ORIGINAL PAPER

ON GENERATING FUNCTIONS OF MODIFIED LAGUERRE POLYNOMIALS

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Abstract: *In the present paper, we obtain a theorem on bilateral generating functions of* $f_n^{\beta+n}(x)$*, a modification of* $f_n^{\beta}(x)$ *by introducing a novel linear partial differential operator obtained by the suitable interpretations of the index n and the parameter* β *of the polynomial under consideration.*

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1. INTRODUCTION

Special functions are the solutions of a wide class of mathematically and physically relevent functional equations. Generating functions play a large role in the study of special functions. The study of special functions, in particular, generating functions by group-theoretic method was originally introduced by L. Weisner's [1 -3], which is lucidly presented in the monograph „Obtaining Generating Functions" by E.D. McBride [4].

In this article we obtain some novel results on bilateral generating functions of $f_n^{\beta+n}(x)$, a modification of $f_n^{\beta}(x)$ by group-theoretic method, where $f_n^{\beta}(x)$ is defined by

$$f_n^{\beta}(x) = \frac{(\beta)_n}{n!}\,{}_1F_1\left[\begin{array}{c} -n; \\ \quad\quad x \\ 1-\beta-n; \end{array}\right]. \tag{1.1}$$

The main result of our investigation is given in the form of the following theorem:

Theorem: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, f_n^{\beta+n}(x)\, w^n \tag{1.2}$$

then

$$(1+tz)^{\beta-1} exp\left\{\frac{txz}{1+tz}\right\} G\left(\frac{x}{1+tz}, (1+tz)t\right) = \sum_{n=0}^{\infty} t^n \sigma_n(x,z) \tag{1.3}$$

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where

$$\sigma_n(x, z) = \sum_{p=0}^{n} a_p \binom{n}{p} f_n^{\beta-n+2p}(x)\, z^{n-p} \tag{1.4}$$

The importance of the above theorem lies in the fact that when one knows a generating relation of the form (1.2), the corresponding bilateral generating relation can at once be written down from (1.3) by attributing different values to a_n in (1.2).

To prove the above theorem, we shall introduce the following linear partial differential operator

$$R = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_0$$

and the extended form of the group generated by R.

2. DERIVATION OF THE OPERATOR AND THE EXTENDED GROUP

Let us now seek first order linear partial differential operator,

$$R = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_0 \tag{2.1}$$

such that

$$R\left(f_n^{\beta+n}(x)\, y^n z^{\beta}\right) = C_{n,\beta}\, f_{n+1}^{\beta+n-1}(x)\, y^{n+1} z^{\beta-2}, \tag{2.2}$$

where $A_i\ (i = 0, 1, 2, 3)$ are functions of x, y, z but independent of n, β and $C_{n,\beta}$ is a function of n, β.

Using (2.2) and the following differential recurrence relation

$$xD\, f_n^{\beta+n}(x) = (x + \beta + n - 1)\, f_n^{\beta+n}(x) - (n+1)\, f_{n+1}^{\beta+n-1}(x)\ ,$$

we easily get

$$R = xyz^{-2} \frac{\partial}{\partial x} - y^2 z^{-2} \frac{\partial}{\partial y} - yz^{-1} \frac{\partial}{\partial z} + (1-x)yz^{-2} \tag{2.3}$$

$$R\left(f_n^{\beta+n}(x)\, y^n z^{\beta}\right) = -(n+1)\, f_{n+1}^{\beta+n-1}(x)\, y^{n+1} z^{\beta-2}. \tag{2.4}$$

In order to find the extended form of the group generated by R i.e. $e^{wR} f(x, y, z)$, where $f(x, y, z)$ is arbitrary function and w is arbitrary constant real or complex, let $\phi(x, y, z)$ be a function such that

$$R\phi(x, y, z) = 0\,.$$

On solving, $R\phi(x, y, z) = 0$, we get $\phi(x, y, z) = x\, y\, z\, e^x$.

