ORIGINAL PAPER

SOME IDENTITIES AND INEQUALITIES ON TRIANGLES

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Abstract. *In this paper, we use cubic polynomial, Viet's theorem and equation transformations to achieve some new identities and inequalities on triangles. To fix notations, suppose we are given a triangle ABC with sidelengh a, b, c. Denote the center and radius of the circumcircle by O, R, the center and radius of incircle by I, r, the area of triange ABC by S, the semiperimeter as P, the radii of the excircles as r_1, r_2, r_3, and the altitudes from sides a, b, c respectively, as h_a, h_b, h_c and ℓ_a, ℓ_b, ℓ_c are the lengths of bisector of triangle ABC; m_a, m_b, m_c are the lengths of median line of triangle ABC.*

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1. CONSTRUCTING CUBIC POLYNOMIAL ON TRIANGLES

In this paper, we use two known following results:

Example 1. [USA 1996] Given three positive real a,b,c, we have inequality

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leqslant\frac{1}{abc}.$$

Example 2. Given triangle ABC, we have inequality $\cos A+\cos B+\cos C\leqslant\frac{3}{2}$. We deduce, a,b,c are three solutions of $x^3-2px^2+(p^2+r^2+4Rr)x-4Rrp=0$; r_1,r_2,r_3 are three solutions of $x^3-(4R+r)x^2+p^2x-p^2r=0$ and x,y,z are three solutions of

$$t^3-(R+r)t^2+\frac{-4R^2+r^2+p^2}{4}t-\frac{R(p^2-(2R+r)^2)}{4}=0.$$

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Proposition 3. Using the above notations, we have a,b,c are three solutions of

$$x^3-2px^2+(p^2+r^2+4Rr)x-4Rrp=0.$$

and we have the following results:

(1) $\dfrac{1}{R^2}\leqslant\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}.$

(2) $\dfrac{1}{R^2}\leqslant\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\leqslant\dfrac{1}{4r^2}.$

(3) $\dfrac{S}{(p^2+r^2+4Rr-bc)(p^2+r^2+4Rr-ca)(p^2+r^2+4Rr-ab)}=\dfrac{\ell_a\ell_b\ell_c}{32Rrabcp^2}.$

Deduce we have inequality of triangles

$$\frac{\ell_a\ell_b\ell_c}{abc}\geqslant\frac{64S^3}{(p^2+r^2+4Rr-bc)(p^2+r^2+4Rr-ca)(p^2+r^2+4Rr-ab)}.$$

(4) $\dfrac{1}{a^3+b^3+4RS}+\dfrac{1}{b^3+c^3+4RS}+\dfrac{1}{c^3+a^3+4RS}\leqslant\dfrac{1}{4r^2(a+b+c)}.$

Proof. From $\tan\dfrac{A}{2}=\dfrac{r}{p-a}$, $a=2R\sin A$ we have

$$a=2R\frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$ Namely $$a=4R\frac{\frac{r}{p-a}}{1+(\frac{r}{p-a})^2}=4Rr\frac{p-a}{r^2+(p-a)^2}.$$

Hence, we have relation $a(a^2-2pa+p^2+r^2)=4Rr(p-a)$

Namely $a^3-2pa^2+(p^2+r^2+4Rr)a-4Rrp=0.$

Hence a is a solution of $x^3-2px^2+(p^2+r^2+4Rr)x-4Rrp=0.$

Similary b and c are solutions of this equation.

(1) Because $\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}$ are solutions of $-1+2px-(p^2+r^2+4Rr)x^2+4Rrpx^3=0$ we deduce

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{2p}{4Rrp}=\frac{1}{2Rr}\geqslant\frac{1}{R^2}.$$

(2) Denoted $f(x)=x^3-2px^2+(p^2+r^2+4Rr)x-4Rrp=(x-a)(x-b)(x-c).$

The derivative twice we have $3x-2p=(x-a)+(x-b)+(x-c).$

Deduce $\dfrac{3x-2p}{f(x)}=\dfrac{1}{(x-a)(x-b)}+\dfrac{1}{(x-b)(x-c)}+\dfrac{1}{(x-c)(x-a)}.$

Choosing $x=p$ we have $\dfrac{1}{(p-a)(p-b)}+\dfrac{1}{(p-b)(p-c)}+\dfrac{1}{(p-c)(p-a)}=\dfrac{p}{r^2p}=\dfrac{1}{r^2}.$

Deduce that $\frac{1}{4r^2}=\frac{1}{4(p-a)(p-b)}+\frac{1}{4(p-b)(p-c)}+\frac{1}{4(p-c)(p-a)}\geqslant\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$

We have $\frac{1}{R^2}\leqslant\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\leqslant\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$

