ORIGINAL PAPER

CONTINUOUS FRACTIONAL GENERALIZED HANKEL-CLIFFORD TRANSFORMATION

V. R. LAKSHMI GORTY[1]

Manuscript received: 20.12.2014; Accepted paper: 09.01.2015; Published online: 30.03.2015.

Abstract. *The main objective of this paper is to study the continuous fractional generalized Hankel-Clifford transformation and continuous fractional generalized Hankel-Clifford wavelet transformation and some of their basic properties. Graphical representations for continuous fractional generalized Hankel-Clifford transformations have been demonstrated for some values. Applications to the continuous fractional generalized Hankel-Clifford transformation in solving generalized* n^{th} *order linear non-homogeneous ordinary differential equations are given. The continuous fractional generalized Hankel-Clifford transformation wavelet transformation, its inversion formula and Parseval's are also studied.*

Keywords: *generalized Hankel-Clifford transformation; fractional generalized Hankel-Clifford transformation; continuous fractional generalized Hankel-Clifford wavelet transformation.*

1. INTRODUCTION

Akhilesh Prasad et.al, introduced the continuous fractional Bessel wavelet transformation, in [1]. Pathak and Dixit [2] introduced continuous and discrete Bessel wavelet transformations. Upadhyay et al. [3] studied the continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator. Namias in [4] studied fractionalization of Hankel transform. In [5], author studied similarity theorems for fractional Fourier transforms and fractional Hankel transforms. The author in [6] analyzed self-fractional Hankel functions and their properties. Kerr studied the fractional powers of Hankel transforms in the Zemanian spaces in [7]. The author in [8] presented wavelet transforms and their applications, in 2002. Shi in [9], a novel fractional wavelet transform and its applications. Lakshmi Gorty [10] studied continuous generalized Hankel-Clifford wavelet transformation.

Let $L_p(0,\infty), 1 \le p \le \infty,$ as the space of real measurable function ϕ on $(0,\infty)$ for which

$$\|\phi\|_{\beta,p} = \left(\int_0^\infty \left| x^{-\beta}\phi(x) \right|^p \frac{dx}{x} \right)^{1/p}, 1 \le p < \infty.$$

$$\|\phi\|_\infty = \underset{0<x<\infty}{ess\sup} \left| x^{-\beta}\phi(x) \right| < \infty.$$

For each $\phi \in L_1(0,\infty),$ generalized Hankel-Clifford transformation of ϕ is defined by

$$\hat{\phi}(x) = t^{-\alpha-\beta} \int_0^\infty (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt} \right] \phi(t)dt \quad, 0 < t < \infty.$$

[1] SVKM's NMIMS University, Department of Mathematics, Mumbai 400056, India.
E-mail: lakshmi_gorty@rediffmail.com.

it is observed $\hat{\phi}(x)$ is bounded and continuous on $(0,\infty)$ and $\left\|\hat{\phi}(x)\right\|_{\infty} \le \|\phi\|_1$.

Malgonde and Bandewar [11] investigated the variant of the generalized Hankel-Clifford transform defined by

$$h_{\alpha,\beta}\left(f(y)\right)=F(y)=y^{-\alpha-\beta}\int_0^\infty \mathcal{J}_{\alpha,\beta}(xy)\,f(x)\,dx\ ,\ x\in\mathbb{R}_+,(\alpha-\beta)\ge -1/2$$

where $\mathcal{J}_{\alpha,\beta}(x)=(x)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\sqrt{x}\right)$, $J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions.

Its inversion formula is given by

$$f(x)=h^{-1}_{\alpha,\beta}\left(F(y)\right)=y^{-\alpha-\beta}\int_0^\infty \mathcal{J}_{\alpha,\beta}(xy)\,F(y)\,dy,\ y\in\mathbb{R}_+$$

