ORIGINAL PAPER

A NOTE ON SAIGO'S GENERALIZED FRACTIONAL INTEGRAL OPERATORS AND GENERALIZED MITTAG-LEFFLER TYPE FUNCTIONS

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Abstract. *This paper is devoted to investigate the Saigo's generalized fractional integral operators of the generalized Mittag-Leffler type functions. Several special cases of interest are mentioned. Results given recently by Sharma[9], Saxena et al.[18] and Saigo et al.[20] follow as special cases of the theorems established here.*

Keywords*: Wright generalized hypergeometric function* ${}_p\Psi_q$, *Riemann-Liouville fractional integral operators, generalized Riemann-Liouville and Erdlyi-Kober fractional integral operators, K-function.*

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1. INTRODUCTION

The various Mittag-Leffler functions discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering. The importance of Mittag-Leffler functions in physics is steadily increasing. It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power law). Currently more and more such phenomena are discovered and studied. It is particularly important for the disciplines of stochastic systems, dynamical systems theory and disordered systems. Eventually, it is believed that all these new research results will lead to the discovery of truly non-equilibrium statistical mechanics. This is statistical mechanics beyond Boltzmann and Gibbs. This non-equilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion, and so forth, and may be governed by fractional calculus. Right now, fractional calculus and generalization of Mittag-Leffler functions are very important in research in physics.

Mittag-Leffler function was defined by [6, 7] in terms of the power series

$$E_\alpha(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)}, \quad (\alpha > 0, z \in C) \tag{1.1}$$

A generalization of this series in the following form

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$$E_{\alpha,\beta}(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)},\ \ (\alpha,\beta>0,z\in C) \tag{1.2}$$

has been studied by several authors notably by Mittag-Leffler [6, 7], Wiman [3], Agrawal [19], Humbert and Agrawal [17] and Dzrbashjan [11 – 13]. A detailed account of the basic properties of these two functions are given in the third volume of Bateman manuscript project [1] and an account of their various properties can be found in [12, 19].

The Wright generalized hypergeometric function [5] is given by

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix};z\right]=\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+rA_i)\,z^r}{\prod_{j=1}^{q}\Gamma(\beta_j+rB_j)r!} \tag{1.3}$$

[$A_i>0(i=1,2,\ldots,p),B_j>0(j=1,2,\ldots,q);\alpha_i,\beta_j\in C;1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\geq 0$ (Equality only for approximately bounded z)].

It is provided that the Riemann-Liouville fractional integral and derivative of the Wright function is also the Wright function but of greater order. Conditions for the existence of the series (1.1) together with its presentation in terms of the Mellin-Barnes integral and of the H-function were established in [2].

When $A_1=\ldots=A_p=B_1=\ldots=B_q=1$, (1.3) reduces to ${}_pF_q(.)$:

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,1),\ldots,(\alpha_p,1)\\(\beta_1,1),\ldots,(\beta_q,1)\end{matrix};z\right]=\frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\,{}_pF_q\ (\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) \tag{1.4}$$

where

$$p\leq q,|z|<\infty;\ p=q+1;|z|<1;\ p=q+1;|z|=1,\mathrm{Re}(\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_i)>0.$$

The K-function was defined by Sharma[8] as

$$ {}_pK_q^{\alpha,\beta;\gamma}(a_1,\ldots,a_p;b_1,\ldots,b_q;x)={}_pK_q^{\alpha,\beta;\gamma}(x)=\sum_{r=0}^{\infty}\frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r}\frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)} \tag{1.5}$$

where $\alpha,\beta,\gamma\in C,\mathrm{Re}(\alpha)>0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols.

The series(1.5) is defined when none of the parameters $b_j s,\ j=1,2,\ldots,q,$ is a negative integer or zero. If any numerator parameter a_{jr} is a negative integer or zero, then the series terminates to a polynomial in x. From the ratio test it is evident that the series is convergent for all x if p > q + 1. When p = q + 1 and $|x|=1,$ the series can converge in some cases. Let $\gamma=\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j$. It can be shown that when p = q + 1 the series is absolutely convergent for

$|x| = 1$ if $(R(\gamma) < 0$, conditionally convergent for x = -1 if $0 \leq R(\gamma) < 1$ and divergent for $|x| = 1$ if $1 \leq R(\gamma)$. Some new properties of this function are recently obtained by Sharma[10].