(3) Because $\ell_a=\frac{2\sqrt{bc(p-a)p}}{b+c}$, $\ell_b=\frac{2\sqrt{ca(p-b)p}}{c+a}$, $\ell_c=\frac{2\sqrt{ab(p-c)p}}{a+b}$ we deduce

$$\ell_a\ell_b\ell_c=\frac{32Rrabcp^2S}{(p^2+r^2+4Rr-bc)(p^2+r^2+4Rr-ca)(p^2+r^2+4Rr-ab)}.$$

Because $R\geqslant 2r$ we always have inequality of triangles ABC

$$\frac{\ell_a\ell_b\ell_c}{abc}\geqslant\frac{64S^3}{(p^2+r^2+4Rr-bc)(p^2+r^2+4Rr-ca)(p^2+r^2+4Rr-ab)}.$$

(4) From $\frac{1}{a^3+b^3+4RS}+\frac{1}{b^3+c^3+4RS}+\frac{1}{c^3+a^3+4RS}\leqslant\frac{1}{4Rrp}$ with examples 1 and $\frac{1}{4Rrp}\leqslant\frac{1}{4r^2(a+b+c)}$ we have infer the proof.

Proposition 4. We have r_1,r_2,r_3 are three solutions of $x^3-(4R+r)x^2+p^2x-p^2r=0.$

Furthermore, we have

(1) $r_1+r_2+r_3=4R+r$ [Steiner].

(2) $r_1r_2+r_2r_3+r_3r_1=p^2.$

(3) $r_1r_2r_3=p^2r.$

(4) $(r_1-r)(r_2-r)(r_3-r)=4Rr^2.$

(5) $8r^3\leqslant(r_1-r)(r_2-r)(r_3-r)\leqslant R^3.$

(6) $81r^2\leqslant r_1^2+r_2^2+r_3^2+2\frac{r_1r_2r_3}{r}\leqslant\frac{81}{4}R^2.$ The equality holds if and only if triangle ABC is equilateral.

(7) $ab+bc+ca\geqslant 4r(r_1+r_2+r_3).$

(8) $r_1^3+r_2^3+r_3^3=(4R+r)^3-3R(a+b+c)^2.$

(9) $729r^3-3R(a+b+c)^2\leqslant r_1^3+r_2^3+r_3^3\leqslant\frac{729}{8}R^3-6r(a+b+c)^2.$

(10) $\frac{1}{r_1^3+r_2^3+Sp}+\frac{1}{r_2^3+r_3^3+Sp}+\frac{1}{r_3^3+r_1^3+Sp}\leqslant\frac{1}{r_1r_2r_3}.$

Proof. From $\tan\frac{A}{2}=\frac{r_1}{p}$, $a=2R\sin A$ we have $a=2R\dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

Namely $a=4R\dfrac{\frac{r_1}{p}}{1+\frac{r_1^2}{p^2}}=4Rr_1\dfrac{p}{r_1^2+p^2}$.

Because $r_1(p-a)=S=rp$ thus we have relation $\dfrac{(r_1-r)p}{r_1}=a=4Rr_1\dfrac{p}{r_1^2+p^2}$

Namely $(r_1-r)(r_1^2+p^2)=4Rr_1^2$. Deduce r_1 is a solution of $x^3-(4R+r)x^2+p^2x-p^2r=0.$

Similary, r_2 and r_3 are solutions of this equation.

(1), (2), (3) Can be infered from $x^3-(4R+r)x^2+p^2x-p^2r=0$ with Viet's theorem.

(4) From $x^3-(4R+r)x^2+p^2x-p^2r=(x-r_1)(x-r_2)(x-r_3)$ we deduce

$$(r_1-r)(r_2-r)(r_3-r)=4Rr^2.$$

(5) From $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$ and $R\geqslant 2r$ we deduce $8r^3\leqslant(r_1-r)(r_2-r)(r_3-r)\leqslant R^3$.

(6) Because $r_1^2+r_2^2+r_3^2=(r_1+r_2+r_3)^2-2(r_1r_2+r_2r_3+r_3r_1)$

we deduce $r_1^2+r_2^2+r_3^2=(4R+r)^2-2\dfrac{r_1r_2r_3}{r}$ Namely $r_1^2+r_2^2+r_3^2+2\dfrac{r_1r_2r_3}{r}=(4R+r)^2$.

Thus we have $\dfrac{81}{4}R^2\geqslant r_1^2+r_2^2+r_3^2+2\dfrac{r_1r_2r_3}{r}\geqslant 81r^2$. The equality holds if and only if triangle ABC is equilateral.

(7) Because $4(ab+bc+ca)=4p^2+4r^2+16Rr$ and $4p^2\geqslant 3(ab+bc+ca)$ we deduce

$$ab+bc+ca\geqslant\geqslant 4r(4R+r)=4r(r_1+r_2+r_2).$$

(8) Because $r_1^3+r_2^3+r_3^3-3r_1r_2r_3=(r_1+r_2+r_3)(r_1^2+r_2^2+r_3^2-r_1r_2-r_2r_3-r_3r_1)$

we deduce $r_1^3+r_2^3+r_3^3-3p^2r=(4R+r)[(4R+r)^2-3p^2]$.

