ORIGINAL PAPER

ON AN INEQUALITY FOR THE MEDIANS OF A TRIANGLE

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Abstract. *In this paper, we give two new simpler proofs of a sharp inequality for the medians of a triangle. We also establish two new inequalities by using this sharp inequality. Some related conjectures checked by the computer are put forward, which include two conjectures related to the famous Erdös-Mordell inequality.*

Keywords: *triangle, median, inequality, Erdös-Mordell inequality*

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1. INTRODUCTION

In 2000, X.G. Chu and X.Z.Yang [1] established the following geometric inequality: Let *ABC* be a triangle with medians m_a, m_b, m_c, circumradius *R*, inradius *r* and semi-perimeter *s*. Then the following inequality holds:

$$(m_a + m_b + m_c)^2 \le 4s^2 - 16Rr + 5r^2, \tag{1.1}$$

with equality if and only if ΔABC is equilateral.

This is a strong inequality and has some applications (see e.g. [1], [2]). In my recent paper [3], I have shown that the combinational coefficients in (1.1) is the best possible. In fact, by Theorem 2 in [3] it is easy to prove the following conclusion: For all inequalities in the form

$$(m_a + m_b + m_c)^2 \le k_1 s^2 + k_2 Rr + k_3 r^2, \tag{1.2}$$

inequality (1.1) is the best possible, where k_1, k_2, k_3 are constants and satisfy $27k_1 + 2k_2 + k_3 = 81$.

On the other hand, it is interesting that there exists the following sharp inequality (1.3) which is stronger than (1.1):

Theorem 1. *In any triangle ABC with sides a, b, c, medians m_a, m_b, m_c, inradius r, and circumradius R, the following inequality holds:*

$$\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \le 2 + \frac{r^2}{R^2}, \tag{1.3}$$

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with equality if and only if triangle ABC is equilateral.

Remark 1.1. If ΔABC might be a degenerate triangle, then the equality in (1.3) would also arrive at the case when $A = 0, B = C = \frac{\pi}{2}$. This fact shows inequality (1.3) is sharp.

H.Y.Yin first posed an equivalent form of (1.3) when the inequality (1.1) just had been set up (see [3], [4]). Until recently, (1.3) has been proved by the author in [3]. However, this proof is very complicated. The author used a lemma in [1], i.e. the inequality:

$$4m_b m_c \le 2a^2 + bc - \frac{4s(s-a)(b-c)^2}{2a^2+bc}, \tag{1.4}$$

with equality if and only if $b = c$.

In this paper, we give two simpler proofs of Theorem 1, both of which do not depend on (1.4). We also give two applications of Theorem 1. One of them is a beautiful linear inequality involving the medians and the altitudes of a triangle. Another result is about the acute-angled triangle. In the last section, we will propose some related conjectures.

2. NEW PROOFS OF THEOREM 1

In this section, we will give two proofs of Theorem 1. To simplify matter, we denote cyclic sums and cyclic products by Σ, Π respectively.

Proof 1: (The method of $R - r - s$) By Cauchy inequality, we have

$$\left(\sum m_a\right)^2 \le \sum\left(b^2+c^2\right)\sum\frac{m_a^2}{b^2+c^2},$$

i.e.

$$\left(\sum m_a\right)^2 \le 2\sum a^2 \sum\frac{m_a^2}{b^2+c^2}. \tag{2.1}$$

Therefore, to prove inequality (1.3) we need to prove that

$$\sum\frac{m_a^2}{b^2+c^2} \le 1 + \frac{r^2}{2R^2}. \tag{2.2}$$

Using the known formula $4m_a^2 = 2\left(b^2+c^2\right) - a^2$, it is easily known that inequality (2.2) is equivalent to

$$\sum\frac{a^2}{b^2+c^2} + \frac{2r^2}{R^2} \ge 2. \tag{2.3}$$

Since

$$\begin{aligned}&\sum\frac{a^2}{b^2+c^2}+3\\&=\sum\frac{a^2+b^2+c^2}{b^2+c^2}=\sum a^2\sum\frac{1}{b^2+c^2}\\&=\frac{\sum a^2\sum\left(c^2+a^2\right)\left(a^2+b^2\right)}{\prod\left(b^2+c^2\right)}=\frac{\left(\sum a^4+3\sum b^2c^2\right)\sum a^2}{\prod\left(b^2+c^2\right)},\end{aligned}$$

hence (2.3) is equivalent to

$$\frac{\left(\sum a^4+3\sum b^2c^2\right)\sum a^2}{\prod\left(b^2+c^2\right)}+\frac{2r^2}{R^2}-5\ge 0.$$