* Relations with Another Special Functions:

- From (1.5), we have

$$ {}_rK_s^{\alpha,\beta;\gamma}(x) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{s}(b_j)_n} \frac{(\gamma)_n x^n}{n!\Gamma(\alpha n+\beta)} = \frac{\prod_{j=1}^{s}(b_j)_n}{\Gamma(\gamma)\prod_{j=1}^{r}(a_j)_n} \times {}_{r+2}\Psi_{s+2}\left[\begin{matrix}(a_1,1),\ldots,(a_r,1),(\gamma,1),(1,1)\\(b_1,1),\ldots,(bs,1),(1,1),(\beta,\alpha)\end{matrix}\middle| x\right] \tag{1.6} $$

- When there is no upper and lower parameters in (1.5), we get

$$ {}_0K_0^{\alpha,\beta;\gamma}(-;-;x) = \sum_{r=0}^{\infty} \frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)} = E_{\alpha,\beta}^{\gamma}(x) \tag{1.7} $$

which reduces to the generalization of the Mittag-Leffler function[3] and its generalized form introduced by Prabhakar[22].

- If we put $\gamma = 1$ in (1.7), we get

$$ {}_0K_0^{\alpha,\beta;1}(-;-;x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r+\beta)} = E_{\alpha,\beta}^{1}(x) = E_{\alpha,\beta}(x) \tag{1.8} $$

which is the generalized Mittag-Leffler function[3].

- If we take $\beta = 1$ in (1.8), we get

$$ {}_0K_0^{\alpha,1;1}(-;-;x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r+1)} = E_{\alpha,1}^{1}(x) = E_{\alpha,1}(x) = E_{\alpha}(x) \tag{1.9} $$

which is the Mittag-Leffler function[6].

- If we take $\alpha = 1$ in (1.9), we get

$$ {}_0K_0^{\alpha,1;1}(-;-;x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r+1)} = E_{1,1}^{1}(x) = E_{1,1}(x) = E_1(x) \tag{1.10} $$

which is the Exponential function[4] denoted by e^x.

Following Section 2 of the book by Samko, Kilbas and Marichev[20], the fractional Riemann-Liouville(R-L) integral operators are given by

$$ I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt \tag{1.11} $$

$$ I_{-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\,dt \tag{1.12} $$

where $x > 0, \alpha \in C$ and $R(\alpha) > 0$.

An interesting and useful generalization of the Riemann-Liouville and Erdlyi-Kober fractional integral operators has been introduced by Saigo [16] in terms of Gauss hypergeometric function as given below:

$$(I_{0+}^{\alpha,\beta,\gamma})f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\right)f(t)dt \tag{1.13}$$

$$(I_{-}^{\alpha,\beta,\gamma})f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\right)f(t)dt \tag{1.14}$$

where $x \in R_{+};\alpha,\beta,\gamma \in C$ and $R(\alpha) > 0$.

2. LEFT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE K-FUNCTION

In this section, we derive a theorem, which give rise to the Saigo's left-sided generalized fractional integration formula of the K-function.

Theorem 2.1 Let $\alpha,\beta,\gamma,\eta \in C$ such that $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\eta+\gamma-\beta) > 0, \xi,\upsilon > 0$ and $c \in R$ and $I_{0+}^{\alpha,\beta,\gamma}$ be the Saigo's left-sided operator of the generalized fractional integration then there holds the relation:

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pK_q^{\xi,\eta;\upsilon}[ct^{\xi}]))(x) = \frac{x^{\eta-\beta-1}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(\upsilon)\Gamma(a_1)...\Gamma(a_p)}$$

$$\times {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1.1),...,(a_p.1),(\eta-\beta+1,\xi),(\upsilon,1) \\ (b_1.1),...,(b_q.1),(\eta-\beta,\xi),(\eta+\alpha+1,\xi)\end{matrix};cx^{\xi}\right] \tag{2.1}$$

provided each member of the equation exists.

Proof: In view of (1.5) and (1.13), we have

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pK_q^{\xi,\eta;\upsilon}[ct^{\xi}]))(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\right)(t^{\eta-1}{}_pK_q^{\xi,\eta;\upsilon}[ct^{\xi}])dt \tag{2.2}$$

Interchanging the order of integration and summations, evaluating the inner integral with the help of Beta function and using Gauss summation theorem, it becomes

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pK_q^{\xi,\eta;\upsilon}[ct^{\xi}]))(x) = \frac{x^{\eta-\beta-1}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(\upsilon)\Gamma(a_1)...\Gamma(a_p)}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k)...\Gamma(a_p+k)\Gamma(\upsilon+k)\Gamma(\eta+\gamma-\beta+\xi k)(cx^{\xi})^k}{\Gamma(b_1+k)...\Gamma(b_q+k)\Gamma(\eta-\beta+\xi k)\Gamma(\eta+\alpha+\gamma+\xi k)k!} \tag{2.3}$$

or equivalently

$$(I_{0+}^{\alpha,\beta,\gamma}(t^{\eta-1}{}_pK_q^{\xi,\eta;\upsilon}[ct^{\xi}]))(x) = \frac{x^{\eta-\beta-1}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(\upsilon)\Gamma(a_1)...\Gamma(a_p)}$$

$$\times {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1.1),...,(a_p.1),(\eta-\beta+1,\xi),(\upsilon,1)\\ \\ (b_1.1),...,(b_q.1),(\eta-\beta,\xi),(\eta+\alpha+1,\xi)\end{matrix};cx^{\xi}\right] \qquad (2.4)$$

This proves theorem(2.1).

