ORIGINAL PAPER

SOME PROPERTIES OF TWO LINEARLY INDEPENDENT MEROMORPHIC SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS*

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Abstract. *This paper is devoted to studying the growth and oscillation of the polynomial of combinaison of two linearly independent meromorphic solutions to the complex differential equation*

$$f''+A(z)f'+B(z)f=0,$$

where A(z) and B (z) are meromorphic functions of finite iterated p-order.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we assume that readers are familiar with the fundamental results and standard notations of the Nevanlinna's theory of meromorphic functions [8, 12]. In order to describe the growth of order of meromorphic functions more precisely, we introduce some notations about finite iterated order. For the definition of the iterated order of a meromorphic function, we use the same definition as in [3, 9, 10]. For all $r\in\mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p\in\mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1 [9, 10]: Let *f* be a meromorphic function. Then the iterated *p*−order $\rho_p(f)$ of *f* is defined as

$$\rho_p(f)=\limsup_{r\to\infty}\frac{\log_p T(r,f)}{\log r} \qquad (p\geq 1 \text{ is an integer}),$$

where $T(r,f)$ is the Nevanlinna characteristic function of *f* [8, 12].

For $p=1$, this notation is called order and for $p=2$ hyper-order.

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Definition 1.2 [9]: The finiteness degree of the order of a meromorphic function f is defined as

$$i(f)=\begin{cases}0 & \text{, for } f \text{ rational,}\\ \min\left\{j\in\mathbb{N}:\ \rho_j(f)<\infty\right\} & \text{, for } f \text{ transcedental for which some } j\in\mathbb{N} \text{ with } \rho_j(f)<\infty \text{ exists,}\\ \infty & \text{, for } f \text{ with } \rho_j(f)=\infty \text{ for all } j\in\mathbb{N}\end{cases}$$

Definition 1.3 [5, 6]: The type of a meromorphic function f of iterated p–order $\rho(0<\rho<\infty)$ is defined as

$$\tau_p(f)=\limsup_{r\to\infty}\frac{\log_{p-1}T(r,f)}{r^{\rho}}\qquad (p\geq 1 \text{ is an integer}).$$

Definition 1.4 [9]: Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as

$$\lambda_p(f)=\limsup_{r\to\infty}\frac{\log_p N\left(r,\frac{1}{f}\right)}{\log r}\qquad (p\geq 1 \text{ is an integer}),$$

where $N\left(r,\frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z:|z|\leq r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of zeros and for $p=2$ hyper-exponent of convergence of the sequence of zeros [7]. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_p(f)=\limsup_{r\to\infty}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log r}\qquad (p\geq 1 \text{ is an integer}),$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z:|z|\leq r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$ hyper-exponent of convergence of the sequence of distinct zeros [7].

It is well-known that the study of the properties of solutions of complex differential equations is an interesting topic. The oscillation theory for complex differential equations in the complex plane was firstly investigated by Bank and Laine in 1982-1983 [1, 2]. In [7], Chen pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of the linear differential equation

$$f''+A(z)f=0 \qquad (1.1)$$

when A is a polynomial and A is a transcendental entire function with finite order. Recently, the author and Latreuch [11] have investigated the relations between the polynomial of solutions of (1.1) and small functions. They showed that $w=d_1f_1+d_2f_2$ keeps the same properties of the growth and oscillation of $f_j\,(j=1,2)$, where f_1 and f_2 are two linearly independent solutions of (1.1) and obtained the following results.

Theorem A [11]: Let $A(z)$ be a transcendental entire function of finite order. Let $d_j(z)\,(j=1,2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1),\rho(d_2)\}<\rho(A)$. If f_1 and f_2 are two linearly independent solutions of (1.1), then the polynomial of solutions $w=d_1f_1+d_2f_2$ satisfies

$$\rho(w)=\rho(f_j)=\infty \qquad (j=1,2)$$

and

$$\rho_2(w)=\rho_2(f_j)=\rho(A) \qquad (j=1,2).$$

Theorem B [11]: Under the hypotheses of Theorem A, let $\varphi(z)\not\equiv 0$ be an entire function with finite order such that $\psi(z)\not\equiv 0$. If f_1 and f_2 are two linearly independent solutions of (1.1), then the polynomial of solutions $w=d_1f_1+d_2f_2$ satisfies

