ORIGINAL PAPER

A SCHEMA FOR OBTAINING THE SUM OF THE ALTERNATING SERIES

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Abstract. *We recall some classical methods for obtaining the sum of the alternating series and we give a special attention to one of these methods, generalizing the schema based on the identity of Catalan.*

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1. INTRODUCTION

Consider a convergent alternating series. To calculate it sum, we may consider several methods. We mention:

(a) The use of the power series;

(b) The integration of certain adequate identities;

(c) The use of the trigonometric series;

(d) The use of the residues theorem;

(e) The use of the identities of Catalan type.

In this paper, after a short remembering of the methods (a)-(d), with some illustra-ting examples, we propose to consider deeply the method (e).

2. A SHORT REMEMBERING OF THE METHODS (a)-(d)

(a) The use of the power series

Of principle, the method consists in to obtain and to use the expansions in Maclaurin power series of some indefinite differentiable adequate functions, which admit such expansions $f(x)=\sum_{n=0}^{\infty} a_n x^n$, where $a_n = \frac{f^{(n)}(0)}{n!}$. These expansions are valid on the convergence domain, defined by a certain convergence radius, given by the celebrated theo-rem of Abel and which can be calculated by the theorem of Cauchy-Hadamard. The use of the power series consists in to give particular values to the variable x, or, in other cases, in the integration on an adequate interval. The expansions in power series offers the advantage of

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the uniform convergence on any compact interval included in the convergence domain and can be differentiated, respectively integrated. We don't repeat here the well-known, usual examples, but we will present two more specific ones.

Let

$$\Omega_n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n} \tag{2.1}$$

be.

Example (a) 1. (according to [14]). The sum of the alternating series

$$\sum_{n=1}^{\infty}(-1)^n \Omega_n \frac{1}{2n+1}.$$

Consider the expansion in Maclaurin series

$$\frac{1}{\sqrt{1+x^2}} = 1+\sum_{n=1}^{\infty}(-1)^n \Omega_n x^{2n} \tag{2.2}$$

(a consequence of the expansion in Newton type series of the binomial function $(1+t)^{\alpha}$, with $t\in(-1,)$, for $\alpha=-1/2$ and $t=x^2$, $x\in(-1,1)$. The series on the right part is uniformly convergent on any compact interval included in the interval $(-1,1)$. For $x\in(-1,1)$, $x\neq 0$, we have

$$\frac{1}{x\sqrt{1+x^2}} = \frac{1}{x}+\sum_{n=1}^{\infty}(-1)^n \Omega_n x^{2n-1} \ . \tag{2.3}$$

Now we will perform a integration. In order to do this easily, we will use the identity

$$\frac{1}{x\sqrt{1+x^2}} = \frac{1}{x} - \frac{D\left(1+\sqrt{1+x^2}\right)}{1+\sqrt{1+x^2}}, \tag{2.4}$$

where we denoted by D the differentiation operator. (The formula is reminiscent of the decomposition of rational functions into simple fractions.) So we rewrite the formula (2.3) as

$$\frac{D\left(1+\sqrt{1+x^2}\right)}{1+\sqrt{1+x^2}} = \sum_{n=1}^{\infty}(-1)^{n-1} \Omega_n x^{2n-1} \tag{2.5}$$

By integrating both sides on an arbitrary compact subinterval $[\alpha,\beta]$ of, $(0,1)$ we obtain

$$\ln\left(1+\sqrt{1+\beta^2}\right) - \ln\left(1+\sqrt{1+\alpha^2}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Omega_n \beta^{2n}}{2n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Omega_n \alpha^{2n}}{2n} \ . \tag{2.6}$$

The function $f:[0,1]\to\mathbf{R}$ defined by $f(x)=\sum_{n=1}^{\infty}(-1)^n \frac{\Omega_n x^{2n}}{2n}$, for $x\in[0,1)$ and $f(1)=\sum_{n=1}^{\infty}(-1)^n \frac{\Omega_n}{2n}$ is continuous and equal to 0 at $x=0$. On the other hand, by a well-known theorem of Abel, f is also continuous at $x=1$. These remarks allow us to take limits in (2.6) as $\alpha\to 0$ and $\beta\to 1$, which yields

$$\sum_{n=1}^{\infty}(-1)^n \Omega_n \frac{1}{2n+1} = 2\left(\ln\left(1+\sqrt{2}\right)-\ln 2\right).$$

Example (a) 2. The sum of the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\Omega_n \frac{1}{n+1}$.

