***ORIGINAL PAPER***

# OPERATION TRANSFORM FORMULAE FOR GENERALIZED FRACTIONAL HILBERT TRANSFORM

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***Abstract.*** *The generalized fractional Hilbert transform has many applications in several areas including signal processing. In this paper we have proved some Operation Transform Formulae for the generalized fractional Hilbert transform which will be useful for solving the partial differential equations.*

***Keywords:*** *Fractional Fourier Transform, Generalized Fractional Hilbert Transform, Signal Processing.*

## 1. INTRODUCTION

The Fourier transform (F.T.) is a most popular transform in the theory of optics, signal processing and many other branch of engineering. The concept of fractional Fourier transform (FrFT) which is generalization of F.T. was first introduced by Namias [3] in 1980 which was supplemented by Mc-Bride and Kerr [2] who gave a rigorous mathematical justification to Namias work. In the past decade FrFT has attracted much attention due to its vast range of applications in imaging, signal processing, filtering, lense designing etc.

The linear operator Hilbert Transform, which associates a function $f(x)$ with the function $(H(x))(t)$ with the same domain is a powerful tool in signal processing. Its relation with F. T. is

$$\left[F\left(H(x)\right)\right](t)=\left(-i\,\mathrm{sgn}(t)\right)\left[F(x)\right](t)$$

where sgn is signum function and hence

$$\left[F\left(H(x)\right)\right](t)=\begin{cases} i\left[F(x)\right](t) \ , & \text{if } t<0 \\ 0 \ , & \text{if } t=0 \\ -i\left[F(x)\right](t), & \text{if } t>0 \end{cases}$$

The analytic part of a signal $f(x)$ is defined in [6] as $F(x)=f(x)+i(Hf)(x)$.

One of the most important properties of the analytic signal is that it contains no negative frequency components of the real signal. On the similar manner Zayed in [5]

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obtained the analytic part of a signal which is associated with its FrFT and thus defined fractional Hilbert transform (FrHT) for the function $f(x)$ as

$$H_\alpha\left[f(x)\right](t)=f(t)=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

for $t\in\mathbb{R}$, $\alpha\neq 0,\ \frac{\pi}{2},\ \pi$.

When the integral exist, where the integral is a Cauchy Principal value.

Notice that generalized fractional Hilbert transform reduces to standard one as in [1] when $\alpha=\frac{\pi}{2}$.

In [4] we have obtained the inversion of fractional Hilbert transform and proved some properties. In this paper we have concentrated on some more operation transform formulae for generalized Fractional Hilbert Transform.

## 2. OPERATION TRANSFORM FORMULAE FOR FRACTIONAL HILBERT TRANSFORM

**Result 2.1:** If *a* and *b* are any constants then

$$H_\alpha\left[f(ax+b)\right](t)=e^{ib\cot\beta(at+b)}H_\beta\left[f(x)e^{-ib\cot\beta x}\right](at+b)$$

where $\cot\beta=\frac{\cot\alpha}{a^2}$.

*Proof:*

$$H_\alpha\left[f(ax+b)\right](t)=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(ax+b)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi a}\int_{-\infty}^{\infty}\frac{f(X)}{\left(t-\frac{X-b}{a}\right)}e^{i\frac{\cot\alpha}{2}\left(\frac{X-b}{a}\right)^2}dX$$

$$=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(X)}{(at+b)-X}e^{i\frac{\cot\alpha}{2a^2}\left(X^2-2Xb+b^2\right)}dX$$

$$=e^{i\frac{\cot\alpha\ b^2}{2a^2}}\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\frac{b\cot\alpha}{a^2}X}f(X)}{(at+b)-X}e^{i\frac{\cot\alpha}{2a^2}X^2}dX$$

$$=\frac{e^{i\frac{b\cot\alpha}{a^2}(at+b)}e^{-i\frac{\cot\alpha}{2a^2}(at+b)^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\frac{b\cot\alpha}{a^2}X}f(X)}{(at+b)-X}e^{i\frac{\cot\alpha}{2a^2}X^2}dX$$

$$=\frac{e^{ib\cot\beta(at+b)}e^{-i\frac{\cot\beta}{2}(at+b)^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-ib\cot\beta X}f(X)}{(at+b)-X}e^{i\frac{\cot\beta}{2}X^2}dX$$

$$H_\alpha\left[f(ax+b)\right](t)=e^{ib\cot\beta(at+b)}H_\beta\left[f(x)e^{-ib\cot\beta x}\right](at+b)$$

where $\cot\beta=\dfrac{\cot\alpha}{a^2}$.