We define a one-dimensional continuous fractional generalized Hankel-Clifford transformation with parameter θ of $f(x)$ for $(\alpha-\beta)\ge -1/2$ and $0<\theta<\pi$ as follows:

$$h^{\theta}_{\alpha,\beta}\left(f(y)\right)=F(y)=y^{-\alpha-\beta}\int_0^\infty C^{\theta}_{\alpha,\beta}(x,y)\,f(x)\,dx\ , \tag{3}$$

where the kernel

$$C^{\theta}_{\alpha,\beta}(x,y)=\begin{cases} e^{|\cot\theta|\,i\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}c^{\theta}_{\alpha,\beta}\left(\dfrac{xy}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\dfrac{xy}{|\sin\theta|}\right)^{1/2}\right), & \theta\ne n\pi \\ (xy)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2(xy)^{1/2}\right) & ,\theta=\dfrac{\pi}{2} \\ \delta(x-y) & ,\theta=n\pi,\forall n\in\mathbb{Z}. \end{cases}$$

and

$$c^{\theta}_{\alpha,\beta}=\frac{e^{i(\alpha-\beta+1)\left(\frac{\pi}{2}-\theta\right)}}{\sin\theta}.$$

Its inversion formula of (3) is given by

$$f(x)=\left(\left(h^{\theta}_{\alpha,\beta}\right)^{-1}F\right)(x)=y^{-\alpha-\beta}\int_0^\infty \overline{C^{\theta}_{\alpha,\beta}(x,y)}\left(h^{\theta}_{\alpha,\beta}f\right)(y)\,dy,\ y\in\mathbb{R}_+$$

where

$$\begin{aligned}\overline{C^{\theta}_{\alpha,\beta}(x,y)}&=e^{-|\cot\theta|\,i\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}e^{-i(\alpha-\beta+1)\left(\frac{\pi}{2}-\theta\right)}\left(\frac{xy}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{|\sin\theta|}\right)^{1/2}\right)\\ &=\overline{c^{\theta}_{\alpha,\beta}}\sin\theta\, e^{-|\cot\theta|\,i\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}\left(\frac{xy}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{|\sin\theta|}\right)^{1/2}\right).\end{aligned}$$

$h^{0}_{\alpha,\beta}f=h^{\pi}_{\alpha,\beta}f=f$,

$h^{\theta+2\pi}_{\alpha,\beta}f=h^{\theta}_{\alpha,\beta}f,\theta\in\mathbb{R}$.

Throughout this paper, $\theta\ne n\pi$, $\forall n\in\mathbb{Z}$

2. PROPERTIES OF FRACTIONAL GENERALIZED HANKEL-CLIFFORD TRANSFORMATION

The $\hbar_{1,\alpha,\beta}$ transformation was studied by Malgonde and Bandewar [11] who extended it to a space of generalized functions. They constructed the space H_β and H_α of testing functions of all the complex-valued smooth functions $\phi(x)$ defined on $I=(0,\infty)$, such that $\gamma^{\beta}_{m,k}(\phi)=\sup_{x\in I}\left|x^m D^k x^{\beta}\phi(x)\right|<\infty$, for every $m,k\in N$ and $\gamma^{\alpha}_{m,k}(\phi)=\sup_{x\in I}\left|x^m D^k x^{-\alpha}\phi(x)\right|<\infty$, for every $m,k\in N$ respectively. $\hbar_{1,\alpha,\beta}$ is an automorphism onto H_β for $\alpha-\beta\geq-\frac{1}{2}$.

Definition (2.1): (Testing function space $H_{\alpha,\beta,\theta}(\mathbb{R}_+)$) The space $H_{\alpha,\beta,\theta}(\mathbb{R}_+)$ is defined as follows: f is a member of $H_{\alpha,\beta,\theta}(\mathbb{R}_+)$ if and only if it is a complex-valued C^∞ -function on $\mathbb{R}_+$ and for every choice of k of non-negative integers, let a denote positive real number and α,β be arbitrary real parameters with $\alpha-\beta\geq-1/2$. Let the space of testing functions that consists of all smooth complex-valued functions $\phi(x)$ defined on $I=(0,\infty)$, such that

$$\gamma^{\alpha,\beta,a}_{k}(\phi)=\sup_{I}\left|e^{-2a\sqrt{x}}x^{\beta}\Delta^{k}_{\alpha,\beta}\phi(x)\right|<\infty\ ,k\in N \tag{2.1}$$

where $\Delta_{\alpha,\beta,x}=\dfrac{d^2}{dx^2}+(1-\alpha-\beta)ix\cot\theta\dfrac{d}{dx}+\left\{\dfrac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}+i\cot\theta-x\cot^2\theta\right\}$

and

and $\Delta^{k}_{\alpha,\beta,x}=\sum_{r=0}^{2k}a_{r,k}x^{k-r}D^{2k-r}$, where the constants $a_{r,k}$ is a constant depends only on α and β and parameter θ for $k=0,1,2,3,\ldots$. On $H_{\alpha,\beta,\theta}(\mathbb{R}_+)$, considering the topology generated by the family $\left\{\gamma^{\theta}_{\alpha,\beta,k}\right\}_{k\in\mathbb{N}_0}$ of seminorms.

