ORIGINAL PAPER

NONNEGATIVE SOLUTIONS IN TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract. *We consider the two point boundary value problem:*

$$-u''(x) = \lambda f(u(x)); \qquad x \in (-1,1),$$
$$u(-1) = 0 = u(1),$$

where f:[0,∞)→R is a twice differentiable function and $\lambda > 0$ *is a positive parameter. We discuss the cases when* $f(0) > 0$ *(positone) and* $f(0) < 0$ *(semipositone). We obtain exact number of positive solutions in any case.*

Keywords: *positone, semipositone, autonomous problem.*

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1. INTRODUCTION

Here we consider the autonomous two point boundary value problem

$$-u''(x) = \lambda f(u(x)); \qquad x \in (-1,1), \tag{1}$$
$$u(-1) = 0 = u(1), \tag{2}$$

where λ is a positive parameter and f is a smooth function. We define g by $g(t) = f(t)/t$ and F by $F(t) = \int_0^t f(s)\,ds$.

We analyse in detail the nonnegative solutions to (1), (2). In this paper we employ the Quadrature Method in [1, 2]. Our results on positive solutions are in contrast to the case of semipositone (see [1, 2]) where $f'' < 0$ guaranteed existence and multiplicity. Our methods are based on building a Quadrature Method for such explosive solutions.

Furthermore we produce a very easy solution way for to obtain the exact number of positive solutions in turning points and also for any $\lambda > 0$.

We will discuss the Quadrature Method in section 2, the statements and discussion of the main results in section 3, proofs of main results in section 4, and finally the discussion of the complete bifurcation curve of nonnegative solutions for the special case $f(u) = e^{-u}$, in section 5.

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2. QUADRATURE METHOD

First, note that any solution $u(x)$ of (1), (2) is symmetric about any point $x_0 \in (-1,1)$ such that $u'(x_0)=0$. That is, $u(x)$ must achieve its maximum at $x=0$. Multiplying (1) by $u'(x)$ and integrating, we obtain

$$-\left[u'(x)\right]^2/2 = \lambda F\left(u(x)\right)+c. \tag{3}$$

Since positive solutions are known to be symmetric with respect to $x=0$ and $u'(x)>0$ for $x\in(-1,0)$ we have $\rho := \sup_{x\in(-1,1)} u(x) = u(0)$. Taking $x=0$ in (3) implies that

$$u'(x)=\sqrt{2\lambda\left[F(\rho)-F(u)\right]}; \qquad x\in[-1,0]. \tag{4}$$

Now integrating (4) over $[-1,x]$, we obtain

$$\int_0^{u(x)} \frac{du}{\sqrt{F(\rho)-F(u)}} = \sqrt{2\lambda}\,(x+1); \qquad x\in[-1,0], \tag{5}$$

which in turn implies that

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}}\int_0^{\rho} \frac{du}{\sqrt{F(\rho)-F(u)}} := G(\rho), \tag{6}$$

by taking $x=0$ in (5). Hence for any $\lambda>0$ if there exists a $\rho\in(0,+\infty)$ with $G(\rho)=\sqrt{\lambda}$, then (1), (2) has a positive solution $u(x)$ given by (5) satisfying

$$\sup\left\{u(x)/\,x\in(-1,1)\right\} = u(0) = \rho.$$

In fact, $G(\rho)$ is a continuous function which is differentiable over $(0,+\infty)$ with

$$\frac{d}{d\rho}G(\rho) = \frac{1}{\sqrt{2}}\int_0^1 \frac{H(\rho)-H(\rho\nu)}{\left[F(\rho)-F(\rho\nu)\right]^{3/2}}\,d\nu, \tag{7}$$

where

$$H(t)=F(t)-(t/2)f(t). \tag{8}$$

For $\rho\in(0,+\infty)$, we recall from (6) that

$$G(\rho)=\frac{1}{\sqrt{2}}\frac{\rho}{\sqrt{F(\rho)}}\int_0^1 \frac{d\nu}{\sqrt{1-\left[F(\rho\nu)/F(\rho)\right]}}. \tag{9}$$