Thus, we have to prove that

$$X_1\equiv R^2\left(\sum a^4+3\sum b^2c^2\right)\sum a^2+\left(2r^2-5R^2\right)\prod\left(b^2+c^2\right)\ge 0. \tag{2.4}$$

Using the following known identities (see e.g. [5]):

$$abc=4Rrs, \tag{2.5}$$
$$\sum a^2=2s^2-8Rrs-2r^2, \tag{2.6}$$
$$\sum b^2c^2=s^4-2\left(4R-r\right)s^2r+\left(4R+r\right)^2r^2, \tag{2.7}$$
$$\sum a^4=2s^4-4\left(4R+3r\right)s^2r+2\left(4R+r\right)^2r^2, \tag{2.8}$$
$$\prod(b^2+c^2)=2s^6-2(12R-r)s^4r+2(40R^2+8Rr-r^2)s^2r^2-2(4R+r)^3r^3, \tag{2.9}$$

we obtain

$$X_1=4r^2X_2, \tag{2.10}$$

where

$$\begin{aligned}X_2=&s^6-\left(8R^2+12Rr-r^2\right)s^4+\left(20R^4+32R^3r+48R^2r^2+8Rr^3-r^4\right)s^2\\&-\left(4R+r\right)^3r^3\end{aligned} \tag{2.11}$$

Obviously, the proof of $X_1\ge 0$ is changed to $X_2\ge 0$. If we put

$$G_2=4R^2+4Rr+3r^2-s^2,$$
$$T_0=-s^4+2\left(2R^2+10Rr-r^2\right)s^2-r\left(4R+r\right)^3,$$

then it is easy to verify the following identity:

$$X_2=G_2T_0+X_3\left(s^2\right), \tag{2.12}$$

where

$$\begin{aligned}X_3\left(s^2\right)=&2\left(6R+r\right)rs^4+\left(4R^4-128R^3r-84R^2r^2-56Rr^3+4r^4\right)s^2\\&+2\left(2R^2+2Rr+r^2\right)\left(4R+r\right)^3r.\end{aligned}$$

By identity (2.12) , Gerretsen inequality $G_2 \geq 0$ and the fundamental inequality $T_0 \geq 0$ of triangles (see [5], [6]), to prove $X_2 \geq 0$ it remains to prove that

$$X_3(s^2) \geq 0 \tag{2.13}$$

Let $K \equiv 4(6R+r)rs^2 + (4R^4 - 128R^3r - 84R^2r^2 - 56Rr^3 + 4r^4)$, then it is easy to show that K may be non-negative and also be negative by giving examples. So we can divide the proof of $X_3(s^2) \geq 0$ into the following two cases, i.e. $K \geq 0$ and $K < 0$.

Case 1. Assuming $K \geq 0$.

In this case, according to the property of parabolas and the Gerretsen inequalities:

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \ , \tag{2.14}$$

$X_3(s^2)$ is strictly increasing on the interval $[16Rr - 5r^2 , 4R^2 + 4Rr + 3r^2]$. So we only need to prove that $X_3(16Rr - 5r^2) \geq 0$, but

$$\begin{aligned} X_3\left(16Rr-5r^2\right) = 2r\ &(6R+r)\left(16Rr-5r^2\right)^2 + \\ &\left(4R^4-128R^3r-84R^2r^2-56Rr^3+4r^4\right)\left(16Rr-5r^2\right)+ \\ &2\left(2R^2+2Rr+r^2\right)(4R+r)^3 r \\ &=4r\left(80R^3-85R^2r+24Rr^2+2r^3\right)(R-2r)^2 \geq 0. \end{aligned}$$

The latter inequality follows from Euler inequality $R \geq 2r$. Hence $X_3(s^2) \geq 0$ is proved under the first case.

Case 2. Assuming $K < 0$. For this case, it is easy to know that $X_3(s^2)$ is decreasing on $[16Rr - 5r^2 , 4R^2 + 4Rr + 3r^2]$. Thus we only need to show $X_3(4R^2 + 4Rr + 3r^2) \geq 0$. Simple computations give

$$X_3(4R^2 + 4Rr + 3r^2) = 4(4R^4 + 4R^3r + 7R^2r^2 + 4Rr^3 + 2r^4)(R-2r)^2 \geq 0 .$$

Therefore $X_3(s^2) \geq 0$ is valid under the second case.

Combing with the arguments of the two cases, $X_3(s^2) \geq 0$ holds for all triangle *ABC*. Therefore, (2.4), (2.3), (2.2) and (1.3) are all proved. From the deductions above, it is clear that the equality in (1.3) holds only when ΔABC is equilateral. The proof of Theorem 1 is complete.