Thus we have $r_1^3+r_2^3+r_3^3=(4R+r)^3-3R(a+b+c)^2$.

(9) From $r_1^3+r_2^3+r_3^3=(4R+r)^3-3R(a+b+c)^2$ and $R\geqslant 2r$

we deduce $729r^3-3R(a+b+c)^2\leqslant r_1^3+r_2^3+r_3^3\leqslant\dfrac{729}{8}R^3-6r(a+b+c)^2$.

(10) Can be infered from example 1.

Corollary 5. We have

(1) $4(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2})=\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}$.

(2) $h_ah_b+h_bh_c+h_ch_a=\dfrac{2r}{R}(r_1r_2+r_2r_3+r_3r_1)$.

Proof. (1) Because r_1, r_2, r_3 are three solutions of $x^3 - (4R+r)x^2 + p^2x - p^2r = 0$ we deduce $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are three solutions of $p^2rx^3 - p^2x^2 + (4R+r)x - 1 = 0.$

Thus, we have $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{1}{r^2} - 2\frac{4R+r}{p^2r}.$

Because $a, b, c,$ are three solutions of $x^3 - 2px^2 + (p^2 + r^2 + 4Rr)x - 4Rrp = 0.$

We deduce $a^2 + b^2 + c^2 = 4p^2 - 2(p^2 + r^2 + 4Rr) = 2p^2 - 2r^2 - 8Rr$

Thus, we get $4(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2}) = \frac{2p^2 - 2r^2 - 8Rr}{p^2r^2} = \frac{2}{r^2} - 2\frac{4R+r}{p^2r}.$

From two above identily, we deduce $4(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2}) = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}.$

(2) Using proposition 4 we get $h_ah_b + h_bh_c + h_ch_a = r\frac{2S^2}{Rr^2} = 2r\frac{p^2}{R} = \frac{2r}{R}(r_1r_2 + r_2r_3 + r_3r_1)$

Proposition 6. We have $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are three solutions of $x^3 - \frac{4R+r}{p}x^2 + x - \frac{r}{p} = 0.$

Here we deduce the following result:

(1) $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{4R+r}{p}.$

(2) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$

Here we deduce the following result

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geqslant \sqrt{3} \text{ and } \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \leqslant \frac{1}{3\sqrt{3}}.$$

(3) $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r}{p}.$

(4) $\tan^3\frac{A}{2} + \tan^3\frac{B}{2} + \tan^3\frac{C}{2} = (\frac{4R+r}{p})^3 - 12\frac{R}{p}.$

(5) $\frac{1}{\tan^3\frac{A}{2} + \tan^3\frac{B}{2} + \frac{r}{p}} + \frac{1}{\tan^3\frac{B}{2} + \tan^3\frac{C}{2} + \frac{r}{p}} + \frac{1}{\tan^3\frac{C}{2} + \tan^3\frac{A}{2} + \frac{r}{p}} \leqslant \frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}.$

Proof. Because $r_1 = p\tan\frac{A}{2}$ and r_1 is a solution of $x^3 - (4R+r)x^2 + p^2x - p^2r = 0$ we deduce $p^3\tan^3\frac{A}{2} - (4R+r)p^2\tan^2\frac{A}{2} + p^2p\tan\frac{A}{2} - p^2r = 0$

Namely $\tan^3\frac{A}{2}-\frac{4R+r}{p}\tan^2\frac{A}{2}+\tan\frac{A}{2}-\frac{r}{p}=0.$

We deduce $\tan\frac{A}{2}$ is a solution of $x^3-\frac{4R+r}{p}x^2+x-\frac{r}{p}=0.$

Similary $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$ are solutions of this equation. The result (1), (2) and (3) can be infered from Viet's theorem.

(4) We have $\tan^3\frac{A}{2}+\tan^3\frac{B}{2}+\tan^3\frac{C}{2}-3\frac{r}{p}=\frac{4R+r}{p}[(\frac{4R+r}{p})^2-3].$

We get $\tan^3\frac{A}{2}+\tan^3\frac{B}{2}+\tan^3\frac{C}{2}=(\frac{4R+r}{p})^3-12\frac{R}{p}.$

(5) can be infered from example 1.

Proposition 7. We deduce $\cos A,\cos B,\cos C$ are three solutions of

$$x^3-\frac{R+r}{R}x^2+\frac{-4R^2+r^2+p^2}{4R^2}x-\frac{p^2-(2R+r)^2}{4R^2}=0.$$

Furthermore we have

(1) $\cos A+\cos B+\cos C=1+\frac{r}{R}.$

We deduce $r\leqslant\frac{R}{2}$ and $\cos A\cos B\cos C\leqslant\frac{1}{8}.$

(2) $P=(1-\cos A)(1-\cos B)(1-\cos C)=\frac{r^2}{2R^2}$ and $P\leqslant\frac{1}{8}.$

(3) $(\frac{r}{R}-\cos A)(\frac{r}{R}-\cos B)(\frac{r}{R}-\cos C)\leqslant 1-\frac{p^2+2r^2}{8R^2}.$

(4) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$

(5) $\cos A\cos B+\cos B\cos C+\cos C\cos A-\cos A\cos B\cos C\leqslant\frac{5}{8}.$

(6) $\cos^3 A+\cos^3 B+\cos^3 C+\cos A\cos B\cos C\geqslant\frac{1}{2}$ where triangles ABC is acute triangles.

