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DIGIT-AMICABLE NUMBERS

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Abstract. *The aim of this paper is to establish some interesting arithmetic properties of a class of integers.*

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1. INTRODUCTION

The starting point of this paper is a problem of Kordemski [5] who investigated whether there are four digit numbers which are equal to the cube of their digit sum, that is numbers $\overline{abcd}$ such that

$$\overline{abcd} = (a+b+c+d)^3 \tag{1.1}$$

Subsequently, Prokofiev [7] became preoccupied for finding numbers $\overline{a_1 a_2 ... a_n}$, with $n \geq 2$ such that

$$\overline{a_1 a_2 ... a_n} = (a_1 + a_2 + ... + a_n)^3 \tag{1.2}$$

Kordemski [5] found the unique solution of (1.1)

$$4913 = (4+9+1+3)^3$$

and Nesterciuk [6] solved (2) in the case of $n = 5$, by indicating the solutions

$$19683 = (1+9+6+8+3)^3 \ , \ \ 17576 = (1+7+5+7+6)^3$$

Every solution $\overline{a_1 a_2 ... a_n}$ of problem (1.2) is a cube. As a consequence, the cases $n = 2$ and $n = 3$ can be verified directly. Thus (1.2) has no solution in the case $n = 2$, and the unique solution in the case $n = 3$ is

$$512 = (5+1+2)^3$$

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Equation (1.2) has no solutions for $n \geq 7$, since

$$\overline{a_1 a_2 ... a_n} \geq 10^{n-1} > (9n)^3 \geq (a_1 + a_2 + ... + a_n)^3.$$

The middle inequality $10^{n-1} > (9n)^3$ is true for $n = 7$ and it can be proved by induction, for $n \geq 7$, using

$$10\,(9n)^3 > (9(n+1))^3.$$

Since, with $n = k + 7$, we get

$$10\,(9n)^3 - (9\,(n+1))^3 = 2127222 + 931662k + 135594k^2 + 6561k^3 > 0$$

This type of problem were further studied and extended by many authors. Starting from the surprising equality

$$2401 = (2 + 4 + 0 + 1)^4,$$

Nesterciuk [6] studied the equation

$$\overline{abcd} = (a + b + c + d)^4 \tag{1.3}$$

Kajibaev [4] found by trials the following new solutions

$$234256 = (2 + 3 + 4 + 2 + 5 + 6)^4$$
$$390625 = (3 + 9 + 0 + 6 + 2 + 5)^4$$

but (1.3) was completely solved by Acu [1] who found the last two solutions

$$614656 = (6 + 1 + 4 + 6 + 5 + 6)^4$$
$$1679616 = (1 + 6 + 7 + 9 + 6 + 1 + 6)^4$$

of the following generalization of (1.3)

$$\overline{a_1 a_2 ... a_n} = (a_1 + a_2 + ... + a_n)^4$$

It is proven in Matematika v skole, 2, 76, 1974, that the problem $\overline{a_1 a_2 ... a_n} = (a_1 + a_2 + ... + a_n)^n$, $n \geq 2$, has only the solutions

$$81 = (8 + 1)^2$$
$$512 = (5 + 1 + 2)^3$$
$$2401 = (2 + 4 + 0 + 1)^4.$$

This type of problem has attracted the attention of several authors. We cite for example the work of Berinde [3], who asked whether there exist numbers $\overline{abc}$ such that both $\overline{abc}$ and $\overline{cba}$ are divisible by $(a+b+c)^2$.

2. DIGIT-AMICABLE NUMBERS

We prove the following

Theorem 1. *The equation*

$$\overline{a_1a_2...a_n} = (a_1 + a_2 + ... + a_n)^2 \tag{2.1}$$

has a unique solution, namely $81 = (8+1)^2$.

Proof: For every $n \geq 5$ we have

$$\overline{a_1a_2...a_n} \geq 10^{n-1} > (9n)^2 \geq (a_1 + a_2 + ... + a_n)^2$$

The middle inequality $10^{n-1} > (9n)^2$ is true for $n = 5$ and it can be proved by induction, for $n \geq 5$, using

$$10\,(9n)^2 > (9(n+1))^2 .$$

Indeed, with $n = k + 5$, we get

$$10\,(9n)^2 - (9\,(n+1))^2 = 17334 + 7128k + 729k^2 .$$

Consequently, the given equation has no solutions for $n \geq 5$.