Proof 2: (The method of the Difference Substitution) Firstly, we can turn the proof of (1.3) into the inequality (2.3) as above. Since

$$\frac{r}{R} = \frac{\prod(b+c-a)}{2abc} \ , \tag{2.15}$$

thus inequality (2.3) is equivalent to

$$\sum\frac{a^2}{b^2+c^2}+\frac{\prod\left(b+c-a\right)^2}{2\left(abc\right)^2}\geq 2, \tag{2.16}$$

i.e.,

$$\begin{aligned} Y_0 \equiv 2\ \left(abc\right)^2\sum a^2\left(c^2+a^2\right)\left(a^2+b^2\right)+ \\ +\prod\left(b^2+c^2\right)\prod\left(b+c-a\right)^2-4\ \left(abc\right)^2\prod\left(b^2+c^2\right)\geq 0. \end{aligned} \tag{2.17}$$

Let $b+c-a=2x$, $c+a-b=2y$, $a+b-c=2z$, then $a=y+z$, $b=z+x$, $c=x+y$, and we have

$$\begin{aligned} Y_0 = 2\ \prod\left(y+z\right)^2\sum\left(y+z\right)^2\left[\left(x+y\right)^2+\left(y+z\right)^2\right]\left[\left(y+z\right)^2+\left(z+x\right)^2\right] \\ +\prod\left[\left(z+x\right)^2+\left(x+y\right)^2\right]\prod x^2 \\ -\ 4\prod\left(y+z\right)^2\prod\left[\left(z+x\right)^2+\left(x+y\right)^2\right]. \end{aligned} \tag{2.18}$$

Because of symmetry, we assume without loss of generality that $x\geq y\geq z$ and let

$$\begin{cases} y=z+m \\ x=z+m+n\ . \end{cases} \tag{2.19}$$

where $m \geq 0$ and $n \geq 0$. Substituting (2.19) into (2.18), with help of the mathematical software we obtain the following identity:

$$\begin{aligned} Y_0 =\ & 1024\ (m^2+mn+n^2)^2z^8+256\ (2m+n)(m^2+mn+n^2)(13m^2+13mn+6n^2)z^7+ \\ & +(18816\ m^6+56448\ m^5n+79680\ m^4n^2+65280\ m^3n^3+31296\ m^2n^4+8064\ mn^5+ \\ & +896\ n^6)z^6+128\ (2m+n)(118\ m^6+354\ m^5n+451\ m^4n^2+312\ m^3n^3+121\ m^2n^4+ \\ & +24\ mn^5+2\ n^6)z^5+30112\ m^8+120448\ m^7n+202496\ m^6n^2+185920\ m^5n^3+ \\ & +101792\ m^4n^4+3240\ m^3n^5+6976\ m^2n^6+768\ mn^7+32\ n^8)z^4+ \\ & +64\ (2m+n)(m+n)(149\ m^6+447\ m^5n+513\ m^4n^2+281\ m^3n^3+79\ m^2n^4+ \\ & +13\ mn^5+n^6)z^3m+16\ (468\ m^6+1404\ m^5n+1627\ m^4n^2+914\ m^3n^3+261\ m^2n^4+ \\ & +38\ mn^5+3\ n^6)(m+n)^2z^2m^2+16\ (13\ m^2+13\ mn+n^2)(2m+n)^3(m+n)^3zm^3+ \\ & +10\ (2m+n)^4(m+n)^4m^4\ . \end{aligned} \tag{2.20}$$

So inequality $Y_0 \geq 0$ holds obviously by $m \geq 0$, $n \geq 0$, and $z > 0$. Hence (2.17), (2.16) and then (1.3) are proved. The equality in $Y_0 \geq 0$ holds if and only if $m = n = 0$. Further, it is known that the equalities in (2.17) and (1.3) occurs only when $a = b = c$, i.e. ΔABC is equilateral. This completes the proof of Theorem 1.

Remark 2.1. From inequality (2.1), using previous methods to prove Theorem 1 we can also prove the following inequality:

$$\frac{(m_a+m_b+m_c)^4}{b^2c^2+c^2a^2+a^2b^2}\le 4-\frac{13r^2}{4R^2}, \tag{2.21}$$

which is posed by the author in [3].

3. TWO APPLICATIONS OF THEOREM 1

In this section, we will apply Theorem 1 to establish two new triangle inequalities, which are not both proved by using inequality (1.1).

We first prove the following beautiful linear inequality:

Theorem A. *For all ΔABC holds*:

$$m_a+m_b+m_c-(h_a+h_b+h_c)\le 2(R-2r), \tag{3.1}$$

with equality if and only if ΔABC is equilateral.