Note that for $\alpha=1$, we get special case (4.2) already proved in [4].

**Result 2.2:** If *a* is any constant then

$$H_\alpha\left[(x+a)\,f(x)\right](t)=(t+a)H_\alpha\left[f(x)\right](t)-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}f(x)e^{i\frac{\cot\alpha}{2}x^2}d$$

*Proof:* $H_\alpha\left[(x+a)\,f(x)\right](t)=\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{(x+a)\,f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx=$

$$=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{\left[(t+a)-(t-x)\right]f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H_\alpha\left[(x+a)\,f(x)\right](t)=(t+a)H_\alpha\left[f(x)\right](t)-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}f(x)e^{i\frac{\cot\alpha}{2}x^2}d$$

**Result 2.3:** If *a* is any constant and for any integer $n\geq 0$ then

$$H_\alpha\left[(x+a)^n\,f(x)\right](t)=(t+a)^nH_\alpha\left[f(x)\right](t)-\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\int_{-\infty}^{\infty}\left(t^k-x^k\right)f(x)e^{i\frac{\cot\alpha}{2}x^2}d\quad x$$

*Proof:* $H_\alpha\left[(x+a)^n\,f(x)\right](t)=\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{(x+a)^n\,f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx=$

$$=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{\left[(t+a)^n-\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\left(t^k-x^k\right)\right]f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H_\alpha\left[(x+a)^n\,f(x)\right](t)=(t+a)^nH_\alpha\left[f(x)\right](t)-\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\int_{-\infty}^{\infty}\left(t^k-x^k\right)f(x)e^{i\frac{\cot\alpha}{2}x^2}d\quad x$$

**Result 2.4:** Prove that $H_\alpha\left[\dfrac{f(x)}{x}\right](t)=\dfrac{1}{t}\left\{H_\alpha\left[f(x)\right](t)-e^{-i\frac{\cot\alpha}{2}}H_\alpha\left[f(x)\right](0)\right\}$.

*Proof:* From result 2.2, by assuming $f(x)=h(x)$ and $\alpha=0$, we write

$$tH_\alpha\left[h(x)\right](t)=H_\alpha\left[xh(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}h(x)e^{i\frac{\cot\alpha}{2}x^2}d\ .$$

Let $h(x)=\dfrac{f(x)}{x}$, we get

$$tH_\alpha\left[\frac{f(x)}{x}\right](t)=H_\alpha\left[f(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$=H_\alpha\left[f(x)\right](t)-e^{-i\frac{\cot\alpha}{2}t^2}H_\alpha\left[f(x)\right](0)$$

$$H_\alpha\left[\frac{f(x)}{x}\right](t)=\frac{1}{t}\left\{H_\alpha\left[f(x)\right](t)-e^{-i\frac{\cot\alpha}{2}}H_\alpha\left[f(x)\right](0)\right\}$$

**Result 2.5:** Show that for any integer $n>0$

$$H_\alpha\left[\frac{f(x)}{x^n}\right](t)=\frac{1}{t^n}\left\{H_\alpha\left[f(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}t^k\int_{-\infty}^{\infty}x^{-1-k}f(x)e^{i\frac{\cot\alpha}{2}x^2}dx\right\}$$

*Proof:* From result 2.3, by assuming $f(x)=h(x)$ and $\alpha=0$, for any integer $n>0$, we write

$$t^nH_\alpha\left[h(x)\right](t)=H_\alpha\left[x^nh(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}t^k\int_{-\infty}^{\infty}x^{-1-k}h(x)e^{i\frac{\cot\alpha}{2}x^2}d\ x$$

Let $h(x)=\dfrac{f(x)}{x^n}$, for any integer $n>0$ , we get

$$tH_\alpha\left[\frac{f(x)}{x^n}\right](t)=H_\alpha\left[f(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}t^k\int_{-\infty}^{\infty}x^{-1-k}\frac{f(x)}{x^n}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H_\alpha\left[\frac{f(x)}{x^n}\right](t)=\frac{1}{t^n}\left\{H_\alpha\left[f(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}t^k\int_{-\infty}^{\infty}x^{-1-k}f(x)e^{i\frac{\cot\alpha}{2}x^2}dx\right\}$$

**Result 2.6:** If $f^n(x)$ is the n[th] derivative of $f(x)$ then

$$H_\alpha\left[f^n(x)\right](t)=-i\cot\alpha H_\alpha\left[xf^{n-1}(x)\right](t)-H_\alpha\left[\frac{f^{n-1}(x)}{t-x}\right](t)$$

