ORIGINAL PAPER

ON SOME GENERATING FUNCTIONS OF MODIFIED GEGENBAUER POLYNOMIALS BY GROUP-THEORETIC METHOD

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Abstract. *In this article, we have obtained some novel theorems on generating functions (both bilateral and mixed trilateral) of modified Gegenbauer polynomials by introducing a partial differential operator obtained by double interpretations to the index (n) and the parameter (λ) of the polynomials under consideration in Weisner's group-theoretic method. Some applications of our results are also discussed.*

Keywords: *Gegenbauer polynomials, Weisner's group-theoretic method. differential operator*

1. INTRODUCTION

The Gegenbauer polynomials, $C_n^{\lambda}(x)$ is defined in hypergeometric form by [5]:

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n\,!}\ {}_2F_1\left[\begin{matrix} -n, & n+2\lambda; & \\ & & \frac{1}{2}-\frac{x}{2} \\ & \lambda+\frac{1}{2}; & \end{matrix}\right].$$

In the present paper, we shall obtain an extension of a bilateral generating function and mixed trilateral generating function by Weisner's group theoretic method [3]. The main results of our investigation are stated in the form of the following theorems.

Theorem 1. If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x)\, w^n \tag{1.1}$$

Then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,v), \tag{1.2}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)!\,(1-\lambda-k)_{n-k}} C_{2n+r-k}^{\lambda-n+2k}(x)\, v^k.$$

Theorem 2. If there exists a bilateral generating relation of the form

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$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x) g_n(u) w^n \qquad (1.3)$$

then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, u, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v), (1.4)$$

where

$$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k} \left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)!\,(1-\lambda-k)_{n-k}} C_{2n+r-k}^{\lambda-n+2k}(x)\, g_k(u) v^k.$$

Theorem 3. If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda}(x)\, w^n \qquad (1.5)$$

then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, \frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x,v), (1.6)$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k} \left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!(1-\lambda)_{n-k}} C_{2n+r-k}^{\lambda-n+k}(x)\, v^k.$$

Theorem 4. If there exists a unilateral generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda}(x) g_n(u)\, w^n \qquad (1.7)$$

then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, u, \frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v), (1.8)$$

where

$$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k} \left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!(1-\lambda)_{n-k}} C_{2n+r-k}^{\lambda-n+k}(x) g_k(u) v^k.$$

Here, we would like to mention that to derive the above theorems we define a novel partial differential operator and the corresponding extended group which do not seem to appear before. Moreover, at the end of the paper, with the help of our results we obtain two new theorems (Theorem 5, Theorem 6) on generating functions. Finally, we like to point it out that some applications of our theorems are also given in the paper.

2. DERIVATION OF OPERATOR AND EXTENDED FORM OF THE GROUP

At first we seek the following first order linear partial differential operator:

$$R = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_0 \qquad (2.1)$$

such that

$$R\left(C_{n+r}^{\lambda+n}(x)\ y^n z^\lambda\right) = \Omega_n C_{n+r+2}^{\lambda+n-1}(x) y^{n+2} z^{\lambda-3}. \tag{2.2}$$

where $Ai\ (i\ = 0,1,2,3)$ are functions of *x*, *y*, *z* but independent of *n*, λ and Ω_n is a function of *n*, λ but independent of *x*, *y*, *z*.

Using (2.2) and with the help of the differential recurrence relation:

$$x(1-x^2)\frac{d}{dx}C_{n+r}^{\lambda+n}(x) = \frac{(n+r+1)(n+r+2)}{2(1-\lambda-n)}C_{n+r+2}^{\lambda+n-1}(x) + [(2\lambda+3n+r)x^2-(n+r+1)]\,C_{n+r}^{\lambda+n}(x) \tag{2.3}$$

we easily obtain the following linear partial differential operator:

$$R = x(1-x^2)\frac{y^2}{z^3}\frac{\partial}{\partial x} + (1-3x^2)\frac{y^3}{z^3}\frac{\partial}{\partial y} - \frac{2x^2y^2}{z^2}\frac{\partial}{\partial z} + (1+r-rx^2)\frac{y^2}{z^3} \tag{2.4}$$

such that

$$R\left(C_{n+r}^{\lambda+n}(x)\ y^n z^\lambda\right) = \frac{(n+r+1)(n+r+2)}{2(1-\lambda-n)}C_{n+r+2}^{\lambda+n-1}(x) y^{n+2} z^{\lambda-3}. \tag{2.5}$$