$$\overline{\lambda}(w-\varphi)=\lambda(w-\varphi)=\rho(f_j)=\infty \qquad (j=1,2)$$

and

$$\overline{\lambda}_2(w-\varphi)=\lambda_2(w-\varphi)=\rho_2(f_j)=\rho(A) \qquad (j=1,2),$$

where

$$\psi(z)=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\varphi^{(3)}+\phi_2\varphi''+\phi_1\varphi'+\phi_0\varphi,$$

$$\phi_2=\frac{3d_2^2d_1''-3d_1d_2d_2''}{h},$$

$$\phi_1=\frac{2d_1d_2d_2'A+6d_2d_1'd_2''-6d_2d_2'd_1''-2d_2^2d_1'A}{h},$$

$$\phi_0=\frac{2d_2d_1'd_2'''-2d_1d_2'd_2'''-3d_1d_2d_2''A-3d_2d_1''d_2''+2d_1d_2d_2'A'}{h}$$

$$-\frac{4d_2d_1'd_2'A-6d_1'd_2'd_2''+3d_1\left(d_2''\right)^2+4d_1\left(d_2'\right)^2A+3d_2^2d_1''A}{h}$$

$$+\frac{6\left(d_2'\right)^2d_1''-2d_2^2d_1'A'}{h}$$

In this paper, we extend the results of Theorems A-B from entire solutions to meromorphic solutions by considering the complex differential equation

$$f''+A(z)f'+B(z)f=0, \qquad (1.2)$$

where $A(z)$ and $B(z)$ are meromorphic functions of finite iterated p−order.

Before we state our results we define the functions $h(z)$ and $\psi(z)$ by

$$h=\begin{vmatrix} H_1 & H_2 & H_3 & H_4 \\ H_5 & H_6 & H_7 & H_8 \\ H_9 & H_{10} & H_{11} & H_{12} \\ H_{13} & H_{14} & H_{15} & H_{16} \end{vmatrix},$$

where $H_1 = d_1$, $H_2 = 0$, $H_3 = d_2$, $H_4 = 0$, $H_5 = d_1'$, $H_6 = d_1$, $H_7 = d_2'$, $H_8 = d_2$,
$H_9 = d_1'' - d_1 B$, $H_{10} = 2d_1' - d_1 A$, $H_{11} = d_2'' - d_2 B$, $H_{12} = 2d_2' - d_2 A$,
$H_{13} = d_1''' - 3d_1' B + d_1 AB - d_1 B'$, $H_{14} = 3d_1'' - 2d_1' A - d_1 B + d_1 A^2 - d_1 A'$,
$H_{15} = d_2''' - 3d_2' B + d_2 AB - d_2 B'$, $H_{16} = 3d_2'' - 2d_2' A - d_2 B + d_2 A^2 - d_2 A'$,

$$\psi(z) = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi, \qquad (1.3)$$

where $\varphi \not\equiv 0$, $d_j\,(j = 1,2)$ are meromorphic functions of finite iterated p-order and

$$\phi_2 = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)A - 3d_1 d_2 d_2'' + 3d_2^2 d_1''}{h}, \qquad (1.4)$$

$$\phi_1 = \frac{1}{h}\left[6d_2\left(d_1' d_2'' - d_2' d_1''\right) + 2d_2\left(d_1 d_2' - d_2 d_1'\right)B + 2d_2\left(d_1 d_2' - d_2 d_1'\right)A' + 3d_2\left(d_2 d_1'' - d_1 d_2''\right)A\right], \qquad (1.5)$$

$$\begin{aligned}\phi_0 = \frac{1}{h}\Big[&\left(d_1 d_2' d_2'' - 3d_2 d_2' d_1'' + 2d_2 d_1' d_2''\right)A + \left(4d_1\left(d_2'\right)^2 + 3d_2^2 d_1'' - 3d_1 d_2 d_2'' - 4d_2 d_1' d_2'\right)B \\ &+ 2\left(d_2 d_1' d_2' - d_1\left(d_2'\right)^2\right)A' + 2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)B' + 6\left(d_2'\right)^2 d_1'' - 2d_1 d_2' d_2''' \\ &+ 2d_2 d_1' d_2''' - 3d_2 d_1'' d_2'' - 6d_1' d_2' d_2'' + 3d_1\left(d_2''\right)^2\Big]\end{aligned} \qquad (1.6)$$