A similar integration of (2.2), without a precedent division by x, conducts us to the formula

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Omega_n \frac{1}{n+1} = 3-2\ln 2.$$

(b) The integration of certain adequate identities

This method also constitutes one of the useful resources to calculate the sum of some alternating series. We illustrate this method by integrating (term by term) certain identi-ties related to the summation of the geometric progressions.

Example (b) 1. (The series of Mercator). $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

The sequence of integrals $(I_n)_n$ defined by the equalities $I_n = \int_0^1 \frac{x^n}{1+x}dx$ is convergent to 0. So, integrating the identity

$$1-x+x^2-x^3+...+(-1)^{n-1}x^{n-1} = \frac{1-(-1)^n x^n}{1+x} \quad (x \neq -1)$$

on the compact interval $[0,1]$, we obtain

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...+(-1)^{n-1}\frac{1}{2n-1} = \int_0^1 \frac{dx}{1+x}-(-1)^n I_n,$$

which gives

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = \ln 2.$$

Example (b) 2. (The series of Leibniz) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}$.

The sequence of integrals $(I_n)_n$ defined by the equalities $I_n = \int_0^1 \frac{x^{2n}}{1+x^2}dx$ is also convergent to 0. Integrating the identity

$$1-x^2+x^4-x^6+...+(-1)^{n-1}x^{2n-2} = \frac{1-(-1)^n x^{2n}}{1+x^2}$$

on the compact interval $[0,1]$, we obtain

$$1-\frac{1}{3}+\frac{1}{5}=\frac{1}{7}+...+(-1)^{n-1}\frac{1}{2n-1} = \int_0^1 \frac{dx}{1+x^2}-(-1)^n I_n,$$

That gives the equality

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1} = \frac{\pi}{4}.$$

(c) The use of the trigonometric series

The method is based on the expansion of a function, under certain conditions, in a trigonometric (Fourier) series.

Consider a function $f:(-\pi,\pi)\to\mathbf{R}$. Let

$$S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$$

be the Fourier sum of order n associated to the function f on the interval $(-\pi,\pi)$, where the coefficients are given by the formulas

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx dx,\ k\in\mathbf{N},\ b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx dx,\ k\in\mathbf{N}^*$$

(with the well-known adequate form, in the case of an even or an odd function). If f is of piecewise C^1 class, the following expansion, given by Dirichlet, holds

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),\ \text{for } x\in(-\pi,\pi)$$

$$f(x)=\frac{1}{2}\big(f(-\pi+0)+f(\pi-0)\big),\qquad \text{for } x=\pm\pi.$$

Example (c) 1. (Finding again the series of Leibniz) Consider the odd function $f:(-\pi,\pi)\to\mathbf{R}$, $f(x)=x$. Its expansion in a trigonometric series is

$$x=2\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n},\qquad x\in(-\pi,\pi).$$

Put here $x=\pi/2$. The even terms vanishes and we obtain the equality

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}=\frac{\pi}{4}.$$

Example (c) 2. (The sum of the alternate series of the reciprocal of the squares) Consider the even function $f:(-\pi,\pi)\to\mathbf{R}$, $f(x)=x$. Its expansion is

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2},\qquad x\in(-\pi,\pi)$$

Put here the particular value $x=0$; we obtain

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}=\frac{\pi^2}{12}.$$

Of course, this result implies a solution of the problem of Basel,

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

(d) The use of the residues theorem

The method is based on the use of the following assertion which belongs to the complex analysis: Let f be a meromorphic function on C, which has a finite number of poles $a_1,a_2,a_3,\dots,a_m\in\mathbf{C}\setminus\mathbf{Z}^*$. If it exists a real number $a\in(0,\pi)$ and a function $z\mapsto\varepsilon(z)$ with the property $\lim_{z\to\infty}\varepsilon(z)=0$, such that

$$\left|f(z)\right| \le e^{a\left|\operatorname{Im} z\right|}\varepsilon(\left|z\right|),$$

on the exterior of a sufficiently large disk which doesn't contain no one of the poles a_k, then we have the following equality

$$\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\pi\sum_{k=1}^{m}\operatorname*{Re}_{z=a_k} z\,\frac{f(z)}{\sin \pi z}.$$

(see [1], [3], [6], [8]).