Proof. We have $\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}$ are three solutions of

$$x^3-\frac{4R+r}{p}x^2+x-\frac{r}{p}=0.$$

Because $\cos A=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}=\frac{2}{1+\tan^2\frac{A}{2}}-1$ we deduce $\cos A+1=\frac{2}{1+\tan^2\frac{A}{2}}.$

Consider the system of equations:

$$\begin{cases} y = \dfrac{2}{1+x^2} \\ x^3 - \dfrac{4R+r}{p}x^2 + x - \dfrac{r}{p} = 0. \end{cases}$$

From $\begin{cases} x^2 = \dfrac{2}{y} - 1 \\ x^3 - \dfrac{4R+r}{p}x^2 + x - \dfrac{r}{p} = 0 \end{cases}$ we deduce $\begin{cases} x = \dfrac{4R+r}{p} - \dfrac{2Ry}{p} \\ x^3 - \dfrac{4R+r}{p}x^2 + x - \dfrac{r}{p} = 0 \end{cases}$ and we get $y(\dfrac{4R+r}{p} - \dfrac{2Ry}{p})^2 + y - 2 = 0.$

We deduce $\cos A+1, \cos B+1, \cos C+1$ are three solutions of

$$y^3 - \frac{4R+r}{R}y^2 + \frac{(4R+r)^2+p^2}{4R^2}y - \frac{p^2}{2R^2} = 0.$$

Choosing $y = x+1$ we get $\cos A, \cos B, \cos C$ are three solutions of

$$x^3 - \frac{R+r}{R}x^2 + \frac{-4R^2+r^2+p^2}{4R^2}x - \frac{p^2-(2R+r)^2}{4R^2} = 0.$$

(1) From Viet's theorem we have $\cos A + \cos B + \cos C = \dfrac{R+r}{R} = 1 + \dfrac{r}{R}$. Because example 2 we deduce $r \leqslant \dfrac{R}{2}$. If triangles ABC is a right triangle or obtuse triangle then we have $\cos A \cos B \cos C \leqslant 0.$

If triangles ABC is a acute triangles. From $\cos A + \cos B + \cos C \leqslant \dfrac{3}{2}$ and using Cauchy inequality, we deduce $\cos A \cos B \cos C \leqslant \dfrac{1}{8}$.

(2) From above equation we have $P = (1-\cos A)(1-\cos B)(1-\cos C) = \dfrac{r^2}{2R^2}$.

From $R \geqslant 2r$ deduce $P \leqslant \dfrac{1}{8}$.

(3) We have

$$(\frac{r}{R} - \cos A)(\frac{r}{R} - \cos B)(\frac{r}{R} - \cos C) = 1 + \frac{r(p^2+r^2)}{4R^3} - \frac{p^2+3r^2}{4R^2} \leqslant 1 - \frac{p^2+2r^2}{8R^2}.$$

(4) Set $T = \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C.$ We have

$$T = \frac{(R+r)^2}{R^2} - 2\frac{-4R^2+r^2+p^2}{4R^2} + 2\frac{p^2-(2R+r)^2}{4R^2} = 1.$$

(5) From proposition 7 we deduce $\cos A, \cos B, \cos C$ are three solutions of

$$x^3 - \frac{R+r}{R}x^2 + \frac{-4R^2+r^2+p^2}{4R^2}x - \frac{p^2-(2R+r)^2}{4R^2} = 0.$$

We have

$$U = \cos A\cos B + \cos B\cos C + \cos C\cos A = \frac{-4R^2 + r^2 + p^2}{4R^2}$$

$$V = \cos A\cos B\cos C = \frac{p^2 - (2R+r)^2}{4R^2}$$

We get $U - V = \dfrac{2Rr + r^2}{2R^2} \leqslant \dfrac{5}{8}$.

(6) From (4) we have equivalent inequality

$$2(\cos^3 A + \cos^3 B + \cos^3 C) \geqslant 1 - 2\cos A\cos B\cos C = \cos^2 A + \cos^2 B + \cos^2 C.$$

We proof inequality $2(\cos^3 A + \cos^3 B + \cos^3 C) \geqslant \cos^2 A + \cos^2 B + \cos^2 C$.