Cases $n = 2$ and $n = 3$ can be verified by direct computation, taking into account that $\overline{a_1a_2...a_n}$ is a perfect square. There are no solutions.

In the case $n = 4$, as $\overline{a_1a_2a_3a_4} \geq 1000$, it follows that

$$32 \leq a_1 + a_2 + a_3 + a_4 < 4 \cdot 9 = 36 ,$$

so $a_1 + a_2 + a_3 + a_4 \in \{32, 33, 34, 35, 36\}$.

There are no solutions and the theorem is proved. □

Starting from (2.1) and inspired by the notion of a *pair of amicable numbers*, namely two different numbers such that the sum of the proper divisors of each is equal to the other number, we say that a pair of distinct numbers $\overline{a_1a_2...a_m}$ and $\overline{b_1b_2...b_n}$ are *digit-amicable numbers* if

$$\overline{a_1a_2...a_m} = (b_1 + b_2 + ... + b_n)^2 \text{ and } \overline{b_1b_2...b_n} = (a_1 + a_2 + ... + a_m)^2 . \tag{2.2}$$

In this case, we have

$$10^{m-1} \leq 81n^2 , \text{ and } 10^{n-1} \leq 81m^2 . \tag{2.3}$$

Multiplying and using the AM-GM inequality, we get

$$10^{m+n=2} \leq 81^2 (mn)^2 \leq 81^2 \left(\frac{m+n}{2} \right)^4,$$

thus

$$10^{m+n=2} \leq \frac{6561}{16} (m+n)^4. \qquad (2.4)$$

Lemma 1. For all integers $s \geq 9$ we have

$$10^{s=2} > \frac{6561}{16} s^4.$$

Proof. The lemma is true for $s = 9$. Assuming that it is true for some $s \geq 9$, we have

$$\left(1 + \frac{1}{s}\right)^4 \leq \left(1 + \frac{1}{9}\right)^9 = 1.5242... < 10,$$

or $(s+1)^4 < 10 s^4$, and the lemma is proved. $\square$

As a direct consequence of (2.4) and Lemma 1, if $\overline{a_1 a_2 ... a_m}$ and $\overline{b_1 b_2 ... b_n}$ are *amicable-digit numbers* then $m + n \leq 8$. Moreover, if we assume $m < n$ then

$10^{n=1} - 81m^2 > 10^m - 81m^2 > 0$

for every $m \geq 3$, and the second inequality in (2.3) does not hold in this case.

We prove the following

Lemma 2. *If* $m < n$ *and* $\overline{a_1 a_2 ... a_m}$ *and* $\overline{b_1 b_2 ... b_n}$ *are amicable-digit numbers, then* $m + n \leq 8$ *and* $m \leq 2$

In fact there are only two pairs (m, n) satisfying (2.3) with $m + n \leq 8$ and $m \leq 2$, namely (m, n) = (1, 2) and (m, n) = (2, 3). In this case, the pair a_1 and $\overline{b_1 b_2}$, and the pair $\overline{a_1 a_2}$ and $\overline{b_1 b_2 b_3}$ are digit-amicable numbers if

$$a_1 = (b_1 + b_2)^2 \text{ and } \overline{b_1 b_2} = a_1^2,$$

and

$$\overline{a_1 a_2} = (b_1 + b_2 + b_3)^2 \text{ and } \overline{b_1 b_2 b_3} = (a_1 + a_2)^2$$

By direct trials, taking into account that a_1 and $\overline{a_1 a_2}$ should be perfect squares, we find no solutions. We now search for pairs of digit-amicable numbers having the same number of digits. We give the following

Theorem 2. *The unique pair of digit-amicable numbers are* 169 *and* 256, *with*

$$169 = (2+5+6)^2 \ , \ \ 256 = (1+6+9)^2$$

Proof: As above, by the first relation in (2.2), we get $10^{n=1} \leq (9n)^2$, so $n \leq 4$

In the case $n=2$ the digit sums a_1+a_2 and b_1+b_2 can take values in {4, 5, 6, 7, 8, 9}.

If $a_1+a_2=4$, then $\overline{b_1b_2}=16$, and $(b_1+b_2)^2=(1+6)^2=49=\overline{a_1a_2}$, which contradicts $a_1+a_2=4$.

If $a_1+a_2=5$, then $\overline{b_1b_2}=25$, and $(b_1+b_2)^2=(2+5)^2=49=\overline{a_1a_2}$, contradiction.

