ORIGINAL PAPER

NEW GENERALIZATIONS AND NEW APPROACHES FOR TWO IMO PROBLEMS

DUMITRU M. BATINETU-GIURGIU[1], NECULAI STANCIU[2]

Manuscript received: 18.11.2011; Accepted paper: 14.01.2012;

Published online: 10.03.2012.

„Man muss immer generalisieren”
(Trebuie întotdeauna să generalizăm)
Carl Jacobi (1804 – 1851)

Abstract. *Success in problem solving requires effort. These are not routine exercises. They are problems whose solutions depend of trying something new (like approaches and generalization for IMO problems).This article presents some new approaches for IMO problems.Also we give new generalizations for two IMO problems.In every section, we give example of inequalities by particularization.*

Keywords: *inequality, Radon, Bergström, Bernoulli, IMO.*
2010 Mathematics Subject Classification: *26D15*

1. INTRODUCTION

At 36^{th} IMO(Canada 1995), Russia has proposed the problem:
1) If $a,b,c>0$ and $abc=1$, then

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2} \tag{1.1}$$

At 42^{th} IMO(SUA 2001), South Korea has proposed the problem:
2) If $a,b,c>0$, then

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1 \tag{1.2}$$

Next we plan to give generalizations and new approaches to the problems mentioned above.

[1] Matei Basarab National College, 030674, Bucharest, Romania.
[2] George Emil Palade Secondary School, 120260, Buzau, Romania. E-mail: stanciuneculai@yahoo.com.

2. MAIN RESULTS

We denote $n \in N^* - \{1\}, x_k, y_k \in R_+^*, \forall k = \overline{1,n}, m \in R_+^*,\ p \in [1,\infty)$

$$P_n = \sum_{k=1}^n x_k y_k^{p-1}, R_n = \sum_{k=1}^n y_k^p, S_n = \sum_{k=1}^n x_k^p y_k^{p-1}, T_n = \sum_{k=1}^n x_k^p y_k^p .$$

In support of its purpose, we prove two theorems:

Theorem 2.1.

$$\sum_{k=1}^n \frac{x_k^{m+p}}{y_k^m} \geq \frac{P_n^{m+p}}{R_n^{m+p-1}} \tag{2.1}$$

Proof. The inequality (2.1) is equivalent with the following

$$\sum_{k=1}^n \left(\frac{x_k}{P_n}\right)^{m+p} \frac{R_n^{m+p-1}}{y_k^m} \geq 1 \tag{2.2}$$

which is equivalent with

$$\sum_{k=1}^n \left(\frac{x_k}{P_n} \cdot \frac{R_n}{y_k}\right)^{m+p} \cdot \frac{y_k^p}{R_n} \geq 1 \tag{2.3}$$

From *Bernoulli's* inequality we have

$$\left(\frac{x_k}{P_n} \cdot \frac{R_n}{y_k}\right)^{m+p} = \left(1 + \frac{x_k R_n - y_k P_n}{y_k P_n}\right)^{m+p} \geq 1 + (m+p) \cdot \frac{x_k R_n - y_k P_n}{y_k P_n}, \forall k = \overline{1,n} \tag{2.4}$$

and we obtain

$$\frac{y_k^p}{R_n}\left(\frac{x_k}{P_n} \cdot \frac{R_n}{y_k}\right)^{m+p} \geq \frac{y_k^p}{R_n} + (m+p) \cdot \left(\frac{x_k y_k^{p-1}}{P_n} - \frac{y_k^p}{R_n}\right), k = \overline{1,n} \tag{2.5}$$

By adding the relations (2.5) for $k = \overline{1,n}$ we deduce

$$\sum_{k=1}^n \frac{y_k^p}{R_n} \cdot \left(\frac{x_k}{P_n} \cdot \frac{R_n}{y_k}\right)^{m+p} \geq \frac{1}{R_n} \cdot \sum_{k=1}^n y_k^p + (m+p) \cdot \frac{1}{P_n} \cdot \sum_{k=1}^n x_k y_k^{p-1} - (m+p) \cdot \frac{1}{R_n} \cdot \sum_{k=1}^n y_k^p =$$

$= \frac{1}{R_n} \cdot R_n + (m+p) \cdot \frac{1}{P_n} \cdot P_n - (m+p) \cdot \frac{1}{R_n} \cdot R_n = 1$, which prove (2.3),

so theorem 2.1 is proved. ■

Theorem 2.2.

