***ORIGINAL PAPER***

# A FIXED POINT METHOD IN THE STUDY OF AN INTEGRAL EQUATION

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***Abstract.*** *We prove an existence and uniqueness result for an integral equation involving selfadjoint integral operators in* $L^2_{\mathbb{R}}(0,1)$. *The proof of principal result uses a variational method in Hilbert space, finalized by application of Banach fixed point theorem in a specific context, replacing classical approach of linear integral equations by an approach specific especially to nonlinear analysis.*

***Keywords:*** *integral equation, selfadjoint operator, strongly positive linear operator.*

***Mathematics Subject Classification 2010:*** *47G10.*

## 1. INTRODUCTION

Functional analysis, in fact the function spaces and the operators between this spaces, appeared as necessity of finding a general and strong tool in the study of differential or integral equations. For nearly 100 years, functional analysis has developed fulminantly, having a major contribution in other branches of pure and applied mathematics.

Relatively recent monographs [1,2] of Ralph Edwin Showalter confirm the fundamental role of functional methods in the study of nonlinear partial differential equations and integro-differential equations.

Also, the fixed point theorems in Banach or Hilbert spaces (as notable results of functional analysis) contributed to obtain important results regarding the existence, uniqueness and properties of solutions for some concrete equations or variational inequalities. For example, the variational Stampacchia and Lax-Milgram theorems can be regarded as consequences of the famous Banach fixed point theorem – see monograph [3].

Let $K:[0,1]\times[0,1]\to\mathbb{R}$ be a continuous function and suppose that $K$ is symmetric, i.e. $K(s,t)=K(t,s)$ for all $s,t\in[0,1]$. We consider the integral equation

$$u(x)+\int_0^1\int_0^1 K(x,y)K(y,z)u(z)dydz = f(x); x\in[0,1], \qquad (1)$$

where $f\in L^2_{\mathbb{R}}(0,1)$.

The aim of this paper is to prove, using a fixed point method, the unique solvability of integral equation (1). Also, this paper continues the research on fixed point theorems and applications started by papers [4 – 6].

Finally, some remarks regarding existence of inverse for linear perturbations of the identity in Hilbert spaces are deduced, responding to some questions proposed in [7, 8].

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## 2. RESULTS

Let us denote by $H$ the real Hilbert space $L^2_{\mathbb{R}}(0,1)$. The inner product and the correspondent norm in $H$ are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. Let $A : H \to H$ defined by

$$(Au)(s) = \int_0^1 K(s,t)u(t)dt; s \in [0,1].$$

$K$ as a continuous and symmetric function, attracts that $A$ is a well defined linear bounded selfadjoint operator from *H* to *H*. We have

$$\int_0^1\int_0^1 K(x,y)K(y,z)u(z)dydz = \int_0^1 \left[ K(x,y)\int_0^1 K(y,z)u(z)dz \right] dy =$$
$$\int_0^1 K(x,y)(Au)(y)dy = A(A(u))(x).$$

Now equation (1) can be written in the operatorial form

$$(I + A^2)u = f, \tag{2}$$

where $I$ is the identity of $H$.

**Theorem 2.1.** *Equation (2) has an unique solution in* $H$ *for all* $f \in H$.

*Proof:* Using the symmetry of $A$, we have

$$\left\langle (I + A^2)x, x \right\rangle = \langle x, x \rangle + \left\langle A^2 x, x \right\rangle$$
$$= \|x\|^2 + \langle Ax, Ax \rangle = \|x\|^2 + \|Ax\|^2 \geq \|x\|^2 \text{ for all } x \in H,$$

so $I + A^2$ is a strongly positive linear operator. We also have

$$\left\|(I + A^2)x\right\| \leq \|x\| + \left\|A^2 x\right\| \leq$$
$$\leq (1 + \|A\|^2_{L(H)})\|x\|$$

for all $x \in H$, where $L(H)$ is the Banach space of all linear and bounded operators from $H$ to $H$ and

$$\|A\|_{L(H)} = \sup\left\{\|Ax\| / x \in H, \|x\| \leq 1\right\} \overset{not}{=} \alpha .$$

