_3 + R_3R_1 + R_1R_2 \geq 4\left(w_2w_3 + w_3w_1 + w_1w_2\right) \tag{1.2}$$

where w_1, w_2, w_3 are the internal angle-bisectors of $\angle BPC$, $\angle CPA$, $\angle APB$ respectively. The author [3] generalized further Carlitz's result in 1996:

$$\begin{aligned} &x^2\left(R_2R_3\right)^k + y^2\left(R_3R_1\right)^k + z^2\left(R_1R_2\right)^k \geq \\ &\geq 2^k\left[yz\left(w_2w_3\right)^k + zx\left(w_3w_1\right)^k + xy\left(w_1w_2\right)^k\right] \end{aligned} \tag{1.3}$$

where x, y, z are arbitrary real numbers and exponent k satisfies $0 < k \leq 1$.

On the other hand, A. Oppenheim [4] considered the stronger version of (1.1) from another viewpoint as early as 1961, and he concluded that the following inequality holds without proof at the end of the reference:

[1] East China Jiaotong University, Jiangxi province Nanchang City, 330013, China.
E-mail: china99jian@163.com.

Theorem 1.1. *For any arbitrary point P of ΔABC, we have*

$$R_2R_3 + R_3R_1 + R_1R_2 \geq \geq (r_1 + r_2)(r_3 + r_1) + (r_2 + r_3)(r_1 + r_2) + (r_3 + r_1)(r_2 + r_3) \quad (1.4)$$

with equality if and only if ΔABC is equilateral and P is its center.

Inequality (1.4) is equivalent to

$$R_2R_3 + R_3R_1 + R_1R_2 \geq r_1^2 + r_2^2 + r_3^2 + 3(r_2r_3 + r_3r_1 + r_1r_2) \quad (1.5)$$

It is easy to show that the combination coefficients of the right hand side is the best possible. In other words, (1.5) is the strongest in the following type inequality:

$$R_2R_3 + R_3R_1 + R_1R_2 \geq m(r_1^2 + r_2^2 + r_3^2) + n(r_2r_3 + r_3r_1 + r_1r_2) \quad (1.6)$$

where m, n are positive constants.

Oppenheim tried to prove his inequality in another paper [5] in the same year. His method is as follows: First suppose that $a \geq b \geq c$, then make use of the well known inequalities

$$aR_1 \geq cr_2 + br_3 \quad (1.7)$$

$$aR_1 \geq br_2 + cr_3 \quad (1.8)$$

(where $a=BC$, $b=CA$, $c=AB$) to prove respectively that the inequality holds in the following six cases:

(i) $r_1 \geq r_2 \geq r_3$	*(ii)* $r_1 \geq r_3 \geq r_2$	*(iii)* $r_2 \geq r_3 \geq r_1$
(iv) $r_2 \geq r_1 \geq r_3$	*(v)* $r_3 \geq r_1 \geq r_2$	*(vi)* $r_3 \geq r_2 \geq r_1$

However, he only discussed amply the two cases of *(i)* and *(vi)*, and also pointed out the other four cases can be proved by the same way. The author thinks this proof is not faultless. Every case should be considered respectively. In a recent paper [6], J.M.Hamiton by using Oppenheim's method, finished the proof of all six cases. But the proof is very complicated.

The purpose of this note is to give a simple proof of the Oppenheim inequality (1.4), also prove the following refinement result:

Theorem 1.2. *For an arbitrary interior point P of the triangle ABC, we have*

$$\begin{aligned} &R_2R_3 + R_3R_1 + R_1R_2 \geq \\ &\geq h_ar_1 + h_br_2 + h_cr_3 + r_2r_3 + r_3r_1 + r_1r_2 \\ &\geq (r_1 + r_2)(r_3 + r_1) + (r_2 + r_3)(r_1 + r_2) + (r_3 + r_1)(r_2 + r_3) \end{aligned} \quad (1.9)$$

where h_a, h_b, h_c are three altitudes of the triangle ABC. Both equalities in (1.9) hold if and only if ΔABC is equilateral and P is its center.

In studying Oppenheim inequality, a lot of unsolved geometric inequalities were found by the author. We will state some related conjectures in the last section of this note.