Using Cauchy inequality we deduce $\begin{cases} \cos^3 A + \cos^3 A + \dfrac{1}{8} \geqslant \dfrac{3}{2}\cos^2 A \\ \cos^3 B + \cos^3 B + \dfrac{1}{8} \geqslant \dfrac{3}{2}\cos^2 B \\ \cos^3 C + \cos^3 C + \dfrac{1}{8} \geqslant \dfrac{3}{2}\cos^2 C \end{cases}$ and we deduce

$$\cos^3 A + \cos^3 B + \cos^3 C + \frac{3}{8} \geqslant \frac{3}{2}(\cos^2 A + \cos^2 B + \cos^2 C).$$

Because $\cos^2 A + \cos^2 B + \cos^2 C \geqslant \dfrac{3}{4}$ we deduce

$$\cos^3 A + \cos^3 B + \cos^3 C \geqslant \frac{1}{2}(\cos^2 A + \cos^2 B + \cos^2 C).$$

Corollary 8. Given triangle ABC. Let $x = R\cos A, y = R\cos B, z = R\cos C$. We have x, y, z are three solutions of below equation

$$f(t) = t^3 - (R+r)t^2 + \frac{-4R^2 + r^2 + p^2}{4}t - \frac{R(p^2 - (2R+r)^2)}{4} = 0.$$

Furthermore, we have

(1) $x + y + z = R + r$ [Carnot].

(2) $xy + yz + zx - \dfrac{xyz}{R} = r(R + \dfrac{r}{2})$ and $\dfrac{5r^2}{2} \leqslant xy + yz + zx - \dfrac{xyz}{R} \leqslant \dfrac{5R^2}{8}$.

(3) $(R-x)(R-y)(R-z) = \dfrac{IA.IB.IC}{8}$, $r^3 \leqslant (R-x)(R-y)(R-z) \leqslant \dfrac{R^3}{8}$.

(4) $24r^2 \leqslant ab + bc + ca - 4(xy + yz + zx) \leqslant 6R^2$.

(5) $34r^2 \leqslant ab + bc + ca - 4\dfrac{xyz}{R} \leqslant \dfrac{17}{2}R^2$.

(6) $\dfrac{ab + bc + ca}{R^2} - 2\dfrac{r(r_1 + r_2 + r_3)}{R^2} = 4 + 4\dfrac{xyz}{R^3}$.

(7) $\dfrac{r}{R-x} + \dfrac{r}{R-y} + \dfrac{r}{R-z} - \dfrac{r}{2R} = -4 + 2\dfrac{r_1}{IA}\dfrac{r_2}{IB}\dfrac{r_3}{IC}$.

(8) $\dfrac{r}{R-x} + \dfrac{r}{R-y} + \dfrac{r}{R-z} \leqslant -\dfrac{15}{4} + 2\dfrac{r_1}{IA}\dfrac{r_2}{IB}\dfrac{r_3}{IC}$.

(9) $x^3+y^3+z^3=(R+r)^3-\frac{3}{4}r(ab+bc+ca-2Rr).$

(10) $27r^3-\frac{3}{4}r(ab+bc+ca-2Rr)\leqslant x^3+y^3+z^3\leqslant\frac{27}{8}R^3-\frac{3}{4}r(ab+bc+ca-2Rr).$

(11) $IA^2+IB^2+IC^2=4(xy+yz+zx)+4R^2-8Rr$ and we deduce

$4(xy+yz+zx)\leqslant IA^2+IB^2+IC^2\leqslant 4(xy+yz+zx)+4R^2-16r^2.$

(12) $8(R-x)(R-y)(R-z)=(r_1-r)(r_2-r)(r_3-r).$

(13) $(R-x)(R-y)(R-z)h_ah_bh_c=S^2r^2.$

(14) $\frac{1}{x^3+y^3+xyz}+\frac{1}{y^3+z^3+xyz}+\frac{1}{z^3+x^3+xyz}\leqslant\frac{1}{xyz}$

where triangles ABC is a acute triangle.

(15) $m_a+m_b+m_c\leqslant r_1+r_2+r_3$ where triangles ABC is not a obtuse triangle.

(16) $Q=\sin A\sin B+\sin B\sin C+\sin C\sin A-\cos A\cos B\cos C\leqslant\frac{17}{8}.$

(17) $\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{C}{2}}\leqslant\frac{R}{r}(4\frac{r_1}{IA}\frac{r_2}{IB}\frac{r_3}{IC}-\frac{15}{2}).$

Proof. Choosing $y=x+1$, we have $\cos A,\cos B,\cos C$ are three solutions of equation

$$x^3-\frac{R+r}{R}x^2+\frac{-4R^2+r^2+p^2}{4R^2}x-\frac{p^2-(2R+r)^2}{4R^2}=0.$$

