ORIGINAL PAPER

INVESTIGATION OF PROJECTILE MOTION ON THE INCLINED PLANE

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Abstract. *In this paper the equations and relations of projectile motion on inclined planes are concluded from concepts such as angular momentum and torque. In addition to describing this concept explicitly, we have tried to clarify the analysis of the results by solving several examples. Reading this paper provides a new point of view to projectile motion and can simplify solving related problems.*

Keywords: *Inclined plane, angular momentum, projectile motion.*

1. INTRODUCTION

Many of mechanics books have confined the study of projectile motion on an inclined plane to a few unsolved problems. For example, see [1] that basic and important topics in studying motion of a projectile are maximum height and range and the equation of path. Finding above mentioned parameters are concluded according to the location of the projectile in the defined coordinate system. Our main motivation to use this method to derive the equations was reading [2]. We believe this using and generalizing this method in kinematics problems can be effective and useful.

2. METHOD

A projectile with the mass m is thrown with the speed v_0 at the angle of α from the base of an inclined plane. The angle of inclined plane with the horizontal surface is θ. Now we can conclude the motion equations using angular momentum and torque by considering the coordinate system as the Fig. 1.

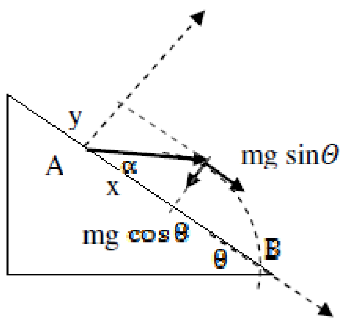

Fig. 1. A projectile with the mass m is thrown with the speed v_0 at the angle of α from the base of an inclined plane. The angle of inclined plane with the horizontal surface is θ.

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Now we consider the $\vec{v} = v_x \hat{i} + v_y \hat{j}$ the velocity vector at A. The angular momentum is $\vec{L} = \vec{r} \times \vec{p}$. According to $\vec{r} = x\hat{i} + y\hat{j}$ we can write

$$L_z = mv_y x - mv_x y \tag{1}$$

According to the torque equation ($\vec{\tau} = \frac{d\vec{L}}{dt}$) and (1) we have

$$\tau_Z = m\frac{dv_y}{dt}x + mv_y\frac{dx}{dt} - m\frac{dv_x}{dt}y - mv_x\frac{dy}{dt} \tag{2}$$

The torque of force around origin is

$$\tau_Z^1 = -mg\cos(\theta)x \tag{3}$$

$$\tau_Z^2 = -mg\sin(\theta)y \tag{4}$$

By comparing equations (2), (3) and (4) we have

$$\frac{dv_y}{dt} = -g\cos(\theta) \quad \& \quad \frac{dv_x}{dt} = g\sin(\theta)$$

Now we can write

$$v_y = -g\cos\theta t + v_{0y} \tag{5}$$

$$v_x = g\sin\theta t + v_{0x} \tag{6}$$

And by definitions $v_x = \frac{dx}{dt}$ and $v_y = \frac{dy}{dt}$ we have

$$y = -\frac{1}{2}g\cos\theta t^2 + v_0\sin\alpha t \tag{7}$$

$$x = \frac{1}{2}g\sin\theta t^2 + v_0\cos\alpha t \tag{8}$$

where $x_0 = y_0 = 0$.

As you see in the Figure (1), at the point of B, y = 0, if we consider this in equation (8), we will have the time of flight of the projectile

$$t = \frac{2v_0\sin\alpha}{g\cos\theta} \tag{9}$$

Now by considering the time of flight (obtained with equation (9) in equation (8)) we obtain the range of the projectile

$$R = \frac{v_0^{\,2}}{g\cos\theta}(\sin\theta + \sin(2\alpha - \theta)) \tag{10}$$

The maximum range is obtained if $\alpha = \frac{\theta}{2} + \frac{\pi}{4}$ which equals to

$$R_{\max} = \frac{v_0^{\,2}}{g(1-\sin\theta)} \tag{11}$$

3. EXAMPLE

Example 1. Once again we consider a projectile with the mass of m which is thrown with the speed v_0 at the angle of α from the surface of an inclined plane (see Fig. 2). To solve this problem we define the coordinate system as shown below.

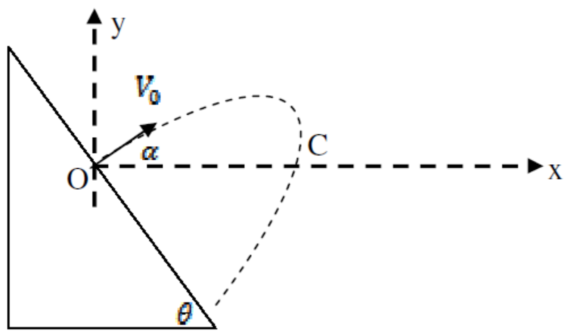

Fig. 2. A projectile with the mass m is thrown with the speed v_0 at the angle of α from the base of an inclined plane. The angle of inclined plane with the horizontal surface is θ. The path of motion, meet x-axis in the point of C.