$$\sum_{k=1}^{n}\frac{x_k^p}{y_k^m}\geq\frac{S_n^{m+p}}{T_n^{m+p-1}} \tag{2.6}$$

Proof. The inequality (2.6) is equivalent with

$$\sum_{k=1}^{n}\frac{x_k^p}{S_n^{m+p}}\cdot\frac{T_n^{m+p-1}}{y_k^m}\geq 1\Leftrightarrow\sum_{k=1}^{n}\left(\frac{x_k}{S_n}\right)^{m+p}\left(\frac{T_n}{x_k y_k}\right)^{m+p-1}(x_k y_k)^{p-1}\geq 1\Leftrightarrow$$

$$\Leftrightarrow\sum_{k=1}^{n}\left(\frac{x_k}{S_n}\cdot\frac{T_n}{x_k y_k}\right)^{m+p}\cdot\frac{(x_k y_k)^p}{T_n}\geq 1\Leftrightarrow\sum_{k=1}^{n}\left(\frac{1}{S_n}\cdot\frac{T_n}{y_k}\right)^{m+p}\cdot\frac{(x_k y_k)^p}{T_n}\geq 1 \tag{2.7}$$

From *Bernoulli's* inequality we have

$$\left(\frac{T_n}{y_k S_n}\right)^{m+p}=\left(1+\frac{T_n-y_k S_n}{y_k S_n}\right)^{m+p}\geq 1+(m+p)\cdot\frac{T_n-y_k S_n}{y_k S_n}=$$

$$=1+(m+p)\cdot\frac{T_n}{y_k S_n}-(m+p),\forall k=\overline{1,n} \tag{2.8}$$

and we obtain

$$\frac{(x_k y_k)^p}{T_n}\cdot\left(\frac{T_n}{y_k S_n}\right)^{m+p}\geq(1-m-p)\cdot\frac{(x_k y_k)^p}{T_n}+(m+p)\cdot\frac{x_k^p y_k^{p-1}}{S_n},\forall k=\overline{1,n} \tag{2.9}$$

By adding the relations (2.9) for $k=\overline{1,n}$ we deduce that

$$\sum_{k=1}^{n}\frac{(x_k y_k)^p}{T_n}\cdot\left(\frac{T_n}{y_k S_n}\right)^{m+p}\geq(1-m-p)\cdot\sum_{k=1}^{n}\frac{(x_k y_k)^p}{T_n}+(m+p)\cdot\sum_{k=1}^{n}\frac{x_k^p y_k^{p-1}}{S_n}=$$

$$=\frac{1-m-p}{T_n}\cdot\sum_{k=1}^{n}(x_k y_k)^p+\frac{m+p}{S_n}\cdot\sum_{k=1}^{n}x_k^p y_k^{p-1}=1,\text{ which prove (2.7),}$$

so theorem 2.2 is proved. ■

Lemma 2.3. *If* $x,y,z\geq 0$, *then*

$$(x+y+z)^3\geq x^3+y^3+z^3+24xyz \tag{2.10}$$

Proof. We have

$$(x+y+z)^3=x^3+3(y+z)x^2+3(y+z)^2x+(y+z)^3=x^3+y^3+z^3+3x^2(y+z)+$$
$$+3x(y+z)^2+3y^2z+3yz^2\geq x^3+y^3+z^3+6x^2\sqrt{yz}+12xyz+6yz\sqrt{yz}=x^3+y^3+$$
$$+z^3+12xyz+6\sqrt{yz}\left(x^2+yz\right)\geq x^3+y^3+z^3+12xyz+12\sqrt{x^2yz}\cdot\sqrt{yz}=$$
$$=x^3+y^3+z^3+24xyz.$$

Remark 2.4. Theorem 2.1 and 2.2 are equivalent.

Theorem 2.1 $\Rightarrow$ *Theorem 2.2*

Indeed, in (2.1) subsitute y_k with $y_k x_k$, $\forall k=\overline{1,n}$, and we deduce that

$$\sum_{k=1}^{n}\frac{x_k^{m+p}}{(x_k y_k)^m}=\sum_{k=1}^{n}\frac{x_k^p}{y_k^m}\geq\frac{\left(\sum_{k=1}^{n}x_k(x_k y_k)^{p-1}\right)^{m+p}}{\left(\sum_{k=1}^{n}(x_k y_k)^p\right)^{m+p-1}}=\frac{S_n^{m+p}}{T_n^{m+p-1}}.$$