Pre-multiplying the two sides with R^3 and set Rx by t deduce x,y,z are three solutions of

$$t^3-(R+r)t^2+\frac{-4R^2+r^2+p^2}{4}t-\frac{R(p^2-(2R+r)^2)}{4}=0.$$

(1) Using Viet’s theorem, we have $x+y+z=R+r.$

(2) Using Viet’s theorem, we have $R(xy+yz+zx)-xyz=Rr(R+\frac{r}{2}).$

(3) Because $(t-x)(t-y)(t-z)=f(t)$ we deduce

$$(R-x)(R-y)(R-z)=\frac{Rr^2}{2}=\frac{IA.IB.IC}{8}.$$

Because $R\geqslant 2r$ we deduce $r^3\leqslant(R-x)(R-y)(R-z)\leqslant\frac{R^3}{8}.$

(4) Using Viet’s theorem and proposition 3, we have $ab+bc+ca=p^2+r^2+4Rr.$

Thus, we deduce $ab+bc+ca-4(xy+yz+zx)=4R(R+r).$

From this identity, we deduce $24r^2\leqslant ab+bc+ca-4(xy+yz+zx)\leqslant 6R^2.$

(5) Using Viet's theorem, we have $ab+bc+ca-4\frac{xyz}{R}=4R^2+8Rr+2r^2$. From this identitiy, we deduce $34r^2\leqslant ab+bc+ca-4\frac{xyz}{R}\leqslant\frac{17}{2}R^2$.

(6) Because $ab+bc+ca-4\frac{xyz}{R}=4R^2+8Rr+2r^2$ and $r_1+r_2+r_3=4R+r$ we deduce $\frac{ab+bc+ca}{R^2}-2\frac{r(r_1+r_2+r_3)}{R^2}=4+4\frac{xyz}{R^3}$.

(7) Because $(t-x)(t-y)(t-z)=f(t)$ we deduce $\frac{1}{t-x}+\frac{1}{t-y}+\frac{1}{t-z}=\frac{f'(t)}{f(t)}$.

Thus, we have $\frac{1}{R-x}+\frac{1}{R-y}+\frac{1}{R-z}=\frac{f'(R)}{f(R)}=\frac{-8Rr+r^2+p^2}{2Rr^2}$.

Because $p^2r=r_1r_2r_3$ and $IA.IB.IC=4Rr^2$. We deduce

$$\frac{r}{R-x}+\frac{r}{R-y}+\frac{r}{R-z}-\frac{r}{2R}=-4+\frac{r_1r_2r_3}{2Rr^2}=-4+2\frac{r_1r_2r_3}{IA.IB.IC}$$

(8) Because $\frac{r}{R-x}+\frac{r}{R-y}+\frac{r}{R-z}-\frac{r}{2R}=-4+\frac{r_1r_2r_3}{2Rr^2}$ and $R\geqslant 2r$

we deduce $\frac{r}{R-x}+\frac{r}{R-y}+\frac{r}{R-z}\leqslant-\frac{15}{4}+\frac{r_1r_2r_3}{IA.IB.IC}$.

(9) From $x^3+y^3+z^3-3xyz=(x+y+z)[(x+y+z)^2-3(xy+yz+zx)]$. We deduce

$x^3+y^3+z^3=(R+r)^3-\frac{3}{4}r(ab+bc+ca-2Rr)$.

(10) From (9) and $R\geqslant 2r$. We get

$$27r^3-\frac{3}{4}r(ab+bc+ca-2Rr)\leqslant x^3+y^3+z^3\leqslant\frac{27}{8}R^3-\frac{3}{4}r(ab+bc+ca-2Rr)$$

(11) Because $IA^2-bc=IB^2-ca=IC^2-ab=-\frac{2abc}{a+b+c}$ and using proposition 3

we have $IA^2+IB^2+IC^2=bc+ca+ab-\frac{6abc}{a+b+c}=p^2+r^2+4Rr-\frac{24Rrp}{2p}$.

Because $4(xy+yz+zx)=p^2+r^2-4R^2$ we deduce

$$IA^2+IB^2+IC^2=4(xy+yz+zx)+4R^2+4Rr-12Rr=4(xy+yz+zx)+4R^2-8Rr.$$

Thus, we get $4(xy+yz+zx)\leqslant IA^2+IB^2+IC^2\leqslant 4(xy+yz+zx)+4R^2-16r^2$.

(12) We have $8(R-x)(R-y)(R-z)=4Rr^2=(r_1-r)(r_2-r)(r_3-r)$,

(13) We have $(R-x)(R-y)(R-z)h_ah_bh_c=\frac{Rr^2}{2}\frac{2S^2}{R}=S^2r^2$.

(14) From Inequality 1, we have the proof.

(15) We have $m_a+m_b+m_c\leqslant R+x+R+y+R+z=r_1+r_2+r_3$ where triangles ABC is not a obtuse triangle.