Theorem 1.1 Let $A(z)$ and $B(z)$ be meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(A) < \rho_p(B)$ and $0 < \tau_p(A) < \tau_p(B) < \infty$ if $\rho_p(B) = \rho_p(A) > 0$. Let $d_j(z)\,(j = 1,2)$ be meromorphic functions that are not all vanishing identically such that $\max\{\rho_p(d_1), \rho_p(d_2)\} < \rho_p(B)$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicity of (1.2), then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies $i(w) = p + 1$,

$$\rho_p(w) = \rho_p(f_j) = \infty \qquad (j = 1,2)$$

and

$$\rho_{p+1}(w) = \rho_{p+1}(f_j) = \rho_p(B) \qquad (j = 1,2).$$

From Theorem 1.1, we can obtain the following result.

Corollary 1.1 Let $f_i(z)$ $(i = 1,2)$ be two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicity of (1.2) , where $A(z)$ and $B(z)$ are meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(A) < \rho_p(B)$ or $\rho_p(B) = \rho_p(A) > 0$ and $0 < \tau_p(A) < \tau_p(B) < \infty$, and let $d_j(z)\,(j = 1,2,3)$ be meromorphic functions satisfying

$$\max\{\rho_p(d_j) : j = 1,2,3\} < \rho_p(B)$$

and

$$d_1(z) f_1 + d_2(z) f_2 = d_3(z).$$

Then $d_j(z) \equiv 0 \ (j = 1,2,3)$.

Proof. Suppose there exists $j = 1,2,3$ such that $d_j(z) \not\equiv 0$ and we obtain a contradiction. If $d_1(z) \not\equiv 0$ or $d_2(z) \not\equiv 0$, then we have

$$\rho_p(d_1 f_1 + d_2 f_2) = \infty = \rho_p(d_3) < \rho_p(B)$$

which is a contradiction. Now if $d_1(z) \equiv 0$, $d_2(z) \equiv 0$ and $d_3(z) \not\equiv 0$ we obtain also a contradiction. Hence $d_j(z) \not\equiv 0 \ (j = 1,2,3)$.

Theorem 1.2 Under the assumptions of Theorem 1.1, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite iterated p–order such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicity of (1.2), then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies

$$\overline{\lambda}_p(w - \varphi) = \lambda_p(w - \varphi) = \rho_p(f_j) = \infty \quad (j = 1,2) \tag{1.7}$$

and

$$\overline{\lambda}_{p+1}(w - \varphi) = \lambda_{p+1}(w - \varphi) = \rho_{p+1}(f_j) = \rho_p(B) \quad (j = 1,2). \tag{1.8}$$

Theorem 1.3 Let $A(z)$ and $B(z)$ be meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(A) < \rho_p(B)$. Let $d_j(z), b_j(z) \ (j = 1,2)$ be finite iterated p-order meromorphic functions such that $d_1(z)b_2(z) - d_2(z)b_1(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicity of (1.2), then

$$\rho_p\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \infty$$

and

$$\rho_{p+1}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \rho_p(B).$$

Remark 1.1 The condition that the multiplicity of poles of the solution f is uniformly bounded can be changed by $\delta(\infty, f) = \liminf_{r \to \infty} \frac{m(r, f)}{T(r, f)} > 0$ for the solution f [6].

2. AUXILIARY LEMMAS

Lemma 2.1 [4] Let $A_0, A_1, \ldots, A_{k-1}, F$ be finite iterated p–order meromorphic functions. If f is a meromorphic solution of the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = F$$

with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho$, then f satisfies

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho.$$

Lemma 2.2 [6] Let $A(z)$ and $B(z)$ be meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(A) < \rho_p(B)$ and $0 < \tau_p(A) < \tau_p(B) < \infty$ if $\rho_p(B) = \rho_p(A) > 0$. Then every meromorphic solution $f \not\equiv 0$ of (1.2) whose poles are of uniformly bounded multiplicity satisfies $i(f) = p+1$, $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(B)$.

Lemma 2.3 Let $A(z)$ and $B(z)$ be meromorphic functions with $i(B) = p \geq 1$, $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ such that $\rho_p(A) < \rho_p(B)$. If f_1 and f_2 are two linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicity of (1.2), then $\frac{f_1}{f_2}$ satisfies $i\left(\frac{f_1}{f_2}\right) = p+1$ and $\rho_p\left(\frac{f_1}{f_2}\right) = \infty$, $\rho_{p+1}\left(\frac{f_1}{f_2}\right) = \rho_p(B)$.