Theorem 2 $\Rightarrow$ *Theorem 1.*

Indeed, if in (2.6) substitute y_k with $\frac{y_k}{x_k}$, $\forall k=\overline{1,n}$, then we deduce that

$$\sum_{k=1}^{n}\frac{x_k^p}{(y_k x_k^{-1})^m}=\sum_{k=1}^{n}\frac{x_k^{m+p}}{y_k^m}\geq\frac{\left(\sum_{k=1}^{n}x_k^p(y_k x_k^{-1})^{p-1}\right)^{m+p}}{\left(\sum_{k=1}^{n}x_k^p(y_k x_k^{-1})^p\right)^{m+p-1}}=\frac{P_n^{m+p}}{R_n^{m+p-1}}. \qquad \blacksquare$$

Remarks 2.5.

If we take $p=1$, then by (2.1) we obtain

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{y_k^m}\geq\frac{\left(\sum_{k=1}^{n}x_k\right)^{m+1}}{\left(\sum_{k=1}^{n}y_k\right)^m}=\frac{X_n^{m+1}}{Y_n^m} \tag{2.11}$$

i.e. *J. Radon's inequality*.

If we take $p=1$, then by (2.6) we deduce that

$$\sum_{k=1}^{n}\frac{x_k}{y_k^m}\geq\frac{\left(\sum_{k=1}^{n}x_k\right)^{m+1}}{\left(\sum_{k=1}^{n}x_k y_k\right)^m}=\frac{X_n^{m+1}}{T_n^m} \tag{2.12}$$

If we take $m=1$, then by (2.11) we obtain *Bergström's inequality,* which is equivalent with *Cauchy-Buniakovski-Schwarz's inequality*(C-B-S).

$$\sum_{k=1}^{n}\frac{x_k^2}{y_k} \geq \frac{\left(\sum_{k=1}^{n}x_k\right)^2}{\sum_{k=1}^{n}y_k} = \frac{X_n^2}{Y_n} \tag{2.13}$$

If we take $m=1$, then by (2.12) we obtain that

$$\left(\sum_{k=1}^{n}\frac{x_k}{y_k}\right)\cdot\left(\sum_{k=1}^{n}x_k y_k\right) \geq \left(\sum_{k=1}^{n}x_k\right)^2 = X_n^2 \tag{2.14}$$

3. APPLICATIONS

Theorem 3.1.

If $a,b,m \in R_+$, $a+b, x_k, u_k \in R_+^*, \forall k=\overline{1,n}, u_{n+1}=u_1, u_{n+2}=u_2$ and $X_n=\sum_{k=1}^{n}x_k$,

then

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{u_{k+1}^m(au_k+bu_{k+2})^m} \geq \frac{X_n^{m+1}}{(a+b)^m\left(\sum_{k=1}^{n}u_k u_{k+1}\right)^m} \tag{3.1}$$

Proof. In we take $y_k = au_k u_{k+1} + bu_{k+1}u_{k+2}$, $\forall k=\overline{1,n}$, then by (2.11) we deduce that

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{u_{k+1}^m(au_k+bu_{k+2})^m} \geq \frac{X_k^{m+1}}{\sum_{k=1}^{n}(au_k u_{k+1}+bu_k u_{k+2})^m} = \frac{X_n^{m+1}}{\left((a+b)\sum_{k=1}^{n}u_k u_{k+1}\right)^m} =$$

$$= \frac{X_n^{m+1}}{(a+b)^m\left(\sum_{k=1}^{n}u_k u_{k+1}\right)^m}.$$

■

Remarks 3.2.

If we take $x_k=\frac{1}{v_k}$, $v_k \in R_+^*, \forall k=\overline{1,n}$, then by (3.1) it results that

$$\sum_{k=1}^{n}\frac{1}{u_{k+1}^m v_k^{m+1}(au_k+bu_{k+2})^m} \geq \frac{\left(\sum_{k=1}^{n}\frac{1}{v_k}\right)^{m+1}}{(a+b)^m\left(\sum_{k=1}^{n}u_k u_{k+1}\right)^m} \tag{3.2}$$