Remarks:

- If we set $\upsilon=1$ in (2.1), we arrive at the well known result given by Sharma[9].
- If we take $\upsilon=1,\eta=1$ in (2.1), we get the Left-sided Generalized Fractional Integration of the M-series given by Sharma [15].
- If we put $\upsilon=1, p=q=0$ and $\beta\to-\alpha$ in (2.1), we arrive at the well known result given by Samko[20].

3. RIGHT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE K-FUNCTION

In this section we shall derive a theorem, which gives a formula involving the K-function, under the Saigo's right-sided generalized fractional integration.

Theorem 3.1. Let $\alpha,\beta,\gamma,\eta\in C$ such that
$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\alpha+\eta)>\max\{-\mathrm{Re}(\beta),-\mathrm{Re}(\gamma)\}$ $\mathrm{Re}(\beta)\neq\mathrm{Re}(\gamma),\xi,\upsilon>0$ and $c\in R$ and $I_{-}^{\alpha,\beta,\gamma}$ be the Saigo's right-sided operator of the generalized fractional integration then there holds the relation:

$$(I_{-}^{\alpha,\beta,\gamma}(t^{-\alpha-\eta}\,{}_pK_q^{\xi,\eta;\upsilon}[ct^{-\xi}]))(x)=\frac{x^{-\eta-\alpha-\beta}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(\upsilon)\Gamma(a_1)...\Gamma(a_p)}$$

$$\times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1.1),...,(a_p.1),(\alpha+\beta+\eta,\xi),(\alpha+\gamma+\eta,\xi),(\upsilon,1)\\ \\ (b_1.1),...,(b_q.1),(\eta,\xi),(\alpha+\eta,\xi),(2\alpha+\beta+\gamma+\eta,\xi)\end{matrix};cx^{-\xi}\right] \qquad (3.1)$$

provided each member of the equation exists.

Proof: By using (1.5) and (1.14), we have

$$(I_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\eta}\,{}_pK_q^{\xi,\eta;\upsilon}[ct^{-\xi}]))(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\right)(t^{-\alpha-\eta}\,{}_pK_q^{\xi,\eta;\upsilon}(ct^{-\xi}])dt \qquad (3.2)$$

Interchanging the order of integration and summations, evaluating the inner integral by the use of Beta function and using Gauss summation theorem, it becomes

$$(I_{-}^{\alpha,\beta,\gamma}(t^{\alpha-\eta}\,{}_pK_q^{\xi,\eta;\upsilon}[ct^{-\xi}]))(x)=\frac{x^{-\eta-\alpha-\beta}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(\upsilon)\Gamma(a_1)...\Gamma(a_p)}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k)...\Gamma(a_p+k)\Gamma(\alpha+\beta+\eta+\xi k)\Gamma(\alpha+\gamma+\eta+\xi k)\Gamma(\upsilon+1)(cx^{\xi})^k}{\Gamma(b_1+k)...\Gamma(b_q+k)\Gamma(\eta-\beta+\xi k)\Gamma(\eta+\alpha+\gamma+\xi k)k!} \qquad (3.3)$$

Or equivalently

$$\left(I_{-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\eta}\,{}_{p}K_{q}^{\xi,\eta;\upsilon}[ct^{-\xi}]\right)\right)(x) = \frac{x^{-\eta-\alpha-\beta}\Gamma(b_1)...\Gamma(b_q)}{\Gamma(\upsilon)\Gamma(a_1)...\Gamma(a_p)}$$
$$\times\,{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1.1),...,(a_p.1),(\alpha+\beta+\eta,\xi),(\alpha+\gamma+\eta,\xi),(\upsilon,1)\\ \\ (b_1.1),...,(b_q.1),(\eta,\xi),(\alpha+\eta,\xi),(2\alpha+\beta+\gamma+\eta,\xi)\end{matrix};cx^{-\xi}\right] \quad (3.4)$$

This completes the proof of the theorem(3.1).

Remarks:

- If we set $\upsilon = 1$ in (3.1), we get the well-known result obtained by Sharma[9].
- If we take $\upsilon = 1, \eta = 1$ in (3.1), we arrive at the Left-sided Generalized Fractional Integration of the M-series given by Sharma[15].
- If we put $\upsilon = 1, p = q = 0$ and $\beta \to -\alpha$ in (3.1), we get the well-known result derived by Saxena[18].

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