Proposition 2.1 Let $C^{\theta}_{\alpha,\beta}(x,y)$ be the kernel of the fractional generalized Hankel-Clifford transformation. Then as in [12]

$$\Delta^{r}_{\alpha,\beta,x}C^{\theta}_{\alpha,\beta}(x,y)=\left(-\frac{y}{|\sin\theta|}\right)^{r}C^{\theta}_{\alpha,\beta}(x,y),\forall r\in\mathbb{N}_{0,}$$

$$h^{\theta}_{\alpha,\beta}\left(\left(\Delta^{*}_{\alpha,\beta,x}\right)^{r}f(x)\right)(y)=\left(-\frac{y}{|\sin\theta|}\right)^{r}\left(h^{\theta}_{\alpha,\beta}f\right)(y)\ \text{ and }\ f\in H_{\alpha,\beta}(\mathbb{R}_+)$$

where $\Delta^{*}_{\alpha,\beta,x}=\dfrac{d^2}{dx^2}-(1-\alpha-\beta)ix\cot\theta\dfrac{d}{dx}+\left\{\dfrac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}-i\cot\theta-x\cot^2\theta\right\}$ and will be known as fractional generalized Hankel-Clifford operator with parameter θ.

Proof: i) Let $r=1$, $\Delta_{\alpha,\beta,x}C^{\theta}_{\alpha,\beta}(x,y)$ is given by

$$\Delta_{\alpha,\beta,x}C_{\alpha,\beta}^{\theta}(x,y)=\left[\frac{d^2}{dx^2}+(1-\alpha-\beta)ix\cot\theta\frac{d}{dx}+\left\{\frac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}+i\cot\theta-x\cot^2\theta\right\}\right]C_{\alpha,\beta}^{\theta}(x,y)$$

$$=-\left(\frac{y}{|\sin\theta|}\right)C_{\alpha,\beta}^{\theta}(x,y).$$

Let $r=2$, $\Delta_{\alpha,\beta,x}^{2}C_{\alpha,\beta}^{\theta}(x,y)$ is

$$\Delta_{\alpha,\beta,x}^{2}C_{\alpha,\beta}^{\theta}(x,y)=\left[\frac{d^2}{dx^2}+(1-\alpha-\beta)ix\cot\theta\frac{d}{dx}+\left\{\frac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}+i\cot\theta-x\cot^2\theta\right\}\right]^2C_{\alpha,\beta}^{\theta}(x,y)$$

$$=(-1)^2\left(\frac{y}{|\sin\theta|}\right)^2C_{\alpha,\beta}^{\theta}(x,y)=\left(\frac{y}{|\sin\theta|}\right)^2C_{\alpha,\beta}^{\theta}(x,y).$$

Thus by Mathematical induction the proof (i) is given.

Proof (ii): For $r=1$, $h_{\alpha,\beta}^{\theta}\left(\left(\Delta_{\alpha,\beta,x}^{*}\right)^{r}f(x)\right)(y)$ can be written as

$$h_{\alpha,\beta}^{\theta}\left(\left(\Delta_{\alpha,\beta,x}^{*}\right)f(x)\right)(y)$$

$$=h_{\alpha,\beta}^{\theta}\left[\left\{\frac{d^2}{dx^2}-(1-\alpha-\beta)ix\cot\theta\frac{d}{dx}+\left\{\frac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}-i\cot\theta-x\cot^2\theta\right\}\right\}f(x)\right](y)$$

$$=-\left(\frac{y}{|\sin\theta|}\right)\left(h_{\alpha,\beta}^{\theta}f\right)(y).$$

Similarly $h_{\alpha,\beta}^{\theta}\left(\left(\Delta_{\alpha,\beta,x}^{*}\right)^{r}f(x)\right)(y)=\left(-\frac{y}{|\sin\theta|}\right)^{r}\left(h_{\alpha,\beta}^{\theta}f\right)(y)$ and $f\in H_{\alpha,\beta}(\mathbb{R}_+)$.