Proof. Suppose that f_1 and f_2 are two linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2). Since $\lambda_p\left(\frac{1}{B}\right) < \rho_p(B)$ and $\rho_p(A) < \rho_p(B)$, then by Lemma 2.2 we have $i(f_1) = i(f_2) = p+1$ and

$$\rho_p(f_1) = \rho_p(f_2) = \infty,\ \rho_{p+1}(f_1) = \rho_{p+1}(f_2) = \rho_p(B). \tag{2.1}$$

On the other hand

$$\left(\frac{f_1}{f_2}\right)' = -\frac{W(f_1, f_2)}{f_2^2}, \tag{2.2}$$

where $W(f_1, f_2) = f_1 f_2' - f_2 f_1'$ is the wronskian of f_1 and f_2. By using (1.2) we obtain that

$$W'(f_1, f_2) = -A(z) W(f_1, f_2) \tag{2.3}$$

which implies that

$$W(f_1, f_2) = K \exp\left(-\int A(z)\right), \tag{2.4}$$

where $\int A(z)$ is the primitive of $A(z)$ and $K \in \mathbb{C} \setminus \{0\}$. By (2.2) and (2.4) we have

$$\left(\frac{f_1}{f_2}\right)' = -K \frac{\exp\left(-\int A(z)\right)}{f_2^2}. \tag{2.5}$$

Since $\rho_p(f_2)=\infty$ and $\rho_{p+1}(f_2)=\rho_p(B)>\rho_p(A)$, then from (2.5) we obtain $i\left(\frac{f_1}{f_2}\right)=p+1$ and $\rho_p\left(\frac{f_1}{f_2}\right)=\infty$, $\rho_{p+1}\left(\frac{f_1}{f_2}\right)=\rho_p(B)$.

Lemma 2.4 [5] Let f and g be meromorphic functions such that $0<\rho_p(f)$, $\rho_p(g)<\infty$ and $0<\tau_p(f)$, $\tau_p(g)<\infty$. Then we have

(i) If $\rho_p(f)>\rho_p(g)$, then we obtain

$$\tau_p(f+g)=\tau_p(fg)=\tau_p(f).$$

(ii) If $\rho_p(f)=\rho_p(g)$ and $\tau_p(f)\neq\tau_p(g)$, then we get

$$\rho_p(f+g)=\rho_p(fg)=\rho_p(f)=\rho_p(g).$$

3. PROOF OF THEOREMS

Proof of Theorem 1.1 In the case when $d_1(z)\equiv 0$ or $d_2(z)\equiv 0$, then the conclusions of Theorem 1.1 are trivial. Suppose that f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicity of (1.2) such that $d_j(z)\not\equiv 0$ $(j=1,2)$ and let

$$w=d_1f_1+d_2f_2 \tag{3.1}$$

Then by Lemma 2.2, we have $i(f_1)=i(f_2)=p+1$ and

$$\rho_p(f_1)=\rho_p(f_2)=\infty,\ \rho_{p+1}(f_1)=\rho_{p+1}(f_2)=\rho_p(B).$$

Suppose that $d_1=cd_2$, where *c* is a complex number. Then, by (3.1) we obtain

$$w=cd_2f_1+d_2f_2.$$

Since $f=cf_1+f_2$ is a solution of (1.2) and $\rho_p(d_2)<\rho_p(B)$, then we have

$$\rho_p(w)=\rho_p(cf_1+f_2)=\infty$$

and

$$\rho_{p+1}(w)=\rho_{p+1}(cf_1+f_2)=\rho_p(B).$$

Suppose now that $d_1\not\equiv cd_2$ where *c* is a complex number. Differentiating both sides of (3.1), we obtain

$$w'=d_1'f_1+d_1f_1'+d_2'f_2+d_2f_2'. \tag{3.2}$$

Differentiating both sides of (3.2) , we have

$$w''=d_1''f_1+2d_1'f_1'+d_1f_1''+d_2''f_2+2d_2'f_2'+d_2f_2''. \tag{3.3}$$

Substituting $f_j''=-A(z)f_j'-B(z)f_j$ $(j=1,2)$ into equation (3.3), we obtain

$$w''=\left(d_1''-d_1B\right)f_1+\left(2d_1'-d_1A\right)f_1'+\left(d_2''-d_2B\right)f_2+\left(2d_2'-d_2A\right)f_2'. \tag{3.4}$$