Now we find the extended form of the group generated by *R* i.e., we shall find $e^{wR}f(x,y,z)$, where $f(x,y,z)$ is arbitrary function and *w* is an arbitrary constant, real or complex. Let $\phi(x,y,z)$ be a function such that $R\ \phi\ =\ 0$. Then on solving $R\ \phi\ =\ 0$, we get a solution as

$$\phi = \frac{yz}{x^{r+2}}\ (1-x^2)^{-\frac{3}{2}}.$$

Let us transform *R* to *E*, where

$$E = x(1-x^2)\frac{y^2}{z^3}\frac{\partial}{\partial x} + (1-3x^2)\frac{y^3}{z^3}\frac{\partial}{\partial y} - \frac{2x^2y^2}{z^2}\frac{\partial}{\partial z} \tag{2.6}$$

then

$$E\ =\ \phi^{-1}(x,y,z)\ R\ \phi(x,y,z),$$

i.e,

$$R\ =\ \phi(x,y,z)\ E\ \phi^{-1}(x,y,z).$$

Now let *X*, *Y*, *Z* be a set of new variables for which

$$EX\ =\ -1,\qquad EY\ =\ 0,\qquad EZ\ =\ 0, \tag{2.7}$$

so that *E* reduces to $D = -\frac{\partial}{\partial X}$.

Now solving (2.7), we get a set of solutions as follows:

$$X = \frac{z^3}{2y^2\,(1-x^2)},\quad Y = \frac{x(1-x^2)}{y},\quad Z = \frac{(1-x^2)}{z} \tag{2.8}$$

from which we get

$$x = \frac{Y}{(2XZ^3)^{\frac{1}{2}}},\qquad y = \frac{2XZ^3-Y^2}{(2XZ^3)^{\frac{3}{2}}},\qquad z = \frac{2XZ^3-Y^2}{2XZ^4}\ . \tag{2.9}$$

Therefore, by Taylor's theorem, we get

$$\begin{aligned}
e^{wR}f(x,y,z) &= e^{w\phi\,E\,\phi^{-1}}f(x,y,z)\\
&= \phi(x,y,z)\ e^{wE}\left(\phi^{-1}(x,y,z)f(x,y,z)\right)\\
&= \phi(x,y,z)\ e^{-w\,D}\left(F(X,Y,Z)\right)\\
&= \phi(x,y,z)\left(F(X-w,Y,Z)\right)\\
&= \phi(x,y,z)g(x,y,z),
\end{aligned}$$

assuming that $F(X-w,Y,Z)$ is transformed into $g(x,y,z)$ by inverse substitution.

On calculation, we have the extended form of the group generated by *R*

$$e^{wR} f(x,\ y,\ z) = \left\{1 - 2w\frac{y^2}{z^3}\right\}^{-\frac{1}{2}} \left\{1 - 2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\frac{r}{2}}$$
$$\times f\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}},\ \frac{y\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}},\ \frac{z\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}}\right). \quad (2.10)$$

3. DERIVATION OF GENERATING FUNCTION

Now writing $f(x, y, z) = C_n^{\lambda+n}(x)\ y^n z^\lambda$ in (2.10), we get

$$e^{wR}\left(C_{n+r}^{\lambda+n}(x)\ y^n z^\lambda\right) = \left\{1 - 2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\left(\frac{3n}{2}+\lambda\right)-\frac{r}{2}}$$
$$\times\left(1 - 2w\frac{y^2}{z^3}\right)^{n+\lambda-\frac{1}{2}} C_{n+r}^{\lambda+n}\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}}\right) y^n z^\lambda. \quad (3.1)$$