Proposition 2.2. Let $f\in L^1(\mathbb{R}_+)$. Then $F_{\alpha,\beta}^{\theta}$ satisfies the following

(i). $F_{\alpha,\beta}^{\theta}\in L^{\infty}(\mathbb{R}_+)$ with $\left\|F_{\alpha,\beta}^{\theta}\right\|_{L^{\infty}}\le K_{\alpha,\beta,\theta}\|f\|_{L^1}$,

(ii). $F_{\alpha,\beta}^{\theta}(y)\to 0$ as $y\to\infty$ or $-\infty$,

(iii). $F_{\alpha,\beta}^{\theta}$ is continuous on $\mathbb{R}_+$,

where $K_{\alpha,\beta,\theta}$ is a positive constant depending on α,β *and* θ.

Proof:

(i) $h_{\alpha,\beta}^{\theta}(f(y))=F(y)=y^{-\alpha-\beta}\int_0^{\infty}C_{\alpha,\beta}^{\theta}(x,y)f(x)dx$, so $\left\|F_{\alpha,\beta}^{\theta}\right\|_{L^{\infty}}\le K_{\alpha,\beta,\theta}\|f\|_{L^1}$.

(ii) From proposition 2.1 (ii), where $r=1$,

$\left|F_{\alpha,\beta}^{\theta}(y)\right|=\left|-\left(\frac{|\sin\theta|}{y}\right)\right|\left|h_{\alpha,\beta}^{\theta}\left(\left(\Delta_{\alpha,\beta,x}^{*}\right)f(x)\right)(y)\right|\to 0$ as $y\to\pm\infty$, although

$F_{\alpha,\beta}^{\theta}(y)\to 0$ as $y\to\pm\infty$ for every $f\in L^1(\mathbb{R}_+)$.

(iii) Let $h>0$, consider

$$\sup_y \left| F^{\theta}_{\alpha,\beta}(y+h) - F^{\theta}_{\alpha,\beta}(y) \right|$$

$$\leq \left| c^{\theta}_{\alpha,\beta} \right| \sup_y \left| y^{-\alpha-\beta} \int_0^{\infty} \begin{Bmatrix} e^{|\cot\theta|\, i\left(\frac{(y+h)^2}{2}+\frac{x^2}{2}\right)} \left(\frac{x(y+h)}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{x(y+h)}{|\sin\theta|}\right)^{1/2}\right) \\ -e^{|\cot\theta|\, i\left(\frac{(y)^2}{2}+\frac{x^2}{2}\right)} \left(\frac{x(y)}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{x(y)}{|\sin\theta|}\right)^{1/2}\right) \end{Bmatrix} \right| \left| f(x) \right| dx$$

$$\leq K_{\alpha,\beta,\theta} \int_0^{\infty} \left| f(x) \right| dx \in L^1(\mathbb{R}_+),$$

and

$$\left| e^{|\cot\theta|\, i\left(\frac{(y+h)^2}{2}+\frac{x^2}{2}\right)} \left(\frac{x(y+h)}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{x(y+h)}{|\sin\theta|}\right)^{1/2}\right) - e^{|\cot\theta|\, i\left(\frac{(y)^2}{2}+\frac{x^2}{2}\right)} \left(\frac{x(y)}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{x(y)}{|\sin\theta|}\right)^{1/2}\right) \right| \to 0$$

as $h \to 0$. So, $\sup_y \left| F^{\theta}_{\alpha,\beta}(y+h) - F^{\theta}_{\alpha,\beta}(y) \right| \to 0$ as $h \to 0$. This proves that $F^{\theta}_{\alpha,\beta}(y)$ is continuous in $\mathbb{R}_+$.