If we take $n=3$ and exists $t\in R_+^*$ such that $\sum_{k=1}^{3}\frac{1}{v_k}\ge t\cdot\sum_{k=1}^{3}\frac{1}{u_k}$, then by (3.2) we obtain

$$\sum_{k=1}^{3}\frac{1}{u_{k+1}^m v_k^{m+1}(au_k+bu_{k+2})^m}\ge\frac{t^{m+1}\left(\sum_{k=1}^{3}\frac{1}{u_k}\right)^{m+1}}{(a+b)^m\left(\sum_{k=1}^{3}u_ku_{k+1}\right)^m}=$$

$$=\frac{t^{m+1}}{(a+b)^m}\cdot\frac{\left(\sum_{k=1}^{3}u_ku_{k+1}\right)^{m+1}}{\left(\sum_{k=1}^{3}u_ku_{k+1}\right)^m}\cdot\frac{1}{(u_1u_2u_3)^{m+1}}=\frac{t^{m+1}}{(a+b)^m}\cdot\frac{(u_1u_2+u_2u_3+u_3u_1)}{(u_1u_2u_3)^{m+1}}\ge$$

$$\ge\frac{3t^{m+1}}{(a+b)^m}\cdot\frac{\sqrt[3]{(u_1u_2u_3)^2}}{(u_1u_2u_3)^{m+1}}=\frac{3t^{m+1}}{(a+b)^m}\cdot(u_1u_2u_3)^{-\frac{3m+1}{3}}\tag{3.3}$$

If we take $u_{k+1}=v_k$, $\forall k=\overline{1,n}$, then by (3.3) it results $t=1$, so we have

$$\sum_{k=1}^{3}\frac{1}{u_{k+1}^{2m+1}(au_k+bu_{k+2})^m}\ge\frac{3}{(a+b)^m}\cdot(u_1u_2u_3)^{-\frac{3m+1}{3}}\tag{3.4}$$

If we take $u_1u_2u_3=1$, then by (3.4) it results that

$$\sum_{k=1}^{3}\frac{1}{u_{k+1}^{2m+1}(au_k+bu_{k+2})^m}\ge\frac{3}{(a+b)^m}\tag{3.5}$$

If we take $a=b=1$, then by (3.5) we deduce that

$$\sum_{k=1}^{3}\frac{1}{u_{k+1}^{2m+1}(u_k+u_{k+2})^m}\ge\frac{3}{2^m}\tag{3.6}$$

The inequality (3.6) is the inequality (6) from $[8]$, and if we take

$m=1,u_2=a,u_1=b,u_3=c$, we solved the problem 1. ■

We denote $a,b,m\in R_+,a+b,t,z_k,u_k,v_k,w_k\in R_+^*,\forall k=\overline{1,n},u_{n+1}=u_1,u_{n+2}=u_2$, $w_{n+1}=w_1,w_{n+2}=w_2$.

Theorem 3.3.

If $\sum_{k=1}^{n}\frac{z_k}{v_k}\ge t\cdot\sum_{k=1}^{n}\frac{w_k}{u_k}$, then

$$\sum_{k=1}^{n} \frac{z_k^{m+1}(u_{k+1}u_{k+2})^m}{v_k^{m+1}(au_{k+2}w_{k+1}+bu_{k+1}w_{k+2})^m} \geq \frac{nt^{m+1}}{(a+b)^m}\sqrt[n]{\prod_{k=1}^{n}\frac{w_k}{u_k}} \tag{3.7}$$

Proof. If we take

$$x_k = \frac{z_k(u_{k+1}u_{k+2})^{\frac{m}{m+1}}}{v_k},\ y_k = au_{k+2}w_{k+1} + bu_{k+1}w_{k+2}, \forall k = \overline{1,n},$$

then by (2.11) we deduce that

$$\sum_{k=1}^{n} \frac{z_k^{m+1}(u_{k+1}u_{k+2})^m}{v_k^{m+1}(au_{k+2}w_{k+1}+bu_{k+1}w_{k+2})^m} = \sum_{k=1}^{n}\frac{\left(\frac{z_k}{v_k}\right)^{m+1}}{\left(a\cdot\frac{w_{k+1}}{u_{k+1}}+b\cdot\frac{w_{k+2}}{u_{k+2}}\right)^m} \geq$$

$$\geq \frac{\left(\sum_{k=1}^{n}\frac{z_k}{v_k}\right)^{m+1}}{\sum_{k=1}^{n}\left(a\cdot\frac{w_{k+1}}{u_{k+1}}+b\cdot\frac{w_{k+2}}{u_{k+2}}\right)^m} = \frac{\left(\sum_{k=1}^{n}\frac{z_k}{v_k}\right)^{m+1}}{(a+b)^m\left(\sum_{k=1}^{n}\frac{w_k}{u_k}\right)^m} \geq$$

$$\geq \frac{t^{m+1}}{(a+b)^m}\cdot\frac{\left(\sum_{k=1}^{n}\frac{w_k}{u_k}\right)^{m+1}}{\left(\sum_{k=1}^{n}\frac{w_k}{u_k}\right)^m} = \frac{t^{m+1}}{(a+b)^m}\cdot\left(\sum_{k=1}^{n}\frac{w_k}{u_k}\right) \tag{3.8}$$