Example (d) 1. (Finding again the alternate series of the reciprocal of the squares)

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = \frac{\pi^2}{12}.$$

Consider the meromorphic function $f : \mathbf{C}^* = \mathbf{C}\setminus\{0\} \to \mathbf{C}$, $f(z) = 1/z^2$. It follows that the function $z \mapsto g(z) = 1/z^2 \sin \pi z$ has a triple pole in the origin and we obtain

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^2} = -\frac{\pi^2}{12},$$

which gives the desired formula.

Example (d) 2. Using a similar way, we can obtain the result

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4} = \frac{7\pi^4}{720}.$$

3. THE IDENTITY OF EUGÈNE CATALAN.

Eugène Charles Catalan (Bruges, May 30, 1814 - Liège, February 14, 1894) was a french mathematician, naturalized in Belgium.

He lived in Paris from 1825 and studied at École Polytechnique, where he met in 1833 Joseph Liouville, the future great mathematician. Initially destined to a carrier of ingenieur, he preferred to be a teacher and worked some time at École des Arts et Métiers of Châlons sur Marne (renamed in 1998 Châlons–en- Champagne) and, a little later, at Lycée Charlemagne of Paris. He formulated in 1844, for Journal für Reine und angewandte Mathematik (shorter Journal de Crelle), in Lettre adréssée à l'éditeur, his celebrated conjecture, namely: the only solution of the equation

$$x^p - y^q = 1 \ ,$$

with $x, y, p, q \in \mathbb{N}^*$ is $x = 3, p = 2, y = 2, q = 3.$ This conjecture was proved in 2002 by the Romanian mathematician Preda Mihăilescu (born in 1955)[see his paper in the same journal, "Primary Cyclotomic Units and a Proof of Catalan's Conjecture", *J. Reine angew. Math.*, **572**, 167, 2004].

Catalan participated at the revolution of 1848 and because of this was visited at his home by the French Police. He left France and return in Belgium, where he obtained the Chair of mathematical analysis of the Uni-vers-ty of Liège. He worked on number theory, differential equations, entire series, the calculation of the multiple integrals and in the differential geometry of the surfaces. His name also is related to the identity which we will discuss immediately, to certain numbers in combinatorics and to a constant in analysis.

Fig. 1. Eugène Charles Catalan.

The identity of Catalan is the following

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}. \tag{3.1}$$

Recall quickly the proof. Denote by A_n and B_n the left part, respectively the right part of the identity; also denote as usually the harmonic sums of order n

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n},\ \ n\in\mathbf{N}^*.$$

Moreover, denote

$$\sigma_n' = 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1},$$

$$\sigma_n'' = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}.$$

Obviously,

$$\sigma_n' + \sigma_n'' = H_{2n};\qquad \sigma_n'' = \frac{1}{2}H_n. \tag{3.2}$$

Write the sum A_n by emphasizing in two ordered sums the positive terms (having the denominators odd) and the negative terms (having the denominators even). We obtain

$$A_n = \sigma_n' - \sigma_n'' = (\sigma_n' + \sigma_n'') - 2\sigma_n'' = H_{2n} - 2\sigma_n'' = H_{2n} - 2\cdot\frac{1}{2}H_n,$$

i. e.

$$A_n = H_{2n} - H_n = B_{n,} \tag{3.3}$$

which completes the proof.

The identity of Catalan allows us to obtain by an elementary way the sum of the alternate series of Mercator, i. e. the result

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\ln 2.$$

To this aim, we will use the asymptotic expression (of the first order) of the harmonic sum H_n, namely

$$H_n=\ln n+\gamma+\varepsilon_n, \tag{3.4}$$

where γ is the well-known constant of Euler (also called the constant of Euler-Mascheroni), $\gamma = \lim_{n\to\infty}(H_n-\ln n) = 0.577215664\ldots$and $(\varepsilon_n)_n$ is a sequence which tends to 0, when $n\to\infty$.