Proof: By Theorem 1, to prove (3.1) we need to prove that

$$(a^2+b^2+c^2)\left(2+\frac{r^2}{R^2}\right)\le [h_a+h_b+h_c+2(R-2r)]^2. \tag{3.2}$$

Multiplying both sides of this inequality by $4R^2$ and using the relation $2Rh_a = bc$ etc., inequality (3.2) becomes the following equivalent form:

$$M_0\equiv[bc+ca+ab+4R\ (R-2r)]^2-4\ (a^2+b^2+c^2)(2R^2+r^2)\ge 0. \tag{3.3}$$

Applying identity (2.6) and the known identity:

$$bc+ca+ab=s^2+4Rr+r^2, \tag{3.4}$$

it is easy to get

$$M_0=(4R^2+4Rr+3r^2-s^2)^2.$$

Thus the claimed inequality $M_0\ge 0$ follows and (3.1) is proved. It is clear that the equality in (3.1) holds only when ΔABC is equilateral. This completes the proof of Theorem A. └

Remark 3.1. By the method to prove Theorem 2 in [3], we can prove that the constant 2 of the right side of (3.1) is the best possible. In addition, from Leuenberger's inequality (see [6]):

$$h_a+h_b+h_c\le 2R+5r, \tag{3.5}$$

we see that inequality (3.1) is stronger than the known result (see [6]):

$$m_a + m_b + m_c \le 4R + r \,. \tag{3.6}$$

Remark 3.2. By using inequality (1.1), it is easy to prove another linear inequality for the sum $m_a + m_b + m_c$:

$$m_a + m_b + m_c \le 2s - \left(6\sqrt{3} - 9\right) r \,. \tag{3.7}$$

This inequality is also stronger than (3.6) since we have the following inequality:

$$s \le 2R + \left(3\sqrt{3} - 4\right) r \,, \tag{3.8}$$

which is due to W.J.Blundon (see [7], [8], [9]).

Next, we prove an inequality for the acute-angled triangle, which was found by the author many years ago, but has not been proved before.

Theorem B. *For acute-angled ΔABC holds:*

$$\frac{h_a + h_b + h_c}{m_a + m_b + m_c} \ge \frac{1}{2} + \frac{r}{R} \,, \tag{3.9}$$

with equality if and only if acute-angled ΔABC is equilateral.

Proof. By Theorem 1, to prove (3.9) we need to show that

$$\left(h_a + h_b + h_c\right)^2 - \left(\frac{1}{2} + \frac{r}{R}\right)^2 \left(2 + \frac{r^2}{R^2}\right)\left(a^2 + b^2 + c^2\right) \ge 0 \,. \tag{3.10}$$

Multiplying both sides of the above by $4R^4$ and then using the relation $2Rh_a = bc$ etc., we see (3.10) is equivalent to

$$N_0 \equiv R^2 \left(bc + ca + ab\right)^2 - \left(R + 2r\right)^2 \left(2R^2 + r^2\right)\left(a^2 + b^2 + c^2\right) \ge 0 \,. \tag{3.11}$$

Substituting (2.6) and (3.4) into the expression of N_0, then (3.11) is equivalent to

$$\begin{aligned} N_0 \equiv\ & s^4 R^2 - 4\ \left(R^4 + 2R^3 r + 4R^2 r^2 + 2Rr^3 + 2r^4\right) s^2 \\ & + \left(4R + r\right)\left(4R^4 + 20R^3 r + 19R^2 r^2 + 8Rr^3 + 8r^4\right) r \ge 0 \,. \end{aligned} \tag{3.12}$$

We rewrite N_0 as follows

$$N_0 = 4r^2 \left(R + 2r\right)\left(2R^2 + r^2\right) e + 8r^4 G_2 + R\left[4r\left(3R + r\right) e + RG_1\right]\ C_0 \ , \tag{3.13}$$

where

$$e = R - 2r$$
$$G_1 = s^2 - 16Rr + 5r^2$$

$$G_2 = 4R^2 + 4Rr + 3r^2 - s^2$$
$$C_0 = s^2 - (2R + r)^2 \ .$$

Therefore, by Euler inequality $e \geq 0$, Gerretsen inequalities $G_1 \geq 0$, $G_2 \geq 0$ (see [5], [6]) and the acute triangle inequality $C_0 \geq 0$ of Ciamberlini (see [10]), we conclude $N_0 \geq 0$ holds for *acute-angled ΔABC*. Hence inequality (3.10) and (3.9) are proved. It is easy to see that the equality in (3.9) holds when ΔABC is equilateral. The proof of Theorem B is completed.