Again, using (2.5), we obtain

$$e^{wR}\left(C_{n+r}^{\lambda+n}(x)\ y^n z^\lambda\right)$$
$$= \sum_{k=0}^{\infty} \left(\frac{2w\, y^2}{z^3}\right)^k \frac{\left(\frac{n+r+1}{2}\right)_k \left(\frac{n+r+2}{2}\right)_k}{k\,!(1-\lambda-n)_k} C_{n+r+2k}^{\lambda+n-k}(x) y^n z^\lambda. \quad (3.2)$$

Equating (3.1) and (3.2) and then substituting $\frac{2w\, y^2}{z^3} = t$, we get

$$\{1 - t(1-x^2)\}^{-\left(\frac{3n}{2}+\lambda\right)-\frac{r}{2}} (1-t)^{n+\lambda-\frac{1}{2}} C_{n+r}^{\lambda+n}\left(\frac{x}{\{1-t(1-x^2)\}^{\frac{1}{2}}}\right)$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{n+r+1}{2}\right)_k \left(\frac{n+r+2}{2}\right)_k}{k\,!(1-\lambda-n)_k} C_{n+r+2k}^{\lambda+n-k}(x)\ t^k, \quad (3.3)$$

which is believed to be new.

Now putting $r = 0$ in (3.3), we get the result found derived in [7] and other publications of authors (see **Corollary 1)**.

Now putting $r = 0$ and replacing λ by $\lambda - n$ in (3.3), we get

$$\{1 - t(1-x^2)\}^{-\left(\frac{n}{2}+\lambda\right)} (1-t)^{\lambda-\frac{1}{2}} C_n^{\lambda}\left(\frac{x}{\{1-t(1-x^2)\}^{\frac{1}{2}}}\right)$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k}{k\,!(1-\lambda)_k} C_{n+2k}^{\lambda-k}(x)\ t^k. \quad (3.4)$$

Again, putting $n = 0$ in (3.4), we get

$$(1-t)^{\lambda-\frac{1}{2}}\{1 - t(1-x^2)\}^{-\lambda} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(1-\lambda)_k} C_{2k}^{\lambda-k}(x)\ t^k, \quad (3.5)$$

which is also found derived in [7].

Now we proceed to prove the Theorem 1.

4. PROOF OF THEOREM 1

Let us consider the generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x)\, w^n. \tag{4.1}$$

Now replacing *w* by *wvy* and multiplying both sides of (4.1) by z^{λ}, we have

$$z^{\lambda}\, G(x,wvy)=\sum_{n=0}^{\infty} a_n \left(C_{n+r}^{\lambda+n}(x)\; y^n z^{\lambda}\right)(wv)^n. \tag{4.2}$$

Operating e^{wR} on both sides of (4.2), we get

$$e^{wR}\left(z^{\lambda}\, G(x,wvy)\right)=e^{wR}\left(\sum_{n=0}^{\infty} a_n \left(C_{n+r}^{\lambda+n}(x)\; y^n z^{\lambda}\right)(wv)^n\right). \tag{4.3}$$

Now the left member of (4.3), with the help of (2.10), reduces to

$$\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\frac{r}{2}-\lambda}\left(1-2w\frac{y^2}{z^3}\right)^{\lambda-\frac{1}{2}} z^{\lambda}$$

$$\times\, G\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}},\; \frac{wvy\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}}\right). \tag{4.4}$$

The right member of (4.3), with the help of (2.5), becomes

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n \frac{(2w)^k}{k\,!}\frac{\left(\frac{n+r+1}{2}\right)_k\left(\frac{n+r+2}{2}\right)_k}{(1-\lambda-n)_k}\, C_{n+r+2k}^{\lambda+n-k}(x)\, y^{n+2k} z^{\lambda-3k}(wv)^n$$

$$=\sum_{n=0}^{\infty}(2w)^n\sum_{k=0}^{n} a_{n-k}\frac{\left(\frac{n+r-k+1}{2}\right)_k\left(\frac{n+r-k+2}{2}\right)_k}{k\,!\,(1-\lambda-n+k)_k}\, C_{n+r+k}^{\lambda+n-2k}(x)\, y^{n+k} z^{\lambda-3k}\left(\frac{v}{2}\right)^{n-k} \tag{4.5}$$