2. PROOFS OF THE THEOREMS

The proofs of the two theorems are both need the following key lemma:

Lemma 2.1. *For any point P of ΔABC, we have*

$$R_2 + R_3 \geq 2r_1 + \frac{(r_2 + r_3)^2}{R_1} \tag{2.1}$$

with equality if and only if $b=c$ and P is the circumcenter of ΔABC.

Proof. Inequality (1.9) is equivalent to

$$R_1 (R_2 + R_3 - 2r_1) - (r_2 + r_3)^2 \geq 0$$

Note that $R_2 + R_3 > 2r_1$, by inequality (1.7) and its two analogues $bR_2 \geq ar_3 + cr_1$, $cR_3 \geq br_1 + ar_2$, it is suffice to prove that

$$\frac{cr_2 + br_3}{a}\left(\frac{ar_3 + cr_1}{b} + \frac{br_1 + ar_2}{c} - 2r_1\right) \geq (r_2 + r_3)^2$$

Namely,

$$\frac{(ar_2r_3 + br_3r_1 + cr_1r_2)(b-c)^2}{abc} \geq 0 \tag{2.2}$$

which is obviously true. We have known that if AO (O is the circumcenter of ΔABC) cuts BC at X then the equality in (1.7) holds if and only if P lies on the segment AX. According to this conclusion and (2.2), we conclude that the equality in (2.1) occurs if and only if $b=c$ and P is its circumcenter. This completes the proof of Lemma 2.1.

Remark 2.1. By the arithmetic-geometric mean inequality, we can easily see that inequality (2.1) is stronger than the following inequality:

$$R_1 (R_2 + R_3)^2 \geq 8r_1 (r_2 + r_3)^2 \tag{2.3}$$

In fact, we have the more stronger inequality:

$$R_1 (R_2 + R_3)^2 \geq 8w_1 (w_2 + w_3)^2 \tag{2.4}$$

which was first posed by the author and was proved by Zhi-Hua Zhang and Yu-Dong Wu [7].

Remark 2.2. The famous Erdös-Mordell inequality [8-20]:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \tag{2.5}$$

can be deduced from Lemma 2.1 as follows: According to (2.1) and Cauchy inequality, we have that

$$\begin{aligned}&2\left(R_1+R_2+R_3\right)\\&\geq 2\left(r_1+r_2+r_3\right)+\frac{\left(r_2+r_3\right)^2}{R_1}+\frac{\left(r_3+r_1\right)^2}{R_2}+\frac{\left(r_1+r_2\right)^2}{R_3}\\&\geq 2\left(r_1+r_2+r_3\right)+\frac{4\left(r_1+r_2+r_3\right)^2}{R_1+R_2+R_3}\end{aligned}$$

Therefore

$$\left(R_1+R_2+R_3\right)^2-\left(R_1+R_2+R_3\right)\left(r_1+r_2+r_3\right)-2\left(r_1+r_2+r_3\right)^2\geq 0$$

Namely,

$$\left(R_1+R_2+R_3+r_1+r_2+r_3\right)\left[R_1+R_2+R_3-2\left(r_1+r_2+r_3\right)\right]\geq 0$$

So we have inequality (2.5).

The proofs of the Erdös-Mordell inequality has been giving constantly [10-20]. Recently, the author gave an alternative proof in [20].

2.1. PROOF OF THEOREM 1.1.

Proof. By Lemma 2.1, we have

$$\begin{aligned}&R_1\left(R_2+R_3\right)+R_2\left(R_3+R_1\right)+R_3\left(R_1+R_2\right)\\&\geq 2\left(R_1r_1+R_2r_2+R_3r_3\right)+\left(r_2+r_3\right)^2+\left(r_3+r_1\right)^2+\left(r_1+r_2\right)^2\end{aligned}$$

which is equivalent to

$$R_2R_3+R_3R_1+R_1R_2\geq R_1r_1+R_2r_2+R_3r_3+\frac{1}{2}\left[\left(r_2+r_3\right)^2+\left(r_3+r_1\right)^2+\left(r_1+r_2\right)^2\right] \quad (2.6)$$

It is easy to see that the equality in (2.6) holds if and only if $a=b=c$ and P is the circumcenter.

