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SOLVING THE KLEIN-GORDON EQUATIONS VIA DIFFERENTIAL TRANSFORM METHOD

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Abstract.*In this paper, a new mathematical technique Differential Transform Method (DTM) is presented for solving variants of nonlinear Klein-Gordon equation. The Differential Transform Method (DTM) provides an effective tool to solve analytically partial differential equations. The method is demonstrated by several examples. The obtained results indicate that this method to be very effective and simple. The method can easily be applied to many problems and is capable of reducing the size of computational work.*

Keywords: *Nonlinear Partial Differential Equations; Klein Gordon Equation; Differential transform Method.*

1. INTRODUCTION

It is well known that many phenomena in scientific fields are nonlinear in nature. Nonlinear phenomena that are encountered in many areas of science such as fluid dynamics, chemical kinetics, crack propagation and quantum mechanics can be modeled by partial differential equations. The Klein–Gordon (KG) equation is known to model many problems in classical and quantum mechanics, particularly in the area of relativity, solitons and condensed matter physics. The non-linear models of real-life problems are still difficult to solve either numerically or theoretically.

In the past several decades, many authors mainly had paid attention to study the solution of such equations by using various developed methods. Recently, the Variational Iteration Method (VIM) [1-3] has been applied to handle various kinds of nonlinear problems, for example, fractional differential equations [4], nonlinear differential equations [5], nonlinear thermo elasticity [6], nonlinear wave equations [7]. In Refs. [8-13] Adomian's Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variation of Parameter Method (VPM) are successfully applied to obtain the exact solution of differential equations.

In this study, a new transformation called differential transform method is introduced to solve nonlinear Klein–Gordon (KG) equation. Jafari [18] used homotopy analysis method to solve the nonlinear Gas Dynamic equation. The concept of differential transform was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by Zhou [22]. Using differential transform, Chen and Ho [16] proposed a method to solve eigenvalue problems. Using differential transformation method, a closed form series

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solution or an approximate solution can be obtained. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. Ayaz developed differential transform method to two-dimensional problem for PDE's initial value problems [14-15]. Kurnaz et al. generalized DTM to n-dimensional case in order to solve PDEs [19]. Recently, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [17, 20-21]. This paper investigates the applicability and effectiveness of differential transform method on variants of nonlinear Klein-Gordon equations.

Our attention will focus on the nonlinear Klein-Gordon equation of the form

$$u_{tt}(x,t) - u_{xx}(x,t) + bu(x,t) + g(u(x,t)) = f(x,t), \quad (1)$$

$$u(x,0) = a_0(x),\ u_t(x,0) = a_1(x). \quad (2)$$

where b is a real number, g is a given nonlinear function, and f is a known function.

The Klein-Gordon equation is one of the more important mathematical models in quantum mechanics. The equation has attracted much attention in studying solitons and condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations. The differential transformation is a numerical method for solving differential equations. The concept of differential transform was introduced by Zhou [22], who was the first one to use differential transform method (DTM) in engineering applications. He employed DTM in solution of initial boundary value problems in electric circuit analysis. In recent years, concept of DTM has broadened to the problems involving partial differential equations and systems of differential equations.

The purpose of this paper is to apply DTM to find the analytical solution of the nonlinear Klein-Gordon equation (1) alongwith (2), to show its ability and efficiency.

2. DIFFERENTIAL TRANSFORM METHOD

The basic definitions and fundamental operations of the two-dimensional differential transform are defined as follows. Consider an analytical function $w(x,t)$ of two variables in domain Ω.Then this function can be represented as a series in $(x_0,\ t_0) \epsilon \Omega$, using the differential transform

$$W(m,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right]_{x=x_0,\ t=t_0}, \quad (3)$$

by

$$w(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right]_{x=0,\ t=0} x^m t^n, \quad (4)$$

where$w(x,t)$ is called the inverse transform of $W(m,n)$.

In the following theorem, we summarize fundamental properties of the two-dimensional differential transform method.

Theorem: Let $U(m,n), V(m,n)$ and $W(m,n)$ be the two dimensional differential transforms of the functions $u(x,t), v(x,t)$and $v(x,t)$ in $(0,0)$, respectively. Then

1. If $u(x,t) = v(x,t) \pm w(x,t)$, then $U(m,n) = V(m,n) \pm W(m,n)$.
2. If $u(x,t) = av(x,t)$, then $U(m,n) = aV(m,n)$.
3. If $u(x,t) = v(x,t)w(x,t)$,then $U(m,n) = \sum_{k=0}^{m} \sum_{l=o}^{n} V(k,n-l)W(m-k,l)$.
4. If $u(x,t) = \frac{\partial^{r+s} v(x,t)}{\partial x^r \partial t^s}$, then $U(m,n) = \frac{(m+r)!}{m!}\frac{(n+s)!}{n!} V(m+r,n+s)$.
5. If $u(x,t) = e^{av(x,t)}$, then

$$U(m,n) = \begin{cases} e^{aV(0,0)}, m = n = 0, \\ a\sum_{k=0}^{m-1}\sum_{l=0}^{n} \frac{m-k}{m} V(m-k,l)U(k,n-l), m \geq 1, \\ a\sum_{k=0}^{m}\sum_{l=0}^{n-1} \frac{n-1}{n} V(k,n-l)U(m-k,l), n \geq 1. \end{cases}$$

6. If $u(x,t) = x^k t^h$, then $U(m,n) = \delta(m-k,n-h) = \begin{cases} 1 \text{ if } m = k.n = h, \\ 0 \quad \text{otherwise.} \end{cases}$
7. If $u(x,t) = x^k e^{at}$, then`$U(m,n) = \delta(m-k)\frac{a^n}{n!}$.