*Proof:* $H_\alpha\left[f^n(x)\right](t)=\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{e^{i\frac{\cot\alpha}{2}x^2}}{t-x}f^n(x)d$

$$=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\left[\frac{i\cot\alpha\ x}{t-x}+\frac{1}{(t-x)^2}\right]e^{i\frac{\cot\alpha}{2}x^2}f^{n-1}(x)dx$$

$$=\frac{-i\cot\alpha\ e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{xf^{n-1}(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{\left[f^{n-1}(x)/(t-x)\right]}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H_\alpha\left[f^n(x)\right](t) = -i\cot\alpha H_\alpha\left[xf^{n-1}(x)\right](t) - H_\alpha\left[\frac{f^{n-1}(x)}{t-x}\right](t)$$

**Result 2.7:** If $f'(x)$ is the first derivative of $f(x)$ then

$$H_\alpha\left[xf'(x)\right](t) = tH_\alpha\left[f'(x)\right](t) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty} f'(x)e^{i\frac{\cot\alpha}{2}x^2}d$$

*Proof:* Consider

$$H_\alpha\left[xf'(x)\right](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{xf'(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\left(\frac{t}{t-x}-1\right)f'(x)e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H_\alpha\left[xf'(x)\right](t) = tH_\alpha\left[f'(x)\right](t) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty} f'(x)e^{i\frac{\cot\alpha}{2}x^2}d$$

**Result 2.8:** If $f^n(x)$ is the n[th] derivative of $f(x)$ for any integer $n \geq 0$ then

$$H_\alpha\left[xf^n(x)\right](t) = tH_\alpha\left[f(x)\right](t) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty} f^n(x)e^{i\frac{\cot\alpha}{2}x^2}d$$

*Proof:* $H_\alpha\left[xf^n(x)\right](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{xf^n(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{\left[t-(t-x)\right]f^n(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H_\alpha\left[xf^n(x)\right](t) = tH_\alpha\left[f(x)\right](t) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty} f^n(x)e^{i\frac{\cot\alpha}{2}x^2}d$$

**Table 1. Tabular form of some operation formulae.**

| **Sr. No.** | **Function** | **Fractional Hilbert Transform H[Function]** |
|---|---|---|
| 1 | $f(ax+b)$ | $e^{ib\cot\beta(at+b)}H_\beta\left[f(x)e^{-ib\cot\beta x}\right](at+b)$ where $\cot\beta = \frac{\cot\alpha}{a^2}$ |
| 2 | $(x+a)f(x)$ | $(t+a)H_\alpha\left[f(x)\right](t) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty} f(x)e^{i\frac{\cot\alpha}{2}x^2}d$ |
| 3 | $(x+a)^n f(x)$ for any integer $n \geq 0$ | $(t+a)^n H_\alpha\left[f(x)\right](t) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}\int_{-\infty}^{\infty}\left(t^k - x^k\right)f(x)e^{i\frac{\cot\alpha}{2}x^2}d\ \ x$ |

| | | |
|---|---|---|
| 4 | $\frac{f(x)}{x}$ | $\frac{1}{t}\left\{H_\alpha\left[f(x)\right](t)-e^{-i\frac{\cot\alpha}{2}}H_\alpha\left[f(x)\right](0)\right\}$ |
| 5 | $\frac{f(x)}{x^n}$ for any integer $n>0$ | $\frac{1}{t^n}\left\{H_\alpha\left[f(x)\right](t)+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}t^k\int_{-\infty}^{\infty}x^{-1-k}f(x)e^{i\frac{\cot\alpha}{2}x^2}dx\right\}$ |
| 6 | $f^n(x)$ | $-i\cot\alpha H_\alpha\left[xf^{n-1}(x)\right](t)-H_\alpha\left[\frac{f^{n-1}(x)}{t-x}\right](t)$ |
| 7 | $xf'(x)$ | $tH_\alpha\left[f'(x)\right](t)-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}f'(x)e^{i\frac{\cot\alpha}{2}x^2}d\quad x$ |
| 8 | $xf^n(x)$ for any integer $n\geq 0$ | $tH_\alpha\left[f^n(x)\right](t)-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}f^n(x)e^{i\frac{\cot\alpha}{2}x^2}d$ |

## 3. CONCLUSIONS

The fractional Hilbert transform is a further generalization of the ordinary Hilbert transform. The operation transform formulae proved above tally with the similar formulae for classical Hilbert transform if we put $\alpha=\frac{\pi}{2}$. The generalized fractional Hilbert transform can produce the image edge enhancement or the image compression in different ways when both parameters (the angle of the fractional Fourier transform and the phase of the fractional Hilbert transform) are varying.

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