4. SEVERAL CONJECTURES

In this section, we will propose some conjectures for the inequalities appeared in this note.

Considering the exponential generalization of Theorem A with help of the computer for verifying, we pose the following three similar conjectures:

Conjecture 1. *If* $0 < k < 1$*, then for any ΔABC we have*

$$\left(m_a + m_b + m_c\right)^k - \left(h_a + h_b + h_c\right)^k \leq \left(2\,R\right)^k - \left(4r\right)^k \tag{4.1}$$

If ΔABC is an acute triangle and $k \geq 1.1$*, then the inequality holds reversed.*

Remark 4.1. It is easy to prove that (4.1) is reversed for all triangles if $k < 0$.

Conjecture 2. *If ΔABC is an acute triangle and* $k \geq 1.1$*, then we have*

$$m_a^k + m_b^k + m_c^k - \left(h_a^k + h_b^k + h_c^k\right) \geq 2\left(R^k - 2^k r^k\right). \tag{4.2}$$

Conjecture 3. *If* $k > 1$ *or* $k < 0$*, then for any* ΔABC *we have*

$$m_a^k + m_b^k + m_c^k - \left(h_a^k + h_b^k + h_c^k\right) \leq \left(2\,R\right)^k - \left(4r\right)^k. \tag{4.3}$$

When $k = -1$, (4.3) is actually equivalent to

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{1}{2R} + \frac{3}{4r}, \tag{4.4}$$

which is clear weaker than the known inequality (see [11]):

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \leq \frac{1}{2R} + \frac{3}{4r}, \tag{4.5}$$

where w_a, w_b, w_c are three internal bisectors of ΔABC. On the other hand, (4.4) can be refined the following:

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\le\frac{2}{3}\left(\frac{1}{R}+\frac{1}{r}\right), \tag{4.6}$$

which is proved by the author in [12].

Considering the lower bound of the left hand side of (3.1), we give

Conjecture 4. *For any ΔABC we have*

$$m_a+m_b+m_c-\left(h_a+h_b+h_c\right)\ge s-3\sqrt{3}r. \tag{4.7}$$

If (4.7) holds true, then Blundon's inequality (3.8) can be obtain from (3.1) and (4.7).

Next, we give a double inequality conjecture which is inspired by Theorem B:

Conjecture 5. *For any ΔABC we have*

$$\frac{k_a+k_b+k_c}{m_a+m_b+m_c}\ge\frac{1}{2}+\frac{r}{R}\ge\frac{k_a+k_b+k_c}{r_a+r_b+r_c}, \tag{4.8}$$

where k_a,k_b,k_c *are symmedians of ΔABC and* r_a,r_b,r_c *are radii of excircles of ΔABC.*

Considering the exponential generalization of inequality (2.3), we present

Conjecture 6. *If* $k>2$*, then for any ΔABC we have*

$$\frac{a^k}{b^k+c^k}+\frac{b^k}{c^k+a^k}+\frac{c^k}{a^k+b^k}+2^{k-1}\frac{r^k}{R^k}\ge 2. \tag{4.9}$$

If $0<k\le\frac{8}{5}$ *, then the inequality is reversed.*

The classical Erdös-Mordell inequality can be stated as follows: Let P be an interior point of Δ*ABC*. Denote by R_1, R_2, R_3 the distances of P from the vertices *A, B, C,* and r_1, r_2, r_3 the distances of P from the sidelines *BC,CA,AB* respectively. Then holds:

$$R_1+R_2+R_3\ge 2\left(r_1+r_2+r_3\right). \tag{4.10}$$

It is well known that there are a few stronger versions of the Erdös-Mordell inequality (see e.g. [5], [13]). Here, we put forward two new stronger inequalities.

Conjecture 7. *For any interior point of* Δ*ABC, we have*

$$\frac{R_1+R_2+R_3}{r_1+r_2+r_3}\ge\frac{2\sqrt{4s^2-16Rr+5r^2}}{m_a+m_b+m_c} \tag{4.11}$$

Inequality (1.1) shows (4.11) is stronger than (4.10).

Conjecture 8. *For any interior point of* Δ*ABC, we have*

$$\frac{R_1+R_2+R_3}{r_1+r_2+r_3} \geq \frac{2\left(h_a+h_b+h_c+2R\right)}{m_a+m_b+m_c+4r} . \tag{4.12}$$

The following equivalent form of (3.1):

$$h_a+h_b+h_c+2R \geq m_a+m_b+m_c+4r \tag{4.13}$$

means again (4.12) is stronger than the Erdös-Mordell inequality (4.10).

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