Proposition 2.3 (Parseval's relation) If $P(y) = h^{\theta}_{\alpha,\beta}(p(y))$ and $Q(y) = h^{\theta}_{\alpha,\beta}(q(y))$ denote the fractional generalized Hankel-Clifford transformation of $p(y)$ and $q(y)$ respectively as in [13], then

$$\int_0^{\infty} p(x)\overline{q(x)}\,dx = \sin\theta \int_0^{\infty} h^{\theta}_{\alpha,\beta}(p(y))\overline{h^{\theta}_{\alpha,\beta}(q(y))}\,dy = \sin\theta \int_0^{\infty} P(y)\overline{Q(y)}\,dy,$$

and $\int_0^{\infty} \left| p(x) \right|^2 dx = \sin\theta \int_0^{\infty} \left| h^{\theta}_{\alpha,\beta}(p(y)) \right|^2 dy.$

Proof: We have

$$\langle p, q \rangle = \int_0^{\infty} p(x)\overline{q(x)}\,dx$$

$$= \int_0^{\infty} p(x) \overline{\left(y^{-\alpha-\beta} \int_0^{\infty} e^{-|\cot\theta|\, i\left(\frac{y^2}{2}+\frac{x^2}{2}\right)} \left(\overline{c^{\theta}_{\alpha,\beta}}\right) \sin\theta \left(\frac{xy}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{xy}{|\sin\theta|}\right)^{1/2}\right) \left(h^{\theta}_{\alpha,\beta}q\right)(y)\,dy \right)} dx$$

$$= c^{\theta}_{\alpha,\beta} \sin\theta \int_0^{\infty} \overline{\left(h^{\theta}_{\alpha,\beta}q\right)(y)} \left(y^{-\alpha-\beta} \int_0^{\infty} e^{-|\cot\theta|\, i\left(\frac{y^2}{2}+\frac{x^2}{2}\right)} \left(\frac{xy}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{xy}{|\sin\theta|}\right)^{1/2}\right) p(x)\,dx \right) dy$$

$$= \sin\theta \int_0^{\infty} \left(h^{\theta}_{\alpha,\beta}p\right)(y) \overline{\left(h^{\theta}_{\alpha,\beta}q\right)(y)}\,dy$$

$$= \sin\theta \int_0^{\infty} P(y)\overline{Q(y)}\,dy.$$

and if $p(y) = q(y)$, then $\int_0^{\infty} \left| p(x) \right|^2 dx = \sin\theta \int_0^{\infty} \left| h^{\theta}_{\alpha,\beta}(p(y)) \right|^2 dy.$

2.1. FRACTIONAL GENERALIZED HANKEL-CLIFFORD TRANSFORMATION OF SOME SIMPLE SIGNALS:

When a unit signal given and if $\sqrt{\frac{y}{|\sin\theta|}} \in R; \sqrt{\frac{y}{|\sin\theta|}} = 0; \& \operatorname{Re}(\alpha) > -1$, then Fractional generalized Hankel-Clifford transformation with angle θ is given by

$$\frac{1}{\sin\theta}\left(y^{-\alpha-\beta}\frac{1}{|\sin\theta|}\right)^{\frac{\alpha+\beta}{2}}.$$

For $\theta = \pi/2$, the fractional generalized Hankel-Clifford transformation becomes $\Gamma(1+\alpha)/y\Gamma(-\beta)$ for $\sqrt{y} \in \mathbb{R} \& \operatorname{Re}(\alpha) > -1; y \geq 0.$

For $\delta(x-y)$ is the signal, then fractional generalized Hankel-Clifford transformation becomes $ie^{|\cot\theta|\, i\left(\frac{y^2}{2}\right)+i(\alpha-\beta+1)\left(\frac{\pi}{2}-\theta\right)} y^{\alpha+\beta} J_{\alpha-\beta}\left(2\frac{y}{|\sin\theta|}\right)$ for $y > 0.$

The fractional generalized Hankel-Clifford transformation is given by

$$\left[\frac{-e^{\frac{1}{2}i\left\{\begin{array}{l}\pi(\alpha-\beta)\\-2(1+\alpha+\beta)\theta\\+y^2|\cot\theta|\end{array}\right\}} 2^{-1-\frac{\beta}{2}} y^{1+\alpha+\beta}\left(\frac{y^2}{2a-i|\cot\theta|}\right)^{-\frac{1}{2}(1+\beta)}\left(|\sin\theta|\right)^{\frac{1}{2}(-\alpha+\beta-2)}}{\left(2a-i|\cot\theta|\right)\Gamma(2+\alpha-\beta)\Gamma(1+\alpha-\beta)}\right]$$