This formula also can be written

$$H_n=\ln n+\gamma+o(1), \tag{3.4'}$$

where $o(1)$ denotes a quantity depending to n (the general term of a sequence) which tends to 0, when tends to infinity.

Let $(S_m)_m$ be the sequence of the partial sums of order *m* of the series. According to the identity of Catalan and to (3.4), we have, for the subsequence of even order of $(S_m)_m$

$$S_{2n} = A_n = H_{2n}-H_n = (\ln 2n+\gamma+\varepsilon_{2n})-(\ln n+\gamma+\varepsilon_n) = \ln 2+o(1)\xrightarrow[(n\to\infty)]{}\ln 2,$$

then

$$\lim_{n\to\infty}S_{2n}=\ln 2. \tag{3.5}$$

By the other hand,

$$S_{2n+1}=S_{2n}+\frac{1}{2n+1},$$

then the sequence $(S_{2n+1})_n$ will be convergent, as a sum of two convergent sequences, namely $(S_{2n})_n$ and $\left(\frac{1}{2n+1}\right)_n$. We obtain

$$\lim_{n\to\infty}S_{2n+1}=\ln 2.$$

This completes the proof.

The limit of the sequence $(B_n)_n$ also can be obtained by using other methods; see [11].

4. THE GENERAL METHOD OF THE SUMMATION OF THE ALTERNATING SERIES USING THE IDENTITIES OF CATALAN TYPE

Let be

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1-a_2+a_3-a_4+a_5-a_6+\ldots \tag{4.1}$$

an alternating series, which satisfies the conditions of the Leibniz criterion, i. e. the sequence $(a_n)_n$ tends decreasingly to 0 (then it has all the terms strictly positive).

We will use the notations:

$$S_m = a_1 + a_2 + a_3 + ... + a_m$$

$$S_m^{[a]} = a_1 - a_2 + a_3 - a_4 + ... + (-1)^{m-1} a_m$$

$$\sigma_n^{'} = a_1 + a_3 + a_5 + ... + a_{2n-1}$$

$$\sigma_n^{''} = a_2 + a_4 + a_6 + ... + a_{2n}.$$

So, by exactly the previous way, we deduce the following:

(a) With the notations we established, we have the general identity of Catalan type

$$S_{2n}^{[a]} = S_{2n} - 2\sigma_n^{''}. \tag{4.2}$$

(b) If the sequence $(S_{2n}^{[a]})_n$ has a limit $S^{[a]}$, then the alternating series (4.1) is convergent and has the sum $S^{[a]}$.

From these two previous facts, it follows that, if for the expression S_m, we know an asymptotic formula (which also implies an corresponding asymptotic formula for S_{2n}) and $\sigma_n^{''}$ can be put in a convenient form, for which we can obtain the limit, then the problem of the summation of the alternating series (4.1) is solved. This is the general summation method of the alternating series, based on the identities of Catalan type.

We illustrate by examples the method in some particular cases.

Example 1. If the general term a_n, for which we intend to obtain the sum of the alternating series (4.1) is $a_n = 1/n$, then the applying of the method consists exactly in all the procedure previously presented, which we don't repeat.

Example 2. Let be now

$$a_n = \frac{\ln n}{n},$$

for which we search the sum of the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}. \tag{4.3}$$

To usefully apply the equality (4.2), we give a more explicit form for all the terms of the sum $2\sigma_n^{''}$ and we do some calculations. So, we obtain

$$2\sigma_n^{''} = 2\left(\frac{\ln 2 + \ln 1}{2 \cdot 1} + \frac{\ln 2 + \ln 2}{2 \cdot 2} + \frac{\ln 2 + \ln 3}{2 \cdot 3} + ... + \frac{\ln 2 + \ln n}{2 \cdot n} \right) =$$
$$= \ln 2 \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \right) + \left(\frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + ... + \frac{\ln n}{n} \right),$$

that is

$$2\sigma_n^{''} = (\ln 2) \cdot H_n + S_n \tag{4.4}$$

Therefore, using the formula (4.2), we have

$$S_{2n}^{[a]} = S_{2n} - \big((\ln 2)H_n + S_n\big), \tag{4.5}$$

Now we will use the fact that for the sum S_n of this example, we posses, as for the harmonic sum H_n, an asymptotic expression, similar to (3.4')

$$S_n = \frac{1}{2}(\ln n)^2 + A + o(1), \tag{4.6}$$

where A is a constant, namely

$$A = \lim_{n\to\infty}\left(S_n - \frac{1}{2}(\ln n)^2\right) = \lim_{n\to\infty}\left(\frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + \ldots + \frac{\ln n}{n} - \frac{1}{2}(\ln n)^2\right)$$