Let us transform the operator R to E, where

$$E = xyz^{-2}\frac{\partial}{\partial x} - y^2 z^{-2}\frac{\partial}{\partial y} - yz^{-1}\frac{\partial}{\partial z}$$

and R is given in (2.3), then $E = \phi^{-1}\left(X, Y, Z\right) R\phi\left(X, y, Z\right)$

$$i.e. \qquad R = \phi\left(X, Y, Z\right) E\ \phi^{-1}\left(X, y, Z\right)$$

Let X, Y, Z be a set of new variables for which

$$EX = -1, \qquad EY = 0, \qquad EZ = 0 \tag{2.5}$$

So that E reduce to $D = -\dfrac{\partial}{\partial X}$.

Now solving (2.5), we have

$$X = y^{-1} z^2, \qquad Y = x\,y, \qquad Z = x\,z$$

From which, we get

$$x = \frac{Z^2}{XY}, \qquad y = \frac{XY^2}{Z^2}, \qquad z = \frac{XY}{Z}.$$

Using Taylor's theorem, we obtained

$$\begin{aligned} e^{wR} f(x, y, z) &= \phi(x, y, z)\, e^{wD} f(X, Y, Z) \\ &= \phi(x, y, z)\, F(X - w, Y, Z) \\ &= \phi(x, y, z)\, g(x, y, z), \end{aligned}$$

assuming that $F\left(X - w, Y, Z\right)$ is transformed into $g\left(x, y, z\right)$ by inverse substitution.

Therefore we obtain

$$e^{wR}\ f\left(x, y, z\right) = \left(1 - \frac{wy}{z^2}\right)^{-1} exp\left\{\frac{-\dfrac{wxy}{z^2}}{1 - \dfrac{wy}{z^2}}\right\} f\left(\frac{x}{\left(1 - \dfrac{wy}{z^2}\right)}, \left(1 - \frac{wy}{z^2}\right)y, \left(1 - \frac{wy}{z^2}\right)z\right) \tag{2.6}$$

3. APPLICATION OF THE OPERATOR

Now, writing $f(x, y, z) = f_n^{\beta+n}(x)\ y^n z^\beta$ in (2.6), we get

$$e^{wR}\left(f_n^{\beta+n}(x)\,y^n\,z^{\beta}\right)=\left(1-\frac{wy}{z^2}\right)^{-1+n+\beta}\ exp\left\{\frac{-\dfrac{wxy}{z^2}}{1-\dfrac{wy}{z^2}}\right\} f_n^{\beta+n}\left(\frac{x}{1-\dfrac{wy}{z^2}}\right)y^n\,z^{\beta} \tag{3.1}$$

Again, on the other hand we have

$$e^{wR}\left(f_n^{\beta+n}(x)\,y^n\,z^{\beta}\right)=\sum_{p=0}^{\infty}\frac{(-wy)^p}{p\,!}(n+1)_p\ f_{n+p}^{\beta+n-p}\ y^p\ z^{\beta-2p} \tag{3.2}$$

Equating (3.1) and (3.2) and then putting $-\dfrac{wy}{z^2}=t$, we get

$$(1+t)^{n+\beta-1}\ \exp\left\{\frac{xt}{1+t}\right\} f_n^{\beta+n}\left(\frac{x}{1+t}\right)=\sum_{p=0}^{\infty}\frac{(n+1)_p}{p\,!}\ f_{n+p}^{\beta+n-p}\ (x)\ t^p \tag{3.3}$$

We shall now prove the theorem, by using the generating relation (3.3)