(16) From $Q=\frac{p^2+r^2+4Rr}{4R^2}-\frac{p^2-(2R+r)^2}{4R^2}=\frac{4R^2+8Rr+2r^2}{4R^2}$

we deduce $Q \leqslant \dfrac{4R^2 + 4R^2 + \dfrac{R^2}{2}}{4R^2} = \dfrac{17}{8}.$

(17) From $\dfrac{r}{R-x} + \dfrac{r}{R-y} + \dfrac{r}{R-z} \leqslant -\dfrac{15}{4} + 2\dfrac{r_1}{IA}\dfrac{r_2}{IB}\dfrac{r_3}{IC}$

we deduce $\dfrac{1}{\sin^2\dfrac{A}{2}} + \dfrac{1}{\sin^2\dfrac{B}{2}} + \dfrac{1}{\sin^2\dfrac{C}{2}} \leqslant \dfrac{R}{r}(4\dfrac{r_1}{IA}\dfrac{r_2}{IB}\dfrac{r_3}{IC} - \dfrac{15}{2}).$

Example 9. Given $ABCD$ be a convex quadrilateral inscribed in the circle with the center O and radius R. Denoted $a = AB,\ b = BC,\ c = CD,\ d = DA$ and r_1, r_2, r_3, r_4 respectively being are the radius of incircle of circumcircle of triangles $ABC,\ BCD,\ CDA,\ DAB.$ We have identity

$$(1-\frac{ab}{2Rr_1})(1-\frac{bc}{2Rr_2})(1-\frac{cd}{2Rr_3})(1-\frac{da}{2Rr_4}) = \frac{(a+b)(b+c)(c+d)(d+a)}{(ac+bd)^2}.$$

Proof. Set $x = AC, y = BD.$ From $(a+b+x)r_1 = \dfrac{abx}{2R}$ deduce $x = \dfrac{a+b}{\dfrac{ab}{2Rr_1} - 1}.$

Similary, $x = \dfrac{c+d}{\dfrac{cd}{2Rr_3} - 1}, y = \dfrac{b+c}{\dfrac{b+c}{2Rr_2} - 1}, y = \dfrac{d+a}{\dfrac{da}{2Rr_4} - 1}.$

Using Ptolemy identity, we deduce $ac + bd = xy$. Thus, We have

$$(ac+bd)^2 = (\frac{a+b}{\frac{ab}{2Rr_1}-1})(\frac{b+c}{\frac{b+c}{2Rr_2}-1})(\frac{c+d}{\frac{cd}{2Rr_3}-1})(\frac{d+a}{\frac{da}{2Rr_4}-1}).$$

We get identily $(1-\dfrac{ab}{2Rr_1})(1-\dfrac{bc}{2Rr_2})(1-\dfrac{cd}{2Rr_3})(1-\dfrac{da}{2Rr_4}) = \dfrac{(a+b)(b+c)(c+d)(d+a)}{(ac+bd)^2}.$

2. CONSTRUCTING CUBIC POLYNOMIAL BY TRANSFORMATIONS

By using Viet's theorem and transformations of equations, we obtain some identities and inequalities of triangles.

Example 10. Using the above notations, we have

(1) $(\dfrac{r_1}{r} - 1)(\dfrac{r_2}{r} - 1)(\dfrac{r_3}{r} - 1) = 4\dfrac{R}{r}.$

(2) $d_a^2 + d_b^2 + d_c^2 = 11R^2 + 2Rr,$ where d_a, d_b, d_c the distances from O to the three centers of escribed circles of triangle ABC (where O is the center of circumcircle of triangle ABC).

(3) $d_a^2 + d_b^2 + d_c^2 \geqslant 12R^2.$

(4) $(d_a^2 - R^2)(d_b^2 - R^2)(d_c^2 - R^2) = R^2 abc(a+b+c).$

(5) $d_a d_b d_c \leqslant 8R^3.$

Proof. (1) Because $x^3 - (4R+r)x^2 + p^2 x - p^2 r = (x-r_1)(x-r_2)(x-r_3)$ choosing $x = r$ we get $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2.$ We division into two sides by r^3.

Thus we have $(\frac{r_1}{r} - 1)(\frac{r_2}{r} - 1)(\frac{r_3}{r} - 1) = 4\frac{R}{r}.$

(2) Because $d_a^2 = R^2 + 2Rr_1,\ d_b^2 = R^2 + 2Rr_2$ and $d_c^2 = R^2 + 2Rr_3$.

We have $d_a^2 + d_b^2 + d_c^2 = 11R^2 + 2Rr.$

(3) Because $R \geqslant 2r$ we deduce $d_a^2 + d_b^2 + d_c^2 = 11R^2 + 2Rr \geqslant 12R^2.$

(4) We have $(d_a^2 - R^2)(d_b^2 - R^2)(d_c^2 - R^2) = 8R^3 r_1 r_2 r_3 = R^2 abc(a+b+c).$