Now equating (4.4) and (4.5) and then substituting $y=z=1$, $2w=w$ and $\frac{v}{2}=v$, we get

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}}\, G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}},\frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right)=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,v), \tag{4.6}$$

where

$$\sigma_n(x,v)=\sum_{k=0}^{n} a_k\frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda-k)_{n-k}}\, C_{2n+r-k}^{\lambda-n+2k}(x)\, v^k.$$

Corollary 1: Putting $r=0$ in Theorem 1, we get the theorem found derived in [7] and in other publication of the authors accepted for publication (*On generating functions of Gegenbauer polynomials*, in Ultra Scientist, **26**, 2014)

5. PROOF OF THEOREM 2

Let us consider the bilateral generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x)\, g_n(u) w^n, \tag{5.1}$$

where $g_n(u)$ is an arbitrary polynomial of degree n.

Now replacing w by wvy and multiplying both sides of (5.1) by z^{λ}, we have

$$z^{\lambda}\, G(x,u,wvy) = \sum_{n=0}^{\infty} a_n \left(C_{n+r}^{\lambda+n}(x)\, y^n z^{\lambda}\right) g_n(u)(wv)^n. \tag{5.2}$$

Operating e^{wR} on both sides of (5.2), we get

$$e^{wR}\left(z^{\lambda}\, G(x,u,wvy)\right) = e^{wR}\left(\sum_{n=0}^{\infty} a_n \left(C_{n+r}^{\lambda+n}(x)\, y^n z^{\lambda}\right) g_n(u)\,(wv)^n\right). \tag{5.3}$$

Now the left member of (5.3), with the help of (2.10), reduces to

$$\left\{1 - 2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\frac{r}{2}-\lambda}\left(1 - 2w\frac{y^2}{z^3}\right)^{\lambda-\frac{1}{2}} z^{\lambda}$$
$$\times\, G\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}}, u, \frac{wvy\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}}\right). \tag{5.4}$$

The right member of (5.3), with the help of (2.5), becomes

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n \frac{(2w)^k}{k\,!} \frac{\left(\frac{n+r+1}{2}\right)_k \left(\frac{n+r+2}{2}\right)_k}{(1-\lambda-n)_k} C_{n+r+2k}^{\lambda+n-k}(x)\, y^{n+2k} z^{\lambda-3k} g_n(u)(wv)^n$$

$$= \sum_{n=0}^{\infty} (2w)^n \sum_{k=0}^{n} a_{n-k} \frac{\left(\frac{n+r-k+1}{2}\right)_k \left(\frac{n+r-k+2}{2}\right)_k}{k\,!\,(1-\lambda-n+k)_k} C_{n+r+k}^{\lambda+n-2k}(x)\, y^{n+k} z^{\lambda-3k} g_{n-k}(u)\left(\frac{v}{2}\right)^{n-k} \tag{5.5}$$

Now equating (5.4) and (5.5) and then substituting $y = z = 1$, $2w = w$ and $\frac{v}{2} = v$, we get

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, u, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v) \tag{5.6}$$

where

$$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda-k)_{n-k}} C_{2n+r-k}^{\lambda-n+2k}(x)\, g_k(u) v^k.$$

Corollary 2: Putting $r = 0$ in Theorem 2, we get exactly the Theorem 2 found derived in article accepted for publication in *Inter. Jour. Math. Anal.*, 2014, entitled *On mixed trilateral generating functions of Gegenbauer polynomials*.