3. NUMERICAL APPLICATIONS

In this section we consider examples that demonstrate the performance and efficiency of the generalized differential transform method for solving nonlinear Klein-Gordon equations.

Example 3.1 Consider the linear Klein-Gordon equation

$$u_{tt} - u_{xx} = u, \tag{5}$$

$$u(x,0) = 1 + sinx, \;\; u_t(x,0) = 0. \tag{6}$$

Taking the differential transform of Eq. (5) and (6)

$$(h+1)(h+2)U(k,h+2) - (k+1)(k+2)U(k+2,h) = U(k,h), \tag{7}$$

$$U(k,0) = \delta(k) + \frac{1}{k!} sin\frac{k\pi}{2}, \tag{8}$$

$$U(k,1) = 0, \tag{9}$$

The inverse differential transform of *U(k, h)* is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, \tag{10}$$

by substituting Eq. (8), (9) into (7), we get the following values

$$U(0,0) = 1, U(0,2) = \frac{1}{2}, U(0,4) = \frac{1}{24}, U(0,6) = \frac{1}{720}, U(0,8) = \frac{1}{40320}, U(1,0) = 1, U(3,0) = -\frac{1}{6}, U(5,0) = \frac{1}{120}, U(7,0) = -\frac{1}{5040}, \dots$$

and so on. Substituting all $U(k, h)$ into Eq. (10), the following series solution will be obtained

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = 1 + x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \cdots,$$

$$u(x,t) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) + \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right).$$

The closed form solution is

$$u(x,t) = sinx + cosht.$$

Example 3.2 Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6 t^6, \quad (11)$$

$$u(x,0) = 0,\ u_t(x,0) = 0. \quad (12)$$

Taking the differential transform of Eq. (11) and (12)

$$(h+1)(h+2)U(k,h+2) - (k+1)(k+2)U(k+2,h) + \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)U(k-r,s=6\delta k-3\delta h-1-6\delta k-1\delta h-3+\delta k-6\delta h-6, \quad (13)$$

$$U(k,0) = 0,\ U(k,1) = 0, \quad (14)$$

The inverse differential transform of $U(k, h)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, \quad (15)$$

by substituting Eq. (14) into (13), we get the following values

$$U(0,0) = 0, U(0,1) = 0, U(0,3) = 0, U(1,2) = 0, U(3,8) = 0, U(5,5) = 0, U(3,2) = 0, U(3,3) = 0, \ldots$$

and so on. Substituting all $U(k, h)$ into Eq. (15), the following series solution will be obtained

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = x^3 t^3.$$

Example 3.3 Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = -xcost + x^2 cos^2 t, \quad (16)$$

$$u(x,0) = x, u_t(x,0) = 0, \quad (17)$$

Taking the differential transform of Eq. (16) and (17)

$$(h+1)(h+2)U(k,h+2) - (k+1)(k+2)U(k+2,h) + \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)U(k-r,s=-\delta k-11h!cosh\pi 2+\delta k-2(1h!cosh\pi 2)2, \quad (18)$$

$$U(k,0) = \delta(k-1),\ U(k,1) = 0, \quad (19)$$

The inverse differential transform of *U(k, h)* is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, (20)$$

by substituting Eq. (19) into (18), we get the following values

$$U(1,0) = 1, U(1,2) = -\frac{1}{2}, U(1,4) = \frac{1}{24}, U(1,6) = -\frac{1}{720}, U(1,8) = \frac{1}{40320}, \ldots$$

and so on. Substituting all *U(k, h)* into Eq. (20), the following series solution will be obtained

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = x - \frac{xt^2}{2} + \frac{xt^4}{24} - \frac{xt^6}{720} + \frac{xt^8}{40320} - \cdots,$$
$$u(x,t) = x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots\right),$$
$$u(x,t) = xcost.$$

Example 3.4 Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} - 2u = -2sinxsint, \quad (21)$$

$$u(x,0) = 0,\ u_t(x,0) = sinx. \quad (22)$$

Taking the differential transform of Eq. (21) and (22)

$$(h+1)(h+2)U(k,h+2) - (k+1)(k+2)U(k+2,h) - 2U(k,h) = -2\frac{1}{k!}sin\frac{k\pi}{2}\frac{1}{h!}sin\frac{h\pi}{2}, \quad (23)$$

$$U(k,0) = 0,\ U(k,1) = \frac{1}{k!}sin\frac{k\pi}{2}, \quad (24)$$

The inverse differential transform of *U(k, h)* is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, (25)$$

by substituting Eq. (24) into (23), we get the following values

$$U(1,1) = 1, U(1,3) = -\frac{1}{6}, U(1,5) = \frac{1}{120}, U(3,1) = -\frac{1}{6}, U(3,3) = \frac{1}{36}, U(3,5) = -\frac{1}{720}, U(5,1) = \frac{1}{120}, U(5,3) = -\frac{1}{720}, U(5,5) = \frac{1}{14400}, \ldots$$

and so on. Substituting all *U(k, h)* into Eq. (25), the following series solution will be obtained

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$
$$=xt-\frac{xt^3}{6}+\frac{xt^5}{120}-\frac{x^3t}{6}+\frac{x^3t^3}{36}-\frac{x^3t^5}{720}+\frac{x^5t}{120}-\frac{x^5t^3}{720}+\frac{x^5t^5}{14400}+\cdots,$$
$$u(x,t)=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right),$$
$$u(x,t)=sinxsint.$$

4. CONCLUSIONS

In this paper, Differential Transform Method (DTM) has been successfully applied to obtain the approximate analytical solutions of the Klein-Gordon equation. The obtained exact solutions reveal that DTM is very effective and convenient. The present study has confirmed that the differential transform method offers great advantages of straightforward applicability, computational efficiency and high accuracy.

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