If in (3.8) apply AM-GM inequality we obtain

$$\sum_{k=1}^{n} \frac{z_k^{m+1}(u_{k+1}u_{k+2})^m}{v_k^{m+1}(au_{k+2}w_{k+1}+bu_{k+1}w_{k+2})^m} \geq \frac{nt^{m+1}}{(a+b)^m}\sqrt[n]{\prod_{k=1}^{n}\frac{w_k}{u_k}}\ ,\ \text{i.e. (3.7).} \qquad \blacksquare$$

Remarks 3.4.

If we have $u_1 \cdot u_2 \cdot ... \cdot u_n = w_1 \cdot w_2 \cdot ... \cdot w_n$, then by (3.8) we obtain that

$$\sum_{k=1}^{n} \frac{z_k^{m+1}(u_{k+1}u_{k+2})^m}{v_k^{m+1}(au_{k+2}w_{k+1}+bu_{k+1}w_{k+2})^m} \geq \frac{nt^{m+1}}{(a+b)^m} \tag{3.9}$$

If we take $n = 3,\ z_k = w_k = 1, u_k = v_k, \forall k = \overline{1,n}$, then by (3.9) we deduce that $t = 1$ and

$$\sum_{k=1}^{3} \frac{(u_{k+1}u_{k+2})^m}{u_k^{m+1}(au_{k+2}+bu_{k+1})^m} \geq \frac{3}{(a+b)^m} \Leftrightarrow (u_1u_2u_3)^m \sum_{k=1}^{3}\frac{1}{u_k^{2m+1}(au_{k+2}+bu_{k+1})^m} \geq \frac{3}{(a+b)^m},$$

and for $u_1u_2u_3 = 1$ we obtain that

$$\sum_{k=1}^{3}\frac{1}{u_k^{2m+1}\left(au_{k+2}+bu_{k+1}\right)^m} \geq \frac{3}{(a+b)^m} \tag{3.10}$$

If we take $a = b = 1$, then by (3.10) we deduce that

$$\sum_{k=1}^{3}\frac{1}{u_k^{2m+1}\left(u_{k+2}+u_{k+1}\right)^m} \geq \frac{3}{2^m} \tag{3.11}$$

If we take $u_1 = a, u_2 = b, u_3 = c$, then by (3.11) we obtain that

$$\frac{1}{a^{2m+1}(b+c)^m}+\frac{1}{b^{2m+1}(c+a)^m}+\frac{1}{c^{2m+1}(a+b)^m} \geq \frac{3}{2^m} \tag{3.12}$$

The inequality (3.12) is inequality (6) from $[8]$, and if we take $m = 1$ we obtain (1.1), i.e. the problem 1 is again solved. ■

We denote $n \in N^* - \{1\}, a_k, b_k, t_k \in R_+^*, \forall k = \overline{1,n}$.

Theorem 3.5.

If $a_k = b_k \cdot \sum_{k=1}^{n} a_k = s_n b_k, \forall k = \overline{1,n} \left(s_n = \sum_{k=1}^{n} a_k \right)$ then

$$\sum_{k=1}^{n}\frac{a_k}{\left(\sqrt{a_k^2+t_k a_{k+1}a_{k+2}}\right)^m} = s_n^{1-m}\cdot\sum_{k=1}^{n}\frac{b_k}{\left(\sqrt{b_k^2+t_{k+1}b_{k+1}b_{k+2}}\right)^m} \tag{3.13}$$

where is obviously that $\sum_{k=1}^{n} b_k = 1$.

The proof is immediate.