3. MAIN RESULTS

Theorem 3.1. If $f(0)>0$ (positone), $\lim_{t\to+\infty} f(t)=M$ where $0\le M<f(0)$ and $f:[0,+\infty)\to R$ is monotonically decreasing, then (1)-(2) have a unique positive solution for any $\lambda>0$. Also, $\lim_{\lambda\to 0}\rho_\lambda = 0$, and $\lim_{\lambda\to+\infty}\rho_\lambda = +\infty$.

Theorem 3.2. If $f(0)<0$ (semipositone), $f''(s)<0$ for $s\in(0,s_0)$ with $s_0>\theta$ (where β and θ denote the unique positive zeros of *f* and *F* respectively), $f''(s)>0$ for $s>s_0$, $(f(\theta)/\theta)<f'(\theta)$, and if there exists $\sigma>\theta$ such that

$$H(\sigma)=F(\theta)-(\sigma/2)f(\sigma)>0,\ \lim_{s\to+\infty}(f(s)/s)=+\infty \text{ and}$$

$$\lim_{s\to+\infty} f(s)-\lim_{s\to+\infty} sf'(s)<0$$

then there exists λ^*, λ_1, λ_2 such that $0<\lambda_1<\lambda_2$, $\lambda_1\le\lambda^*\le\lambda_2$ and (1)-(2) have a unique positive solution for $0<\lambda<\lambda_1$ and no positive solutions for $\lambda>\lambda_2$. Furthere, there exists a range for λ in (λ_1,λ^*) in which (1)-(2) have exactly three positive solutions. If $\lambda_2=\lambda^*$ then (1)-(2) have exactly two positive solutions for $\lambda=\lambda^*$. If $\lambda_2<\lambda^*$ then (1)-(2) have exactly one positive solution for $\lambda\in(\lambda_2,\lambda^*]$, and finally if $\lambda_2>\lambda^*$ then (1)-(2) have exactly two positive solutions for $\lambda\in(\lambda^*,\lambda_2)$, and in this case (1)-(2) have exactly three positive solutions for $\lambda=\lambda^*$.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1. Firstly, note that from hypotheses we $\lim_{t\to0+} g(t)=+\infty$, $\lim_{t\to+\infty} g(t)=0$, and $g'(t)<0$ for all $t>0$. Hence $H'(t)>0$ for all $t>0$ and $H''(t)<0$ for all $t>0$. Also, we have $H(0)=0$. Consequently $G''(\rho)>0$ for any $\rho\in(0,+\infty)$. Next, let $L(v):=F(\rho v)/F(\rho)$. Hence $L(v)\ge v$ for $v\in[0,1]$. Consequently from (9) we have

$$G(\rho)\ge\left(1/\sqrt{2}\right)\left(\rho/\sqrt{F(\rho)}\right)\int_0^1\frac{dv}{\sqrt{1-v}}=\sqrt{2}\left(\rho/\sqrt{F(\rho)}\right).$$

But since $\lim_{t\to+\infty} f(t)=M$; $0<M\le f(0)$, we have

$$\lim_{\rho\to+\infty}\rho^2/F(\rho)=\lim_{\rho\to+\infty}2\rho/f(\rho)=+\infty.$$

and hence $\lim_{\rho\to+\infty}G(\rho)=+\infty$. Finally, it remains to prove that $\lim_{\rho\to0+}G(\rho)=0$. Since $\lim_{t\to0+} f(t)/t=+\infty$, coseqently we have

$$\lim_{\rho\to0+}G(\rho)=\lim_{\rho\to0+}\left(1/\sqrt{2}\right)\left(\rho/\sqrt{F(\rho)}\right)\int_0^1\frac{dv}{\sqrt{1-v}}=\lim_{\rho\to0+}\sqrt{2}\left(\rho/\sqrt{F(\rho)}\right)=0.$$