For all $\gamma > 0$ we consider the operators $S_\gamma : H \to H$ defined by

$$S_\gamma x = x - \gamma[(I + A^2)x - f].$$

We obtain

$$\begin{aligned}
\left\|S_\gamma x - S_\gamma y\right\|^2 &= \left\langle S_\gamma x - S_\gamma y, S_\gamma x - S_\gamma y\right\rangle = \\
&= \left\langle x - \gamma(I + A^2)x - y + \gamma(I + A^2)y, x - \gamma(I + A^2)x - y + \gamma(I + A^2)y\right\rangle \\
&= \left\langle (x - y) - \gamma(I + A^2)(x - y), (x - y) - \gamma(I + A^2)(x - y)\right\rangle \\
&= \left\|x - y\right\|^2 - 2\gamma\left\langle (I + A^2)(x - y), x - y\right\rangle + \gamma^2\left\|(I + A^2)(x - y)\right\|^2 \\
&\le \left\|x - y\right\|^2 - 2\gamma\left\|x - y\right\|^2 + \gamma^2\left(1 + \alpha^2\right)^2\left\|x - y\right\|^2 \\
&= [1 - 2\gamma + \gamma^2\left(1 + \alpha^2\right)^2]\left\|x - y\right\|^2,
\end{aligned}$$

so

$$\left\|S_\gamma x - S_\gamma y\right\| \le \sqrt{1 - 2\gamma + \gamma^2\left(1 + \alpha^2\right)^2}\left\|x - y\right\|, \text{ for all } x, y \in H.$$

Now remark that if $\gamma \in \left(0, \frac{2}{\left(1 + \alpha^2\right)^2}\right)$, then $S_\gamma$ is a strict contraction, because $\sqrt{1 - 2\gamma + \gamma^2\left(1 + \alpha^2\right)^2} < 1$.

According to Banach fixed point theorem, it follows that $S_\gamma$ has an unique fixed point, i.e. there exists an unique element $u^* \in H$ such that $S_\gamma u^* = u^*$. We obtain that $u^* - \gamma[(I + A^2)u^* - f] = u^*$, and consequently $u^*$ is the unique solution of equation $u + A^2u = f$. So, the proof of *Theorem 2.1.* is complete.

As consequence of *Theorem 2.1.*, we obtain now the unique solvability of integral equation (1).

## 3. REMARKS

*1.* It is easy to observe that, in the said coditions for $K$ and $f$, *Theorem 2.1.* justifies the unique solvability of integral equation

$$u(x) + \lambda\int_0^1\int_0^1 K(x, y)K(y, z)u(z)dydz = f(x); x \in [0,1],$$

for all $\lambda > 0$.

*2.* As a direct consequence of *Theorem 2.1.* we obtain the following:

**Theorem 3.1.** *Let $H$ be a Hilbert space, $I$ the identity of $H$ and $A: H \to H$ a linear selfadjoint operator from $H$ to $H$. Then the operator $T = I + A^{2n}$ has an inverse for all $n \geq 1$.*

It is clear that this last theorem holds even if $H$ is a complex Hilbert space.

If $H$ is a real Hilbert space and $A: H \to H$ is a linear positive selfadjoint operator from $H$ to $H$, then we have

$$\left\langle A^{2n+1}x, x \right\rangle = \left\langle A(A^n x), A^n x \right\rangle \geq 0 \text{ for all } n \geq 0 .$$

Using the same type of reasoning as in proof of *Theorem 2.1.*, we obtain

**Theorem 3.2.** *Let $H$ be a real Hilbert space, $I$ the identity of $H$ and $A: H \to H$ a linear positive selfadjoint operator from $H$ to $H$. Then the operator $T = I + A^{2n+1}$ has an inverse for all $n \geq 0$.*

This last two results respond to the problem of finding iterative methods for approximate in a Hilbert space $H$ the solutions of linear operator equations of type $x + A^k x = y$, problem proposed in monographs [7, 8].

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