6. PROOF OF THEOREM 3

$$R.H.S.=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,v)$$
$$=\sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}} C_{2n+r-k}^{\lambda-n+k}(x)\,v^k$$
$$=\sum_{k=0}^{\infty} a_k\,(wv)^k \sum_{n=0}^{\infty} \frac{\left(\frac{k+r+1}{2}\right)_{n}\left(\frac{k+r+2}{2}\right)_{n}}{n\,!\,(1-\lambda)_{n}} C_{2n+r+k}^{\lambda-n}(x)\,w^n$$
$$=\sum_{k=0}^{\infty} a_k\,(wv)^k \frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\left(\lambda+\frac{k}{2}+\frac{r}{2}\right)}}\; C_{k+r}^{\lambda}\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}\right) \quad [\,using\ (3.4)]$$
$$=\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}}\sum_{k=0}^{\infty} a_k C_{k+r}^{\lambda}\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}\right)\left\{\frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right\}^k$$
$$=\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}},\frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right), \quad [\,using\ \ (1.5)]$$
$$=L.H.S.,$$

which is the Theorem 3.

Corollary 3: If we put $r = 0$ in Theorem 3, then we get the following Theorem:

Theorem 5. If

$$G(x,w)=\sum_{n=0}^{\infty} a_n C_n^{\lambda}(x)\,w^n \tag{6.1}$$

then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}},\frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right)=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,v), \tag{6.2}$$

where

$$\sigma_n(x,v)=\sum_{k=0}^{n} a_k \frac{\left(\frac{k+1}{2}\right)_{n-k}\left(\frac{k+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}} C_{2n-k}^{\lambda-n+k}(x)\,v^k.$$

7. PROOF OF THEOREM 4

$$\sum_{n=0}^{\infty} w^n\,\sigma_n(x,u,v)$$
$$=\sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}} C_{2n+r-k}^{\lambda-n+k}(x)\,g_k(u)v^k$$

$$= \sum_{k=0}^{\infty} a_k\, g_k(u)(wv)^k \sum_{n=0}^{\infty} \frac{\left(\frac{k+r+1}{2}\right)_n \left(\frac{k+r+2}{2}\right)_n}{n\,!\,(1-\lambda)_n} C_{2n+r+k}^{\lambda-n}(x)\, w^n$$

$$= \sum_{k=0}^{\infty} a_k \frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\left(\lambda+\frac{k}{2}+\frac{r}{2}\right)}} C_{k+r}^{\lambda}\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}\right) g_k(u)(wv)^k \;\; [\,using\ (3.4)]$$

$$= \frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} \sum_{k=0}^{\infty} a_k C_{k+r}^{\lambda}\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}\right) g_k(u) \left\{\frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right\}^k$$

$$= \frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, u, \frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right), \quad [\,using\ (1.7)]$$

Corollary 4: If we put $r = 0$ in Theorem 4, then we get the following Theorem:

Theorem 6. If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda}(x) g_n(u)\, w^n \tag{7.1}$$

then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, u, \frac{wv}{\{1-w+wx^2\}^{\frac{1}{2}}}\right) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,u,v), \tag{7.2}$$

where

$$\sigma_n(x,u,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+1}{2}\right)_{n-k} \left(\frac{k+2}{2}\right)_{n-k}}{(n-k)!\,(1-\lambda)_{n-k}} C_{2n-k}^{\lambda-n+k}(x) g_k(u) v^k.$$

8. APPLICATIONS

8.1. APPLICATION OF THEOREM 3

As an application of Theorem 3, we consider the following generating relation [1, 2]:

$$\{1-2wx+w^2\}^{-\left(\frac{r}{2}+\lambda\right)} C_r^{\lambda}\left(\frac{x-w}{\{1-2wx+w^2\}^{\frac{1}{2}}}\right) = \sum_{n=0}^{\infty} \binom{r+n}{n} C_{n+r}^{\lambda}(x)\, w^n. \tag{8.1}$$

If in our theorem, we take

$$a_n = \binom{r+n}{n},$$

then

$$G(x,w) = \{1-2wx+w^2\}^{-\left(\frac{r}{2}+\lambda\right)} C_r^{\lambda}\left(\frac{x-w}{\{1-2wx+w^2\}^{\frac{1}{2}}}\right).$$