Now observe that

$$\begin{aligned}&R_1r_1+R_2r_2+R_3r_3+\frac{1}{2}\left[\left(r_2+r_3\right)^2+\left(r_3+r_1\right)^2+\left(r_1+r_2\right)^2\right]-\\&-\left[\left(r_1+r_2\right)\left(r_3+r_1\right)+\left(r_2+r_3\right)\left(r_1+r_2\right)+\left(r_3+r_1\right)\left(r_2+r_3\right)\right]=\\&=R_1r_1+R_2r_2+R_3r_3-2\left(r_2r_3+r_3r_1+r_1r_2\right)\end{aligned}$$

and the well known result (see [4, 8, 9]):

$$R_1r_1+R_2r_2+R_3r_3\geq 2\left(r_2r_3+r_3r_1+r_1r_2\right) \quad (2.7)$$

Oppenheim inequality (1.4) follows from (2.6) at once. Clearly, the equality in (1.4) holds is the same as (2.6). Theorem 1.1 is proved. □

2.2. PROOF OF THEOREM 1.2.

Proof. We first prove the first inequality of (1.9)

$$R_2R_3+R_3R_1+R_1R_2\geq h_ar_1+h_br_2+h_cr_3+r_2r_3+r_3r_1+r_1r_2 \quad (2.8)$$

According to the known inequality (1.8) and its two analogues, we get

$$R_1r_1+R_2r_2+R_3r_3+\frac{1}{2}\left[(r_2+r_3)^2+(r_3+r_1)^2+(r_1+r_2)^2\right]\ge$$
$$\ge\frac{(br_2+cr_3)r_1}{a}+\frac{(cr_3+ar_1)r_2}{b}+\frac{(ar_1+br_2)r_3}{c}+$$
$$+\frac{1}{2}\left[(r_2+r_3)^2+(r_3+r_1)^2+(r_1+r_2)^2\right]=$$
$$=r_2r_3+r_3r_1+r_1r_2+r_1\left(r_1+\frac{br_2+cr_3}{a}\right)+r_2\left(r_2+\frac{cr_3+ar_1}{b}\right)+r_3\left(r_3+\frac{ar_1+br_2}{c}\right)=$$
$$=r_2r_3+r_3r_1+r_1r_2+(ar_1+br_2+cr_3)\left(\frac{r_1}{a}+\frac{r_2}{b}+\frac{r_3}{c}\right)=$$
$$=h_ar_1+h_br_2+h_cr_3+r_2r_3+r_3r_1+r_1r_2$$

where we used the identity $ar_1+br_2+cr_3=2S=ah_a=bh_b=ch_c$ (S is the area of ΔABC). Therefore, inequality (2.8) follows from (2.6). Clearly, the equality condition in (2.8) is the same as (2.6).

It is easy to check that the second inequality of (1.9) is equivalent to

$$h_ar_1+h_br_2+h_cr_3\ge(r_1+r_2+r_3)^2 \tag{2.9}$$

which follows from Cauchy inequality and the simple identity:

$$\frac{r_1}{h_a}+\frac{r_2}{h_b}+\frac{r_3}{h_c}=1 \tag{2.10}$$

The proof of Theorem 1.2 is completed. └

Remark 2.3. In [5], A.Oppenheim pointed out a set of inequalities equivalent to (1.4) by using various geometric transformations (see [4, 9, 21]). If we apply these transformations to the stronger inequality (2.8), then we can get some new results. For example, applying reciprocation transformation to (2.8), one can obtain the following inequality:

$$\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}-\frac{1}{R_2R_3}-\frac{1}{R_3R_1}-\frac{1}{R_1R_2}\ge\frac{h_a}{r_1R_1^2}+\frac{h_a}{r_1R_1^2}+\frac{h_a}{r_1R_1^2} \tag{2.11}$$

3. SOME RELATED CONJECTURES

In this section we will give some related conjectures which all have been checked by the computer.