4. PROOF OF THE THEOREM

Now the right hand side of (1.3)

$$=\sum_{n=0}^{\infty} t^n\ \sigma_n\ (x,\ z)$$

$$=\sum_{n=0}^{\infty} t^n\ \sum_{p=0}^{n} a_p\binom{n}{p} f_n^{\beta-n+2p}(x)\ z^{n-p} \qquad \text{[using (1.4)]}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} t^{n+p}\ a_p\binom{n+p}{p} f_{n+p}^{\beta-n+p}(x)\ z^n$$

$$=\sum_{p=0}^{\infty} t^p\ a_p\left(\sum_{n=0}^{\infty}\binom{n+p}{p} f_{n+p}^{\beta-n+p}(x)\ (zt)^n\right)$$

$$=\sum_{p=0}^{\infty} t^p\ a_p\ (1+tz)^{p+\beta-1}\ \exp\left\{\frac{txz}{1+tz}\right\} f_p^{\beta+p}\left(\frac{x}{1+tz}\right) \qquad \text{[using (3.3)]}$$

$$=(1+tz)^{\beta-1}\ \exp\left\{\frac{txz}{1+tz}\right\}\sum_{p=0}^{\infty} a_p\quad f_p^{\beta+p}\left(\frac{x}{1+tz}\right)\ \{(1+tz)\,t\}^p$$

$$=(1+tz)^{\beta-1}\ \exp\left\{\frac{txz}{1+tz}\right\} G\left(\frac{x}{1+tz},\ (1+tz)t\right) \qquad \text{[using (1.2)]}$$

This completes the proof.

Now, we would like to point it out that the above theorem can be proved by the direct appliction of the operator R by using the method as discussed in [5].

Consider the generating relation

$$G(x,\ w)=\sum_{n=0}^{\infty} a_n\ f_n^{\beta+n}(x)\ w^n. \tag{4.1}$$

Replacing w by wy and multiplying both sides of (4.1) by z^{β}, we get

$$z^{\beta}\ G(x,\ wy)=\sum_{n=0}^{\infty} a_n\ \left(f_n^{\beta+n}(x)\, y^n\ z^{\beta}\right) w^n. \tag{4.2}$$

Operating $\left(\exp\,(wy)\right)$ on both sides of (4.2), we get

$$\left(\exp\,(wy)\right)\left(z^{\beta}\ G(x,\ wy)\right)=\left(\exp\,(wy)\right)\left(\sum_{n=0}^{\infty} a_n\ \left(f_n^{\beta+n}(x)\, y^n\ z^{\beta}\right) w^n\right). \tag{4.3}$$

The left hand side of (4.3), with the help of (2.6), becomes

$$\left(1-\frac{wy}{z^2}\right)^{\beta-1} \exp\left\{\frac{-\dfrac{wxy}{z^2}}{1-\dfrac{wy}{z^2}}\right\} z^{\beta}\ G\left(\frac{x}{\left(1-\dfrac{wy}{z^2}\right)},\ \left(1-\frac{wy}{z^2}\right)wy\right). \tag{4.4}$$

The right hand side of (4.3), with the help of (2.4), becomes

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\ \frac{w^{n+p}}{p\,!}(-1)^p\ (n+1)_p\ f_{n+p}^{\beta+n-p}(x)\ y^{n+p}\ z^{\beta-2p}$$

$$=z^{\beta}\sum_{n=0}^{\infty}(wy)^n\sum_{p=0}^{n} a_{n-p}\ \frac{(-1)^p\ (n+1)_p}{p\,!}\ f_n^{\beta+n-2p}(x)\ z^{-2p}. \tag{4.5}$$

Equating (4.4) and (4.5) and then putting $wy=t$ and $-z^{-2}=z$, we get

$$\left(1-\frac{wy}{z^2}\right)^{\beta-1} exp\left\{\frac{-\dfrac{wxy}{z^2}}{1-\dfrac{wy}{z^2}}\right\} z^{\beta}\ G\left(\frac{x}{\left(1-\dfrac{wy}{z^2}\right)},\ \left(1-\frac{wy}{z^2}\right)wy,\right)=\sum_{n=0}^{\infty} t^n\ \sigma_n\ (x,z)$$

where $\sigma_n\ (x,\ z)=\sum_{p=o}^{n} a_p \begin{pmatrix} n \\ p \end{pmatrix} f_n^{\beta-n+2p}(x)\ z^{n-p}$, which is the theorem.

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