(5) Because $p^2 \leqslant \frac{27}{4}R^2$ and

$d_a^2 d_b^2 d_c^2 = R^3(R+2r_1)(R+2r_2)(R+2r_3) = R^3(9R^3 + 2R^2 r + 4Rp^2 + 8rp^2)$

we deduce $d_a^2 d_b^2 d_c^2 \leqslant R^3(9R^3 + 2R^2 r + 27R^3 + 54R^2 r)$

Namely $(d_a d_b d_c)^2 \leqslant R^3(36R^3 + 56R^2 r) \leqslant 64R^6.$

We get $d_a d_b d_c \leqslant 8R^3.$

Example 11. Calculate the following sum:

$$2\frac{r_1 - r}{r_1 + r}\frac{r_2 - r}{r_2 + r}\frac{r_3 - r}{r_3 + r} + \frac{r_1 - r}{r_1 + r}\frac{r_2 - r}{r_2 + r} + \frac{r_2 - r}{r_2 + r}\frac{r_3 - r}{r_3 + r} + \frac{r_3 - r}{r_3 + r}\frac{r_1 - r}{r_1 + r}.$$

Proof. r_1, r_2, r_3 are three solutions of $x^3 - (4R+r)x^2 + p^2 x - p^2 r = 0.$

we transformations $y = \frac{x-r}{x+r}.$ We deduce $y \neq 1$ and $x = \frac{r(y+1)}{1-y}.$

From above equation we deduce

$(r^2 + 2Rr + p^2)y^3 + (2r^2 + 2Rr - 2p^2)y^2 + (r^2 - 2Rr + p^2)y - 2Rr = 0.$

Let y_1, y_2, y_3 are three solutions of this equation. Thus, we have

$2y_1 y_2 y_3 + y_1 y_2 + y_2 y_3 + y_3 y_1 = \frac{4Rr + r^2 - 2Rr + p^2}{r^2 + 2Rr + p^2} = 1.$

Example 12. Given triangles ABC, we have

$$\frac{4R}{r_1 - r} + \frac{4R}{r_2 - r} + \frac{4R}{r_3 - r} = 1 - 8\frac{R}{r} + \frac{r_1 r_2 r_3}{r^3}.$$

Then we deduce this inequality $\frac{4R}{r_1-r}+\frac{4R}{r_2-r}+\frac{4R}{r_3-r}\leqslant\frac{r_1r_2r_3}{r^3}-15.$

Proof. Because r_1,r_2,r_3 are three solutions of $x^3-(4R+r)x^2+p^2x-p^2r=0$ we deduce $\frac{1}{x-r_1}+\frac{1}{x-r_2}+\frac{1}{x-r_3}=\frac{3x^2-2(4R+r)x+p^2}{x^3-(4R+r)x^2+p^2x-p^2r}.$

Choosing $x=r$ we deduce $\frac{1}{r_1-r}+\frac{1}{r_2-r}+\frac{1}{r_3-r}=\frac{r^2-8Rr+p^2}{4Rr^2}.$

Because $p^2=\frac{r_1r_2r_3}{r}$, we get $\frac{4R}{r_1-r}+\frac{4R}{r_2-r}+\frac{4R}{r_3-r}=1-8\frac{R}{r}+\frac{r_1r_2r_3}{r^3}.$

Because $8\frac{R}{r}\geqslant 16$ we get $\frac{4R}{r_1-r}+\frac{4R}{r_2-r}+\frac{4R}{r_3-r}\leqslant\frac{r_1r_2r_3}{r^3}-15.$

Example 13. Given triangles ABC. We have

$$(\frac{r_1^2}{p^2}+1)(\frac{r_2^2}{p^2}+1)(\frac{r_3^2}{p^2}+1)=\frac{16R^2}{p^2}.$$

Proof. r_1,r_2,r_3 are three solutions of $x^3-(4R+r)x^2+p^2x-p^2r=0.$

Set $y_1=r_1^2+p^2, y_2=r_2^2+p^2, y_3=r_3^2+p^2$.

From $\begin{cases} x^3-(4R+r)x^2+p^2x-p^2r=0 \\ x^2+p^2-y=0. \end{cases}$ we deduce $\begin{cases} x^3-(4R+r)x^2+p^2x-p^2r=0 \\ x^3+p^2x-yx=0 \end{cases}$ and $(4R+r)x^2-yx+p^2r=0$ Namely $(4R+r)(y-p^2)-yx+p^2r=0.$

Set $T=4R+r,$ we deduce $x=\frac{Ty-4Rp^2}{y}.$ We get $\frac{(Ty-4Rp^2)^2}{y^2}+p^2-y=0$

Namely $y^3-(T^2+p^2)y^2+8RTp^2y-16R^2p^4=0.$ This equation have three solutions $y_1,y_2,y_3.$ Thus, we have $\frac{(r_1^2+p^2)(r_2^2+p^2)(r_3^2+p^2)}{p^6}=\frac{y_1y_2y_3}{p^6}=\frac{16R^2p^4}{p^6}.$

We get $(\frac{r_1^2}{p^2}+1)(\frac{r_2^2}{p^2}+1)(\frac{r_3^2}{p^2}+1)=\frac{16R^2}{p^2}.$

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