We study the effect of the torque in the path OC and interval between t_1 and t_2, so that $y(t_1) = 0$, $x(t_1) = 0$, $y(t_2) = 0$, $x(t_2) = d$. According to the

$$\int_{t_1}^{t_2} d\vec{L} = \int_{t_1}^{t_2} \vec{\tau}\, dt\,,\ dt = \frac{dx}{v_{0x}} \tag{12}$$

Therefore

$$L_z(t_2) - L_z(t_1) = \int_{x(t_1)}^{x(t_2)} \tau_Z \frac{dx}{v_{0x}}$$

In which $\tau_Z = mgx$ (x equals the horizontal distance on path OC)

$$mv_{0y}d - 0 = \int_0^d mgx\frac{dx}{v_{0x}} = \frac{mgd^2}{2v_{0x}}\,.$$

So

$$d = \frac{2v_{0y}v_{0x}}{g} = \frac{2{v_0}^2}{g}\cos(\alpha - \theta)\sin(\alpha - \theta) \qquad (13)$$

By considering $\theta = 0$ in the above mentioned equation, the equation of the range of the projectile will be obtained. To calculate the maximum height we have to consider $v_y = 0$.

In equation (5) which gives us the time at the maximum height, then we can insert it in equation (7) then we have

$$R = \frac{{v_0}^2 \sin^2\alpha}{2g\cos\theta} \qquad (14)$$

4. THE FIRST SPECIAL CONDITION

A projectile is thrown perpendicular to the surface of an inclined plane which makes the angle θ with the horizontal surface. Calculate the range of the projectile.

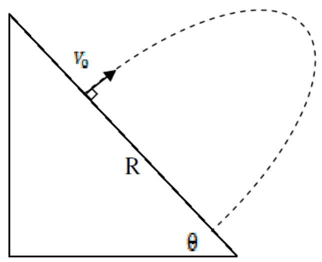

Fig. 3. In this figure R is the range and θ is the angle of inclined plane.

To solve we have to consider $\alpha = \frac{\pi}{2}$ in equation (10). So we will have

$$R = \frac{2{v_0}^2 \sin\theta}{g\cos^2\theta} \qquad (15)$$

5. THE SECOND SPECIAL CONDITION

A projectile is thrown with the speed v_0 at the angle of α from the base of an inclined plane. And the angle of inclined plane with the horizontal surface is θ. The projectile lands perpendicular to the inclined plane. Calculate the range of projectile.

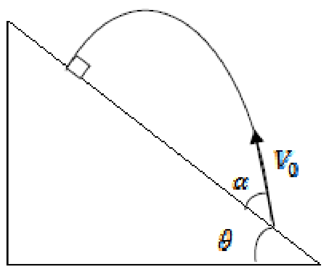

Fig. 4. A projectile is thrown with the speed v_0 at the angle of α from the base of an inclined plane. And the angle of inclined plane with the horizontal surface is θ. The projectile lands at the inclined plane perpendicularly.

We can write the equation of motion

$$x = -\frac{1}{2} g \sin \theta t^2 + v_0 \cos \alpha t \tag{16}$$

$$y = -\frac{1}{2} g \cos \theta t^2 + v_0 \sin \alpha t \tag{17}$$

We know that $v_x = 0$ at landing time so we have

$$t = \frac{v_0 \cos\alpha}{g \sin\theta} \tag{18}$$

By considering this obtained time (equation (18) in equation (16)) we will have

$$R = \frac{v_0^2 \cos^2 \alpha}{2g \sin\theta} \tag{19}$$

We know that $y = 0$, and also we have the time of flight (equation (18)) by inserting them in equation (17) we would have

$$2\tan\alpha = \cot\theta \tag{20}$$

6. THE EQUATION OF PATH

We return to Fig. 1. At first we obtain time from equation of (8) then we have

$$t = \frac{\sqrt{2gx\sin\theta + v_0^2 \cos^2 \alpha}}{g\sin\theta} - \frac{v_0 \cos\alpha}{g\sin\theta} \tag{21}$$

By limit of equation (21), $\theta \to 0$, we obtain time of projectile motion on the horizontal.

Also by the arrangement relation (21) we can write

$$t = \frac{2x}{v_0 \cos\alpha + \sqrt{2gx\sin\theta + {v_0}^2 \cos^2\alpha}} \tag{22}$$

By substitute relation (22) in relation (7) we conclude the equation of path

$$y = \frac{-2g\cos\theta}{\left(v_0 \cos\alpha + \sqrt{2gx\sin\theta + {v_0}^2 \cos^2\alpha}\right)^2} x^2 + \frac{v_0 \sin\alpha}{v_0 \cos\alpha + \sqrt{2gx\sin\theta + {v_0}^2 \cos^2\alpha}} x \tag{23}$$

Again by limit, $\theta \to 0$, on above relation we obtain the equation of path relate to projectile motion on the horizontal.

7. CONCLUSION

The motion of a projectile on an inclined plane is analyzed. We solved some example, in which the concepts such as angular momentum and torque can provide the easiest and the most convenient solutions. We obtain exact relation for the equation of the path.

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[2] Baghchi, B., Holody, P., *Study of projectile motion by angular momentum and torque, The phys. Teach*, **29**, 376, 1991.