First considering the stronger inequality of Lemma 2.1, we propose the following conjecture:

Conjecture 3.1. For any interior point P of ΔABC, we have

$$R_2+R_3\ge 2w_1+\frac{(w_2+w_3)^2}{R_1} \tag{3.1}$$

If the above inequality is valid, using the same way to deduce Erdös-Mordell inequality (see Remark 2.2), we can prove Barrow's inequality [10]:

$$R_1 + R_2 + R_3 \geq 2(w_1 + w_2 + w_3) \tag{3.2}$$

In addition, if (3.1) holds true, then using it we can easily prove that

$$4R_1^2 + (R_2 + R_3)^2 \geq 8R_1 w_1 + 4(w_2 + w_3)^2 \tag{3.3}$$

This inequality inspires the author to put forward the following inequality:
Conjecture 3.2. For any interior point P of ΔABC, we have

$$R_1^2 + R_2 R_3 \geq 2(R_1 w_1 + 2w_2 w_3) \tag{3.4}$$

When the author considered the proof of Conjecture 3.1, we conjectured the following:
Conjecture 3.3. For any interior point P of ΔABC, we have

$$\frac{a^2}{(w_2 + w_3)^2} - \frac{(w_2 + w_3)^2}{R_1^2} \geq \frac{4w_1}{R_1} \tag{3.5}$$

In deed, inequality (3.5) is stronger than (3.1). The following similar inequality has not yet been proved:
Conjecture 3.4. For any interior point P of ΔABC, we have

$$\frac{a^2}{(r_2 + r_3)^2} - \frac{(r_2 + r_3)^2}{R_1^2} \geq \frac{4r_1}{R_1} \tag{3.6}$$

In [22], the author obtained the inequality:

$$R_2 R_3 + R_3 R_1 + R_1 R_2 \geq \left(\frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3} \right)^2 \tag{3.7}$$

This inequality prompts the author again to pose the following stronger inversion of Oppenheim inequality (1.5):
Conjecture 3.5. For any interior point P of ΔABC, we have

$$\left(\frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3} \right)^2 \geq r_1^2 + r_2^2 + r_3^2 + 3(r_2 r_3 + r_3 r_1 + r_1 r_2) \tag{3.8}$$

In addition, it is possible that the Oppenheim inequality (1.4) has a stronger version:

$$R_2 R_3 + R_3 R_1 + R_1 R_2 \geq (w_1 + w_2)(w_3 + w_1) + (w_2 + w_3)(w_1 + w_2) + (w_3 + w_1)(w_2 + w_3) \tag{3.9}$$

We also think it has the following exponential generalization:
Conjecture 3.6. If real number k satisfies $0 < k \leq 2$, then we have

$$\begin{aligned}&(R_2R_3)^k+(R_3R_1)^k+(R_1R_2)^k \ge (w_1+w_2)^k(w_3+w_1)^k+\\&+(w_2+w_3)^k(w_1+w_2)^k+(w_3+w_1)^k(w_2+w_3)^k\end{aligned} \tag{3.10}$$

When $-0.35 \le k < 0$ the inequality is reverse.

On the other hand, we also suppose that the equivalent form of (3.9) can be generalized to the case involving two points:

Conjecture 3.7. For any interior point *P* of ΔABC and arbitrary point *Q*, we have

$$\begin{aligned}&(R_2+R_3)D_1+(R_3+R_1)D_2+(R_1+R_2)D_3 \ge\\&\ge 2(w_1^2+w_2^2+w_3^2)+6(w_2w_3+w_3w_1+w_1w_2)\end{aligned} \tag{3.11}$$

where D_1, D_2, D_3 denote distances from *Q* to the vertices *A*, *B*, *C* respectively.

It seems to be very hard even to prove the following much weaker inequality:

$$(R_2+R_3)D_1+(R_3+R_1)D_2+(R_1+R_2)D_3 \ge 8(r_2r_3+r_3r_1+r_1r_2) \tag{3.12}$$

which is similar the following inequality proved by the authors of [23]:

$$R_1D_1+R_2D_2+R_3D_3 \ge 4(r_2r_3+r_3r_1+r_1r_2) \tag{3.13}$$

When the author studied inequality (3.9), the following interesting inequality was found:

Conjecture 3.8. For any interior point *P* of ΔABC, we have

$$R_1^2+2R_2R_3 \ge w_1^2+w_2^2+w_3^2+3(w_2w_3+w_3w_1+w_1w_2) \tag{3.14}$$

Note that inequality (2.8), we also propose the conjecture:

Conjecture 3.9. For any interior point *P* of ΔABC holds:

$$R_1^2+2R_2R_3 \ge h_ar_1+h_br_2+h_cr_3+r_2r_3+r_3r_1+r_1r_2 \tag{3.15}$$

For inequality (2.8), we pose the following stronger inequality:

Conjecture 3.10. For any interior point *P* of ΔABC holds:

$$R_2R_3+R_3R_1+R_1R_2 \ge w_aw_1+w_bw_2+w_cw_3+w_2w_3+w_3w_1+w_1w_2 \tag{3.16}$$

where w_a, w_b, w_c are the angle bisectors of the ΔABC.