Therefore by the application of our Theorem 3 we get the following generalization of the result (8.1):

$$(1-w)^{\lambda-\frac{1}{2}} \{1-w+wx^2\}^{r} \{1-w+wx^2-2vwx+v^2w^2\}^{-\left(\frac{r}{2}+\lambda\right)}$$

$$\times C_r^{\lambda}\left(\frac{(x-vw)}{\{1-w+wx^2-2vwx+v^2w^2\}^{\frac{1}{2}}}\right)$$
$$=\sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}} C_{2n+r-k}^{\lambda-n+k}(x)\, v^k. \quad (8.2)$$

8.2. APPLICATION OF THEOREM 4

As an application of Theorem 4, we consider the following generating relation [6]:

$$\sum_{n=0}^{\infty}\frac{(r+n)!}{(2\alpha)_n}\, C_{n+r}^{\lambda}(x)\, C_n^{\alpha}(u)\, w^n$$
$$=(2\lambda)_r(x-uw)^{-2\lambda-r}\, F_4\left[\begin{matrix}\lambda+\frac{r}{2},\ \lambda+\frac{r}{2}+\frac{1}{2}; & \\ & \frac{(u^2-1)w^2}{(x-uw)^2},\ \frac{(x^2-1)}{(x-uw)^2} \\ \alpha+\frac{r}{2},\ \lambda+\frac{r}{2}; & \end{matrix}\right]. \quad (8.3)$$

If in our theorem, we take

$$a_n=\frac{(r+n)!}{(2\alpha)_n}, \qquad g_n(u)=C_n^{\alpha}(u),$$

then

$$G(x,u,w)=(2\lambda)_r(x-uw)^{-2\lambda-r}\, F_4\left[\begin{matrix}\lambda+\frac{r}{2},\ \lambda+\frac{r}{2}+\frac{1}{2}; & \\ & \frac{(u^2-1)w^2}{(x-uw)^2},\ \frac{(x^2-1)}{(x-uw)^2} \\ \alpha+\frac{r}{2},\ \lambda+\frac{r}{2}; & \end{matrix}\right].$$

Therefore by the application of our Theorem 4 we get the following generalization of (8.3):

$$(2\lambda)_r(1-w)^{\lambda-\frac{1}{2}}(x-uvw)^{-2\lambda-r}\, F_4\left[\begin{matrix}\lambda+\frac{r}{2},\ \lambda+\frac{r}{2}+\frac{1}{2}; & \\ & \frac{(u^2-1)v^2w^2}{(x-uvw)^2},\ \frac{(x^2-1)(1-w)}{(x-uvw)^2} \\ \alpha+\frac{r}{2},\ \lambda+\frac{r}{2}; & \end{matrix}\right]$$
$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} a_k \frac{\left(\frac{k+r+1}{2}\right)_{n-k}\left(\frac{k+r+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}} C_{2n+r-k}^{\lambda-n+k}(x)\, g_k(u)\, v^k\right\} w^n. \quad (8.4)$$

8.3. APPLICATION OF THEOREM 5

As an application of Theorem 5, we consider the following generating relations [2]:

$$\{1-2wx+w^2\}^{-\lambda-\frac{r}{2}} C_r^{\lambda}\left(\frac{1-wx}{\{1-2wx+w^2\}^{\frac{1}{2}}}\right)=\sum_{n=0}^{\infty}\frac{(2\lambda+n)_r}{r\,!}C_n^{\lambda}(x)\,w^n \quad (8.5)$$

and

$$e^{wx}\;{}_0F_1\left[\begin{matrix} --; & \\ & \dfrac{w^2(x^2-1)}{4} \\ \lambda+\dfrac{1}{2}; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{1}{(2\lambda)_n}C_n^{\lambda}(x)\,w^n. \quad (8.6)$$