Remarks 3.6.

$$\sum_{k=1}^{n}\frac{b_k}{\left(\sqrt{b_k^2+t_k b_{k+1}b_{k+2}}\right)^m} = \sum_{k=1}^{n}\frac{b_k^{m+1}}{\left(b_k\sqrt{b_k^2+t_{k+1}b_{k+1}b_{k+2}}\right)^m} =$$

$$= 2^m\cdot\sum_{k=1}^{n}\frac{b_k^{m+1}}{\left(2\sqrt{b_k}\cdot\sqrt{b_k^3+t_k b_k b_{k+1}b_{k+2}}\right)^m} \tag{3.14}$$

Since $2\sqrt{x}\cdot\sqrt{y} \leq x+y$, it results that

$$2\sqrt{b_k}\cdot\sqrt{b_k^3+t_k b_k b_{k+1}b_{k+2}} \geq b_k + b_k^3 + t_k b_k b_{k+1}b_{k+2}, \forall k = \overline{1,n},$$

and (3.14) becomes

$$=\frac{2^m}{\left(1+\sum_{k=1}^{n} b_k^3+\sum_{k=1}^{n} t_k b_k b_{k+1} b_{k+2}\right)^m} \tag{3.15}$$

If, $t_k \in [0,8], \forall k=\overline{1,n}$ then (3.15) becomes

$$\sum_{k=1}^{n} \frac{b_k}{\left(\sqrt{b_k^2+t_k b_{k+1} b_{k+2}}\right)^m} \geq \frac{2^m}{\left(1+\sum_{k=1}^{n} b_k^3+8 \cdot \sum_{k=1}^{n} b_k b_{k+1} b_{k+2}\right)^m} \tag{3.16}$$

For $n=3$, (3.16) becomes

$$\sum_{k=1}^{3} \frac{b_k}{\left(\sqrt{b_k^2+t_k b_{k+1} b_{k+2}}\right)^m} \geq \frac{2^m}{\left(1+\sum_{k=1}^{3} b_k^3+8 \cdot \sum_{k=1}^{3} b_k b_{k+1} b_{k+2}\right)^m}=$$

$$=\frac{2^m}{\left(1+\sum_{k=1}^{3} b_k^3+24 b_1 b_2 b_3\right)^m} \tag{3.17}$$

From (2.10) we have

$$\left(\sum_{k=1}^{3} b_k\right)^3 \geq \sum_{k=1}^{3} b_k^3+24 b_1 b_2 b_3$$

and the inequality (3.17) becomes

$$\sum_{k=1}^{3} \frac{b_k}{\left(\sqrt{b_k^2+t_k b_{k+1} b_{k+2}}\right)^m} \geq \frac{2^m}{\left(1+\left(\sum_{k=1}^{3} b_k\right)^3\right)^m}=\frac{2^m}{(1+1)^m}=1 \tag{3.18}$$

From (3.13) and (3.18) it results that

$$\sum_{k=1}^{n} \frac{a_k}{\left(\sqrt{a_k^2+t_{k+1} a_{k+1} a_{k+2}}\right)^m} \geq s_3^{1-m}=\left(a_1+a_2+a_3\right)^{1-m} \tag{3.19}$$

For $m=1$, (3.19) becomes

$$\sum_{k=1}^{3} \frac{a_k}{\sqrt{a_k^2+t_k a_{k+1} a_{k+2}}} \geq 1 \tag{3.20}$$

If we take $t_1 = t_2 = t_3 = 8$, then by (3.20) and we deduce that

$$\sum_{k=1}^{3} \frac{a_k}{\sqrt{a_k^2 + 8a_{k+1}a_{k+2}}} \geq 1 \tag{3.21}$$

which is equivalent with (1.2), i.e. the problem 2 is solved. ■

FINAL REMARKS

For the inequality (1.1) were given one solution at [2], other solution at [5], and a generalization at [3, 4, 8] and recently at [10].

For the inequality (1.2) were given six solutions at [1], five solutions at [6] and a generalization at [7 and 9].

REFERENCES

[1] Andreescu, T., *Gaz.Mat.-seria B*, **107**(1), 1, 2002.
[2] Bătineţu– Giurgiu , D.M., *Revista Sinus*, **3**(9), 1, 2007.
[3] Bătineţu– Giurgiu , D.M., *Revista Sinus*, **5**(13), 9, 2009.
[4] Bătineţu– Giurgiu , D.M., *Revista Sinus* ,**5**(14), 3, 2009.
[5] Becheanu, M., Enescu , *Gaz.Mat.-seria B*, **100**(9), 515, 1995.
[6] Becheanu, M., *Gaz.Mat.-seria B*, 106(9-10), 326, 2001.
[7] Dincă, M., *Gaz.Mat.-seria B*, **107**(3), 103, 2002.
[8] Panaitopol, L., *Gaz.Mat.-seria B*, **107**(4), 145, 2002.
[9] Satnoianu, R., *Gaz.Mat.-seria B*, **106**(11), 390, 2001.
[10] Stanciu, N., Bătineţu– Giurgiu , D.M., *Gaz.Mat.-seria B*, **116** (5), 225, 2011.