$$\times\left(\begin{array}{l}\sqrt{2}|\sin\theta|\Gamma(2+\alpha-\beta)\Gamma\left(\frac{1}{2}-\frac{\beta}{2}\right)\,{}^{p}F_{q}\left[\begin{array}{l}\left\{\frac{1}{2}-\frac{\beta}{2}\right\},\\ \left\{\frac{1}{2},\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},1+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\\ \frac{y^2}{8\left(2a-i|\cot\theta|\right)|\sin\theta|^2}\end{array}\right]\\ +\beta\sqrt{\frac{y^2}{\left(2a-i|\cot\theta|\right)}}\Gamma(1+\alpha-\beta)\Gamma\left(1-\frac{\beta}{2}\right)\,{}^{p}F_{q}\left[\begin{array}{l}\left\{1-\frac{\beta}{2}\right\},\\ \left\{\frac{3}{2},1+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\\ \frac{y^2}{8\left(2a-i|\cot\theta|\right)|\sin\theta|^2}\end{array}\right]\end{array}\right)$$

for the signal $x^{-\alpha-\beta}e^{-ax^2}$ when $y \geq 0 \& \left(\sqrt{\frac{y}{|\sin\theta|}}\right) \in R; \operatorname{Re}(a) \geq 0; \operatorname{Re}(\beta) < 1.$

The fractional generalized Hankel-Clifford transformation is given by

$$\frac{2^{-1+\frac{\alpha}{2}} e^{\frac{1}{2}i\left\{\begin{array}{l}\pi(\alpha-\beta)\\ -2(1+\alpha+\beta)\theta\\ +y^2|\cot\theta|\end{array}\right\}} \left(\frac{y^2}{1+\frac{1}{2}i|\cot\theta|}\right)^{\frac{\alpha}{2}} \left(|\sin\theta|\right)^{\frac{1}{2}(-\alpha+\beta-2)}}{y\left(2a-i|\cot\theta|\right)\Gamma(2+\alpha-\beta)\Gamma(1+\alpha-\beta)}$$

$$\times\left(\begin{array}{l} 2y^2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha-\beta)\Gamma\left(1+\frac{\alpha}{2}\right)\\ \times {}^pF_q\left[\left\{1+\frac{\alpha}{2}\right\},\left\{\frac{3}{2},\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{iy^2}{8\left(-2i+|\cot\theta|\right)|\sin\theta|^2}\right]\\ +\sqrt{2}\sin\left(\frac{\pi\alpha}{2}\right)|\sin\theta|\left(-2-i|\cot\theta|\right)\sqrt{-\frac{iy^2}{\left(-2i+|\cot\theta|\right)}}\Gamma(2+\alpha-\beta)\Gamma\left(\frac{1+\alpha}{2}\right)\\ {}^pF_q\left[\left\{\frac{1+\alpha}{2}\right\},\left\{\frac{3}{2},1+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{iy^2}{8\left(-i2+|\cot\theta|\right)|\sin\theta|^2}\right] \end{array}\right)$$

$y \geq 0$ & $\left(\sqrt{\frac{y}{|\sin\theta|}}\right) \in R; \operatorname{Re}(a) \geq 0 \operatorname{Re}(\beta) < 1$ for a signal in the form of $x^{-\alpha-\beta}e^{-ax^2}$.

For the signal e^{x^2}, the fractional generalized Hankel-Clifford transformation is given by if $y \geq 0$ & $\left(\sqrt{\frac{y}{|\sin\theta|}}\right) \in R$ and $\operatorname{Re}(\alpha) > -1.$

For the signal $x^{-\alpha-\beta+1}e^{-x^2}$; if $y \geq 0$ & $\left(\sqrt{\frac{y}{|\sin\theta|}}\right) \in R$ and $\operatorname{Re}(\beta) < 0,$, the fractional generalized Hankel-Clifford transformation is given by

$$\left[\frac{-e^{\frac{1}{2}i\left\{\begin{array}{l}\pi(\alpha-\beta)\\ -2(1+\alpha+\beta)\theta\\ +y^2|\cot\theta|\end{array}\right\}} 2^{-1-\frac{\beta}{2}} y^{\alpha+\beta}\left(\frac{y^2}{2-i|\cot\theta|}\right)^{-\frac{\beta}{2}}\left(|\sin\theta|\right)^{\frac{1}{2}(-\alpha+\beta-2)}}{\Gamma(2+\alpha-\beta)\Gamma(1+\alpha-\beta)}\right]$$