Remark, *en passant*, that, in fact, this constant, which for simplicity was denoted by A, belongs to the family of the constants of Stieltjes, defined by the equalities

$$\gamma_r = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{(\ln k)^r}{k} - \frac{\ln(n)^{r+1}}{r+1}\right), \tag{4.7}$$

(see [5], pp. 32 , 166-169 and [13]). The constant A is obtained from (4.7), for $r = 1$, i. e. $A = \gamma_1$ $(\gamma_1 = -0{,}0728158454\ldots)$. For $r = 0$, we obtain the constant of Euler.

Put the asymptotic expressions (3.4') and (4.6) in (4.4). We obtain

$$2\sigma_n'' = (\ln 2)\cdot(\ln n + \gamma + o(1)) + \frac{1}{2}(\ln n)^2 + A + o(1),$$

which implies, by (4.5)

$$S_{2n}^{[a]} = \left(\frac{1}{2}(\ln 2n)^2 + A + o(1)\right) - \left((\ln 2)\cdot(\ln n + \gamma + o(1)) + \frac{1}{2}(\ln n)^2 + A + o(1)\right).$$

Performing the calculations, we obtain

$$S_{2n}^{[a]} = \frac{1}{2}(\ln 2)^2 - (\ln 2)\cdot\gamma + o(1).$$

Therefore

$$\lim_{n\to\infty} S_{2n}^{[a]} = \frac{1}{2}(\ln 2)^2 - (\ln 2)\cdot\gamma$$

and so the sum of the series (4.3) is

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n} = \frac{1}{2}(\ln 2)^2 - (\ln 2)\cdot\gamma\,.$$

Example 3. Let be

$$a_n = \frac{1}{n^\alpha} \qquad (\alpha > 1).$$

We will calculate the sum of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^\alpha}.$$

As well known, if $\alpha > 1,$ the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ is convergent and defines the very important function ζ , of Riemann, namely

$$\zeta(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}.$$

Denote, for all $n \in$ **N***,

$$\zeta_n(\alpha) = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + ... + \frac{1}{n^{\alpha}}.$$

So, applying the formula (4.2), we obtain

$$S_{2n}^{[a]} = \zeta_{2n}(\alpha) - 2\left(\frac{1}{(2\cdot 1)^{\alpha}} + \frac{1}{(2\cdot 2)^{\alpha}} + \frac{1}{(2\cdot 3)^{\alpha}} + ... + \frac{1}{(2n)^{\alpha}}\right) = \zeta_{2n}(\alpha) - \frac{1}{2^{\alpha-1}}\zeta_n(\alpha). \quad (4.8)$$

But we have the following two sided estimate

$$\frac{1}{(\alpha-1)(n+1)^{\alpha-1}} < \zeta(\alpha) - \zeta_n(\alpha) < \frac{1}{(\alpha-1)n^{1-\alpha}}$$

(see [4], **II**, 262, the formula (11) and, for a simpler proof, see [12]).

This implies the equality

$$\lim_{n\to\infty} n^{1-\alpha}\left(\zeta(\alpha) - \zeta_n(\alpha)\right) = \frac{1}{\alpha-1},$$

which gives an asymptotic formula (of first order) for the partial sum $\zeta_n(\alpha),$ namely

$$\zeta_n(\alpha) = \zeta(\alpha) + o(1).$$

Using two times this formula in (4.8), we obtain

$$S_{2n}^{[a]} = \zeta(\alpha) + o(1) - \frac{1}{2^{\alpha-1}}(\zeta(\alpha) + o(1)),$$

which implies

$$\lim_{n\to\infty} S_{2n}^{[a]} = \left(1 - \frac{1}{2^{\alpha-1}}\right)\zeta(\alpha).$$

So, we have obtained, for $\alpha > 1$, the searched sum of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\alpha}} = \left(1 - \frac{1}{2^{\alpha-1}}\right)\zeta(\alpha).$$

Concerning the particular values of ζ , see [7], 333.