Differentiating both sides of (3.4) and substituting $f_j'' = -A(z) f_j' - B(z) f_j \quad (j = 1,2)$, we get

$$\begin{aligned} w''' &= \left(d_1''' - 3d_1'B + d_1\left(AB - B'\right)\right) f_1 + \left(3d_1'' - 2d_1'A + d_1\left(A^2 - A' - B\right)\right) f_1' \\ &+ \left(d_2''' - 3d_2'B + d_2\left(AB - B'\right)\right) f_2 + \left(3d_2'' - 2d_2'A + d_2\left(A^2 - A' - B\right)\right) f_2' \end{aligned} \tag{3.5}$$

By (3.1) − (3.5) we have

$$\begin{cases} w = d_1 f_1 + d_2 f_2 \\ w' = d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2' \\ w'' = \left(d_1'' - d_1 B\right) f_1 + \left(2d_1' - d_1 A\right) f_1' + \left(d_2'' - d_2 B\right) f_2 + \left(2d_2' - d_2 A\right) f_2' \\ w''' = \left(d_1''' - 3d_1'B + d_1\left(AB - B'\right)\right) f_1 + \left(3d_1'' - 2d_1'A + d_1\left(A^2 - A' - B\right)\right) f_1' \\ \quad + \left(d_2''' - 3d_2'B + d_2\left(AB - B'\right)\right) f_2 + \left(3d_2'' - 2d_2'A + d_2\left(A^2 - A' - B\right)\right) f_2' \end{cases} \tag{3.6}$$

To solve this system of equations, we need first to prove that $h \not\equiv 0$. By simple calculations we obtain

$$h = \begin{vmatrix} H_1 & H_2 & H_3 & H_4 \\ H_5 & H_6 & H_7 & H_8 \\ H_9 & H_{10} & H_{11} & H_{12} \\ H_{13} & H_{14} & H_{15} & H_{16} \end{vmatrix}, \tag{3.7}$$

$$\begin{aligned} &= 2\left(d_1 d_2' - d_2 d_1'\right)^2 B + \left(d_2^2 d_1' d_1'' + d_1^2 d_2' d_2'' - d_1 d_2 d_1' d_2'' - d_1 d_2 d_2' d_1''\right) A \\ &- 2\left(d_1 d_2' - d_2 d_1'\right)^2 A' + 2d_1 d_2 d_1' d_2''' + 2d_1 d_2 d_2' d_1''' - 6d_1 d_2 d_1'' d_2'' - 6d_1 d_1' d_2 d_2'' - 6d_2 d_1' d_2 d_1'' \end{aligned}$$

It is clear that $\left(d_1 d_2' - d_2 d_1'\right)^2 \not\equiv 0$ because $d_1 \not\equiv c d_2$. Since

$$\max\left\{\rho_p\left(d_1\right), \rho_p\left(d_2\right)\right\} < \rho_p\left(B\right)$$

and $\left(d_1 d_2' - d_2 d_1'\right)^2 \not\equiv 0$, then by using Lemma 2.4 we obtain

$$\rho_p\left(h\right) = \rho_p\left(B\right) > 0. \tag{3.8}$$

Hence $h \not\equiv 0$. By Cramer's method we have

$$f_1 = \frac{\begin{vmatrix} w & H_2 & H_3 & H_4 \\ w' & H_6 & H_7 & H_8 \\ w'' & H_{10} & H_{11} & H_{12} \\ w^{(3)} & H_{14} & H_{15} & H_{16} \end{vmatrix}}{h} = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(3)} + \phi_2 w'' + \phi_1 w' + \phi_0 w \tag{3.9}$$

where $\phi_j \ \left(j = 0,1,2\right)$ are meromorphic functions of finite iterated $p-$order which are defined in (1.4) − (1.6). Suppose now $\rho_p\left(w\right) < \infty$. Then by (3.9) we obtain $\rho_p\left(f_1\right) < \infty$ which is a contradiction. Hence $\rho_p\left(w\right) = \infty$. By (3.1) we have $\rho_{p+1}\left(w\right) \le \rho_p\left(B\right)$. Suppose that $\rho_{p+1}\left(w\right) < \rho_p\left(B\right)$. Then by (3.9) we obtain $\rho_{p+1}\left(f_1\right) < \rho_p\left(B\right)$ which is a contradiction. Hence $\rho_{p+1}\left(w\right) = \rho_p\left(B\right)$.