$$\times\left(\begin{array}{l} -\sqrt{2}|\sin\theta|\left(\frac{y^2}{2-i|\cot\theta|}\right)^{1/2}\Gamma(1+\alpha-\beta)\Gamma\left(\frac{1}{2}-\frac{\beta}{2}\right)\\ {}^pF_q\left[\left\{\frac{1}{2}-\frac{\beta}{2}\right\},\left\{\frac{3}{2},\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{y^2}{8\left(2a-i|\cot\theta|\right)|\sin\theta|^2}\right]\\ +|\sin\theta|\Gamma(2+\alpha-\beta)\Gamma\left(-\frac{\beta}{2}\right)\\ {}^pF_q\left[\left\{-\frac{\beta}{2}\right\},\left\{\frac{3}{2},1+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{y^2}{8\left(2a-i|\cot\theta|\right)|\sin\theta|^2}\right] \end{array}\right).$$

When $\sqrt{y} \in \mathbb{R}$ and $\mathrm{Re}(\alpha) > -1; y \geq 0$ and $\theta = \frac{\pi}{2}$, the graphical representation of fractional generalized Hankel-Clifford transformation is given by

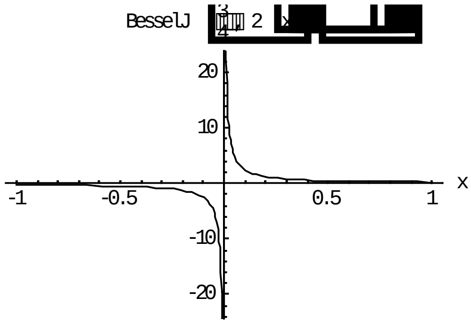

Figure 1: Fractional generalized Hankel-Clifford transformation for

$$f(x) = 1;\ \alpha = \frac{1}{2}; \beta = -\frac{1}{4}; \theta = \frac{\pi}{2}.$$

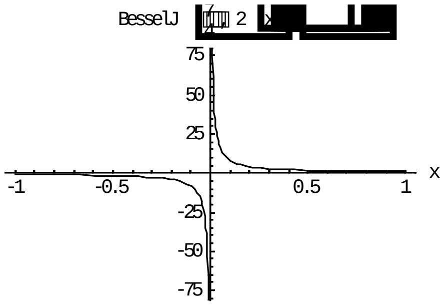

Figure 2: Fractional generalized Hankel-Clifford transformation for $f(x) = 1;\ \alpha = 1; \beta = -\frac{3}{4}; \theta = \frac{\pi}{2}.$

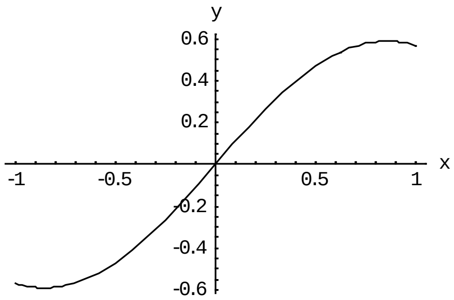

Figure 3: Fractional generalized Hankel-Clifford transformation for

$$f(x) = \delta(x-y);\ \alpha = \frac{1}{2}; \beta = -\frac{1}{4}; \theta = \frac{\pi}{2}; y > 0..$$

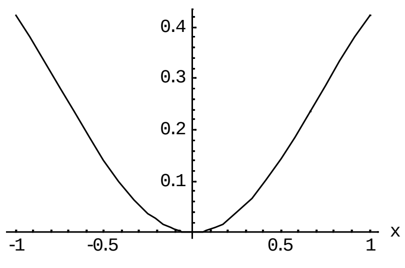

Figure 4: Fractional generalized Hankel-Clifford transformation for

$$f(x) = \delta(x-y);\ \alpha = 1; \beta = -\frac{3}{4}; \theta = \frac{\pi}{2}, y > 0.$$