Example 4. Let be

$$a_n = \frac{1}{n^{\alpha}} \qquad (0 < \alpha < 1)\,.$$

We will calculate the sum of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\alpha}}.$$

We have, with similar notations,

$$S_{2n}^{[a]}(\alpha) = S_{2n}(\alpha) - 2\sigma_n^{''}(\alpha),$$

where

$$S_n(\alpha) = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \ldots + \frac{1}{n^{\alpha}}\,.$$

$$2\sigma_n^{''}(\alpha) = 2\left(\frac{1}{(2\cdot 1)^{\alpha}} + \frac{1}{(2\cdot 2)^{\alpha}} + \frac{1}{(2\cdot 3)^{\alpha}} + \ldots + \frac{1}{(2n)^{\alpha}}\right) = 2^{1-\alpha}\, S_n(\alpha).$$

But the sequence of general term

$$x_n(\alpha) = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \ldots + \frac{1}{n^{\alpha}} - \frac{n^{1-\alpha}}{1-\alpha} = S_n(\alpha) - \frac{n^{1-\alpha}}{1-\alpha}$$

is convergent to a limit $C(\alpha)$ (a generalized constant of Euler type; see[5], 32 and [13]). This implies the following asymptotic formula

$$S_n(\alpha) = \frac{n^{1-\alpha}}{1-\alpha} + C(\alpha) + o(1)\,.$$

Therefore we obtain

$$S_{2n}^{[a]} = \frac{(2n)^{1-\alpha}}{1-\alpha} + C(\alpha) + o(1) - 2^{1-\alpha}\left(\frac{n^{1-\alpha}}{1-\alpha} + C(\alpha) + o(1)\right) = (1-2^{1-\alpha})C(\alpha) + o(1)$$

and so

$$\lim_{n\to\infty} S_{2n}^{[a]} = (1-2^{1-\alpha})C(\alpha).$$

This conducts us to the final result

$$\sum_{n=1}(-1)^{n-1}\frac{1}{n^{\alpha}} = (1-2^{1-\alpha})C(\alpha).$$

REFERENCES

[1] Boccara, N., *Fonctions analytiques*, Ellipses, Paris, 1996.
[2] Bucur, Gh., Campu, E., Gaina, S., *Culegere de probleme de calcul diferenţial şi integral*, vol. III, Ed. Tehnica, Bucharest, 1967.
[3] Evgrafov, M. et all., *Recueil de problèmes sur la théorie des fonctions analytiques*, Ed. Mir, Moscow, 1974.
[4] Fihtenholţ, G. M., *Curs de calcul diferential si integral (3 vols.)*, Ed. Tehnica, Bucharest, 1963, 1964, 1965.
[5] Finch, S. R., *Mathematical Constants*, Cambridge University Press, city, 2003.
[6] Günther, N. M., Cuzmin, R. O., *Culegere de probleme de matematici superioare*, vol. III, Ed. Tehnica, Bucharest, 1950.
[7] Iaglom, A. M., Iaglom, I. M., *Probleme neelementare tratate elementar*, Ed. Tehnica, Bucharest, 1983.
[8] Kessler, P. G., Rosiu, M., *Culegere de probleme si exercitii de analiza complexa*, Ed. Universitaria, Craiova, 2005.
[9] Knopp, K., *Infinte Sequences and Series*, Dover Publ. Inc., New York, 1956.
[10] Knopp, K., *Theory and Applications of Infinite Series*, Dover Publ. Inc., New York, 1989.
[11] Vernescu, A., *Gaz. Mat.*, **102**(10-11), 370, 1997.
[12] Vernescu, A., *Gaz. Mat. Seria A*, 15(3), 186, 1997.
[13] Vernescu, A., *Gaz. Mat. Seria A*, 25(1), 11, 2007.
[14] Vernescu, A., *Amer Math. Monthly*, 115(10), 939, 2008.