Proof of Theorem 1.2 By Theorem 1.1 we have $\rho_p\left(w\right) = \infty$ and $\rho_{p+1}\left(w\right) = \rho_p\left(B\right)$. Set $g\left(z\right) = d_1 f_1 + d_2 f_2 - \varphi$. Since $\rho_p\left(\varphi\right) < \infty$, then we have $\rho_p\left(g\right) = \rho_p\left(w\right) = \infty$ and $\rho_{p+1}\left(g\right) = \rho_{p+1}\left(w\right) = \rho_p\left(B\right)$. In order to prove $\overline{\lambda}_p\left(w-\varphi\right) = \lambda_p\left(w-\varphi\right) = \infty$ and $\overline{\lambda}_{p+1}\left(w-\varphi\right) = \lambda_{p+1}\left(w-\varphi\right) = \rho_p\left(B\right)$, we need to prove only $\overline{\lambda}_p\left(g\right) = \lambda_p\left(g\right) = \infty$ and $\overline{\lambda}_{p+1}\left(g\right) = \lambda_{p+1}\left(g\right) = \rho_p\left(B\right)$. By $w = g + \varphi$ we get from (3.9)

$$f_1 = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g^{(3)} + \phi_2 g'' + \phi_1 g' + \phi_0 g + \psi, \tag{3.10}$$

where

$$\psi = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} \varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi.$$

Substituting (3.10) into equation (1.2), we obtain

$$2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g^{(5)} + \sum_{j=0}^{4} \beta_j g^{(j)} = -\left(\psi'' + A\left(z\right)\psi' + B\left(z\right)\psi\right) = F\left(z\right),$$

where $\beta_j \ \left(j = 0,...,4\right)$ are meromorphic functions of finite iterated $p-$order. Since $\psi \not\equiv 0$ and $\rho_p\left(\psi\right) < \infty$, it follows that ψ is not a solution of (1.2), which implies that $F\left(z\right) \not\equiv 0$. Then by applying Lemma 2.1 we obtain (1.7) and (1.8).

Proof of Theorem 1.3 Suppose that f_1 and f_2 are two nontrivial linearly independent meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2). Then by Lemma 2.3, we have $i\left(\frac{f_1}{f_2}\right) = p+1$ and

$$\rho_p\left(\frac{f_1}{f_2}\right)=\infty\,,\ \rho_{p+1}\left(\frac{f_1}{f_2}\right)=\rho_p\left(B\right).$$

Set $g=\frac{f_1}{f_2}$. Then

$$w(z)=\frac{d_1(z)f_1(z)+d_2(z)f_2(z)}{b_1(z)f_1(z)+b_2(z)f_2(z)}=\frac{d_1(z)g(z)+d_2(z)}{b_1(z)g(z)+b_2(z)}. \tag{3.11}$$

It follows that $i(w)\le p+1$ and $\rho_p(w)\le\rho_p(g)=\infty$,

$$\rho_{p+1}(w)\le\max\left\{\rho_{p+1}(d_j),\rho_{p+1}(b_j)\ (j=1,2),\rho_{p+1}(g)\right\}=\rho_{p+1}(g). \tag{3.12}$$

On the other hand, we have

$$g(z)=-\frac{b_2(z)w(z)-d_2(z)}{b_1(z)w(z)-d_1(z)}$$

which implies that that $i(w)\ge p+1$ and $\rho_p(w)=\rho_p(g)=\infty$,

$$\rho_{p+1}(g)\le\max\left\{\rho_{p+1}(d_j),\rho_{p+1}(b_j)\ (j=1,2),\rho_{p+1}(w)\right\}=\rho_{p+1}(w). \tag{3.13}$$

By using (3.12) and (3.13), we obtain $i(w)=p+1$ and

$$\rho_p(w)=\rho_p(g)=\infty\,,\ \rho_{p+1}(w)=\rho_{p+1}(g)=\rho_p(B).$$

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