Hence Theorem 3.1 is proved. Δ

Proof of Theorem 3.2. Note that hypotheses imply that there exist σ_1, σ_2, σ_3, σ_4 such that $H(\rho)-H(\rho v)<0$ for $v\in[0,1)$ when $\rho\in(\theta,\sigma_1]$ and when $\rho>\sigma_4$, while $H(\rho)-H(\rho v)>0$ for $v\in[0,1)$ when $\rho\in(\sigma_2,\sigma_3]$. Consequently $G'(\rho)<0$ for $\rho\in(\theta,\sigma_1]$ and for $\rho>\sigma_4$, while $G'(\rho)>0$ for $\rho\in(\sigma_2,\sigma_3]$. Now to complete the proof of Theorem 3.2 it remains to prove that $\lim_{\rho\to+\infty}G(\rho)=0$ and $G(\rho)$ has a unique minimum point and a unique maximum point. Since $\lim_{t\to+\infty} f(t)=+\infty$ then $\lim_{t\to+\infty}F(t)/t=+\infty$. Consequently we have

$$\lim_{\rho\to+\infty} G(\rho) = \lim_{\rho\to+\infty}\left(1/\sqrt{2}\right)\left(\rho/\sqrt{F(\rho)}\right)\int_0^1 \frac{d\nu}{\sqrt{1-\nu}} = \lim_{\rho\to+\infty}\sqrt{2}\left(\rho/\sqrt{F(\rho)}\right).$$

But $\lim_{\rho\to+\infty} F(\rho) = +\infty$ and $\lim_{\rho\to+\infty} \rho^2/F(\rho) = \lim_{\rho\to+\infty} 2\rho/f(\rho) = 0$. Thus $\lim_{\rho\to+\infty} G(\rho) = 0$ easily follows.

But $G(\rho)$ has a unique minimum point and a unique maximum point, that is clear from on diagrams $K(\nu) := H(\rho_0) - H(\rho_0\nu)$ where in that ρ_0 can be in any one of following cases:

(1). $\rho_0 \in (\sigma_1, \sigma_2)$ (2). $\rho_0 \in (\sigma_3, \delta)$ (3). $\rho_0 = \delta$ (4). $\rho_0 \in (\delta, \sigma_4)$.

Of course δ is which that $\delta \in (\sigma_3, \sigma_4)$ and $H(\delta) = 0$ [2].

Hence Theorem 3.2 is proved. Δ

5. EXAMPLE

Consider the problem

$$-u'' = \lambda e^{-u}$$
$$u(-1) = 0 = u(1).$$

This example for which $f(u) = e^{-u}$ demonstrates Theorem 3.1 since $f(0) = 1 > 0$, $\lim_{u\to+\infty} f(u)/u = 0$, i.e. $M = 0 < 1 = f(0)$, and $f(u) = e^{-u}$ is decreasing for $u > 0$. Note that $F(u) = -e^{-u} + 1$ implies

$$G(\rho) = \frac{1}{\sqrt{2}}\int_0^1 \frac{du}{\sqrt{e^{-u} - e^{-\rho}}}.$$

Letting $w = e^{-u/2}$ we obtain

$$G(\rho) = -\sqrt{2}\int_{\sec^{-1}\left(e^{\rho/2}\right)}^{0} \frac{e^{-\rho/2}\sec\theta\tan\theta d\theta}{e^{-\rho/2}\sec\theta\sqrt{e^{-\rho}}\tan^2\theta} = \sqrt{2}e^{\rho/2}\int_0^{\sec^{-1}\left(e^{\rho/2}\right)} d\theta = \sqrt{2}e^{\rho/2}\sec^{-1}\left(e^{\rho/2}\right).$$

Hence, $\lim_{\rho\to 0+} G(\rho) = 0$, and $\lim_{\rho\to+\infty} G(\rho) = +\infty$.

Consequently, this example shows truth of Theorem 3.1.

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