If $a_1+a_2=6$, then $\overline{b_1b_2}=36$, and $(b_1+b_2)^2=(3+6)^2=81=\overline{a_1a_2}$, contradiction.

If $a_1+a_2=7$, then $\overline{b_1b_2}=49$, and $(b_1+b_2)^2=(4+9)^2=169=\overline{a_1a_2}$, contradiction.

If $a_1+a_2=8$, then $\overline{b_1b_2}=64$, and $(b_1+b_2)^2=(6+4)^2=100=\overline{a_1a_2}$, contradiction.

If $a_1+a_2=9$, then $\overline{b_1b_2}=81$, and $(b_1+b_2)^2=(8+1)^2=81$, but $\overline{a_1a_2}=\overline{b_1b_2}$.

In the case $n=3$ the digit sums $a_1+a_2+a_3$ and $b_1+b_2+b_3$ can take values in {10, 11, 12,...27}.

If $a_1+a_2+a_3=10$, then $\overline{b_1b_2b_3}=100$, and $(b_1+b_2+b_3)^2=(1+0+0)^2=\overline{a_1a_2a_3}$, contradiction.

If $a_1+a_2+a_3=11$, then $\overline{b_1b_2b_3}=121$, and $(b_1+b_2+b_3)^2=(1+2+1)^2=\overline{a_1a_2a_3}$, contradiction.

If $a_1+a_2+a_3=12$, then $\overline{b_1b_2b_3}=144$, and $(b_1+b_2+b_3)^2=(1+4+4)^2=\overline{a_1a_2a_3}$, contradiction.

If $a_1+a_2+a_3=13$, then $\overline{b_1b_2b_3}=169$, and $(b_1+b_2+b_3)^2=(1+6+9)^2=\overline{a_1a_2a_3}$, so $\overline{a_1a_2a_3}=256$, which is a solution of our problem.

After similiar treatment of the remaining cases $a_1+a_2+a_3\in\{14,15,16,...,27\}$ no other solutions are found. □

3. CONCLUDING REMARKS

The general problem of solving the equation

$$\overline{a_1a_2...a_n}=(a_1+a_2+...+a_n)^k,\ n,k\geq 2, \tag{3.1}$$

remains open and we are convinced that other interesting results can be found.

In this connection, we mention the work of Acu [2] who solved the equations

$$\overline{a_1a_2...a_n}=(a_1+a_2+...+a_n)^5$$

and

$$\overline{a_1a_2...a_n}=(a_1+a_2+...+a_n)^6$$

where $n\geq 2$. The solutions are the following

$$17210368=(1+7+2+1+0+3+6+8)^5$$

$$52521875=(5+2+5+2+1+8+7+5)^5$$

$$60466176=(6+0+4+6+6+1+7+6)^5$$

$$205962976=(2+0+5+9+6+2+9+7+6)^5$$

and, respectively,

$$34012224 = (3+4+0+1+2+2+2+4)^6$$
$$8303765625 = (8+3+0+3+7+6+5+6+2+5)^6$$
$$24794911296 = (2+4+7+9+4+9+1+1+2+9+6)^6$$
$$68719476736 = (6+8+7+1+9+4+7+6+7+3+6)^6$$

A computer program allows us to find the solutions of (3.1) for larger values of k. Below we tabulate the solutions for $7 \le k \le 30$.

Table 1. Solutions for $7 \le k \le 30$

k	*Solutions*
7	612220032, 10460353203, 27512614111, 52523350144, 271818611107, 1174711139837, 2207984167552, 6722988818432
8	20047612231936, 72301961339136, 248155780267521
9	3904305912313344, 45848500718449031, 150094635296999121
10	13744803133596058624, 19687440434072265625, 53861511409489970176, 73742412689492826049
11	8007313507497959524352
13	81920000000000000, 671088640000000000000, 14076019706120526112710656
14	267041951127206120525454504361
17	225179981368524800000000000000000
19	144115188075855872000000000000000000000, 1350851717672992089000000000000000000000
20	121576654590569288010000000000000000000000
21	1094189891315123592090000000000000000000000
22	984770902183611232881000000000000000000000000
28	5233476330273605372135115210000000000000000000000000000

There are no solutions for $k \in \{12, 15, 16, 18, 23, 24, 25, 26, 27, 29, 30\}$.

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