Therefore, by the application of our Theorem 5, we get the following generalization of the results (8.5) and (8.6):

$$\begin{aligned}
&(1-w)^{\lambda-\frac{1}{2}}\{1-w+wx^2\}^{\frac{r}{2}}\{1-w+wx^2-2vwx+v^2w^2\}^{-\left(\frac{r}{2}+\lambda\right)}\\
&\times C_r^{\lambda}\left(\frac{(1-w+wx^2-vwx)}{\{1-w+wx^2\}^{\frac{1}{2}}\{1-w+wx^2-2vwx+v^2w^2\}^{\frac{1}{2}}}\right)\\
&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}a_k\frac{\left(\frac{k+1}{2}\right)_{n-k}\left(\frac{k+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}}C_{2n-k}^{\lambda-n+k}(x)v^k\right\}w^n. \quad (8.7)
\end{aligned}$$

and

$$\begin{aligned}
&(1-w)^{\lambda-\frac{1}{2}}\{1-w+wx^2\}^{-\lambda}\exp\left\{\frac{vwx}{1-w+wx^2}\right\}\\
&\times\;{}_0F_1\left[\begin{matrix} --; & \\ & \dfrac{v^2w^2(1-w)(x^2-1)}{4(1-w+wx^2)^2} \\ \lambda+\dfrac{1}{2}; & \end{matrix}\right]\\
&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}a_k\frac{\left(\frac{k+1}{2}\right)_{n-k}\left(\frac{k+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}}C_{2n-k}^{\lambda-n+k}(x)v^k\right\}w^n. \quad (8.8)
\end{aligned}$$

8.4. APPLICATION OF THEOREM 6

As an application of Theorem 5, we consider the following generating relations [2, 4]:

$$\{1-2wx+w^2\}^{-\lambda}\exp\left\{\frac{-uw(x-w)}{1-2wx+w^2}\right\}\ {}_0F_1\left[\begin{matrix}--; \\ \lambda+\frac{1}{2};\end{matrix}\ \frac{w^2u^2(x^2-1)}{4(1-2wx+w^2)^2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{n!}{(2\lambda)_n}C_n^{\lambda}(x)\,L_n^{(2\lambda-1)}(u)w^n. \tag{8.9}$$

If in our theorem, we take

$$a_n=\frac{n!}{(2\lambda)_n}\quad and \quad g_n(u)=L_n^{(2\lambda-1)}(u),$$

then

$$G(x,u,w)=\{1-2wx+w^2\}^{-\lambda}\exp\left\{\frac{-uw(x-w)}{1-2wx+w^2}\right\}\ {}_0F_1\left[\begin{matrix}--; \\ \lambda+\frac{1}{2};\end{matrix}\ \frac{w^2u^2(x^2-1)}{4(1-2wx+w^2)^2}\right].$$

Therefore, by the application of our Theorem 6, we get the following generalization of (8.9):

$$(1-w)^{\lambda-\frac{1}{2}}\{1-w+wx^2-2vwx+v^2w^2\}^{-\lambda}$$

$$\times\exp\left\{\frac{-uvw(x-vw)}{1-w+wx^2-2vwx+v^2w^2}\right\}\ {}_0F_1\left[\begin{matrix}--; \\ \lambda+\frac{1}{2};\end{matrix}\ \frac{u^2v^2w^2(1-w)(x^2-1)}{4(1-w+wx^2-2vwx+v^2w^2)^2}\right]$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}a_k\frac{\left(\frac{k+1}{2}\right)_{n-k}\left(\frac{k+2}{2}\right)_{n-k}}{(n-k)\,!\,(1-\lambda)_{n-k}}C_{2n-k}^{\lambda-n+k}(x)g_k(u)v^k\right\}w^n. \tag{8.10}$$

9. CONCLUSION

From the above discussion, it is clear that whenever one knows a generating relations of the form (1.1, 1.3, 1.5, 1.7, 6.1, 7.1) then the corresponding bilateral and mixed trilateral generating relations can at once be written down from (1.2, 1.4, 1.6, 1.8, 6.2, 7.2). So one can get a large number of bilateral and mixed trilateral generating relations by attributing different suitable values to a_n in (1.1, 1.3, 1.5, 1.7, 6.1, 7.1).

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