Comparing with (3.15) and (1.2), we suggest that again:

Conjecture 3.11. For any interior point *P* of ΔABC holds:

$$w_aw_1+w_bw_2+w_cw_3 \ge 3(w_2w_3+w_3w_1+w_1w_2) \tag{3.17}$$

If this inequality holds, then it shows that (3.16) is stronger than (1.2).

Finally, we are going to put forward a conjecture involving six segments R_1, R_2, R_3 and r_1, r_2, r_3. It is easy to prove that the following inequality which is similar to the preceding inequality (2.3):

$$R_1\left(R_2+R_3\right)>2r_1\left(r_2+r_3\right) \tag{3.18}$$

where the constant 2 of the right hand side is the best possible. (3.18) is equivalent to

$$\frac{R_2+R_3}{r_2+r_3}-\frac{2r_1}{R_1}>0 \tag{3.19}$$

This strict inequality motivates us to find the following stronger conjecture:

Conjecture 3.12. For any interior point *P* of ΔABC holds:

$$\frac{R_2+R_3}{r_2+r_3}-\frac{2r_1}{R_1}\geq 1 \tag{3.20}$$

It is very interesting that the equality in (3.20) seems to be special. We conjectured that the equality holds if and only if $b=c$ and *P* coincide with a fixed point of the altitude drawn from vertex *A* to the side *BC*. But we do not know what the fixed point is.

REFERENCES

[1] Child, J.M., *Math Gazette*, **23**, 138, 1939.
[2] Carlitz, L., *Amer. Math. Monthly*, **71**, 881, 1964.
[3] Liu, J., *Secondary Education(Math)*, **4**, 29, 1996. (*In Chinese*)
[4] Oppenheim, A., *Amer.Math.Monthly*, **68**, 226, 1961.
[5] Oppenheim, A., *Ann. Math. Pura Appl.*, **53**(4), 157, 1961.
[6] Hamilton, J.M., *An exploration of the the Erdös-Mordell inequality* (Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in the Mathematics Program), 160pp. http://etd.ohiolink.edu/search.cgi?q=inequality.
[7] Zhang, Z.H., Wu, Y.D., *RGMIA Research Report Collection*, **7**(3), Article 8, 2004.
[8] Bottema, O., Djordjevic, R.Z., Janic, R.R., Mitrinovic, D.S., Vasic, P.M., *Geometric Inequalities*, Groningen, 1969.
[9] Mitrinovio, D., Peoaric, J.E., Volenec, V., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1989.
[10] Mordell, L.J., Barrow, D.F., *Amer.Math.Monthly*, **44**, 252, 1937.
[11] Veldkamp, G.R., *Nieuw Tijdschr Wisk*, **45**,193, 1957/1958.
[12] Komornik, V., *Amer.Math.Monthly*, **104**, 57, 1997.
[13] Kazarinoff, D.K., *Mathematical Journal*, **4**, 97, 1957.
[14] Bankoff, L., *Amer.Math.Monthly*, **65**, 521, 1958.
[15] Avez, A., *Amer.Math.Monthly*, **100**, 60, 1993.
[16] Lee, H., *Forum Geom*, **1**, 7, 2001.
[17] Dergiades, N., *Forum Geom*, **4**, 67, 2004.
[18] Alsina, C., Nelsen, R.B., *Forum Geom*, **7**, 99, 2001.
[19] Liu, J., *Journal of Chongqing Normal University (Natural Science Edition)*, **22**(2), 12, 2005. (*in Chinese*)
[20] Liu, J., *RGMIA Research Report Collection*, **14**, Article 13, 2011.
[21] Liu, J., *Mathematics Competition*, **15**, 80, 1992. (*in Chinese*)
[22] Liu, J., *J.Inequal.Pure Appl.Math*, **9**(2), art.58, 2008.
[23] Wu, Y.D., Yu, C.L., Zhang, Z.H., *J.Inequal.Pure Appl.Math*, **10**(4), art.106, 2009.