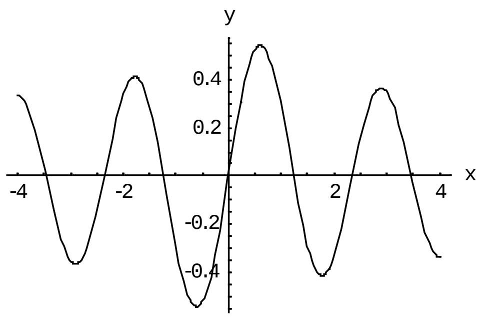

Figure 5: Fractional generalized Hankel-Clifford transformation for
$f\left(x\right)=\delta\left(x-y\right);\ \alpha=\frac{1}{2};\beta=-\frac{1}{4};\theta=\frac{\pi}{6}$ **, considering absolute value of the results.**

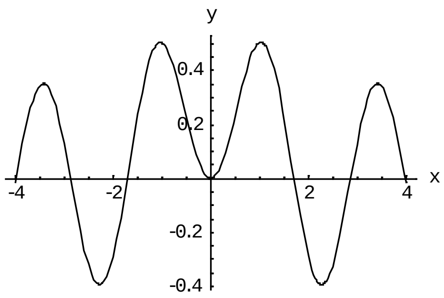

Figure 6: Fractional generalized Hankel-Clifford transformation for
$f\left(x\right)=\delta\left(x-y\right);\ \alpha=1;\beta=-\frac{3}{4};\theta=\frac{\pi}{6}$**, considering absolute value of the results.**

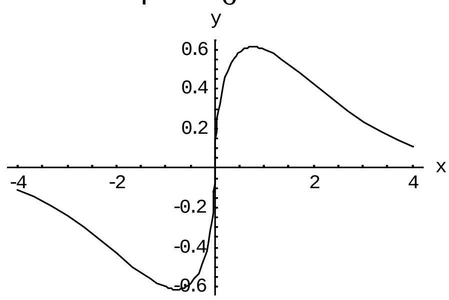

Figure 7: Fractional generalized Hankel-Clifford transformation for
$f\left(x\right)=e^{-x^2};\ \alpha=\frac{1}{2};\beta=-\frac{1}{4};\theta=\frac{\pi}{2}.$

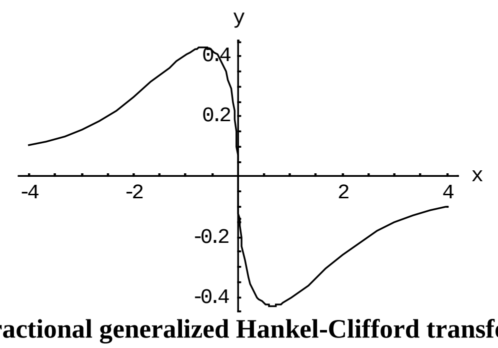

Figure 8: Fractional generalized Hankel-Clifford transformation for
$f\left(x\right)=e^{-x^2};\ \alpha=1;\beta=-\frac{3}{4};\theta=\frac{\pi}{6}$
considering absolute value of the results.

3. APPLICATIONS OF THE FRACTIONAL GENERALIZED HANKEL-CLIFFORD TRANSFORMATION TO GENERALIZED DIFFERENTIAL EQUATIONS

Example 1: Consider the generalized n^{th} order linear non-homogeneous ordinary differential equation

$$L\left(p(x)\right)=f(x), \qquad (15)$$

where L is the generalized n^{th} order differential operator given by $L=a_n\left(\Delta^*_{\alpha,\beta,x}\right)^n+a_{n-1}\left(\Delta^*_{\alpha,\beta,x}\right)^{n-1}+...+a_1\left(\Delta^*_{\alpha,\beta,x}\right)+a_0$, where $a_n, a_{n-1},...,a_0$ are constants and $\Delta^*_{\alpha,\beta,x}$ is given in proposition 2.1. Applying fractional generalized Hankel-Clifford transformation to both sides of the equation (15),

$$\int_0^\infty C^\theta_{\alpha,\beta}(x,y)L\left(p(x)\right)dx=\int_0^\infty C^\theta_{\alpha,\beta}(x,y)f(x)dx\ ,$$

$$\left[a_n\left(-\frac{y^2}{|\sin\theta|^2}\right)^n+a_{n-1}\left(-\frac{y^2}{|\sin\theta|^2}\right)^{n-1}+...+a_1\left(-\frac{y^2}{|\sin\theta|^2}\right)+a_0\right]\left(h^\theta_{\alpha,\beta}p(y)\right)=h^\theta_{\alpha,\beta}\left(f(y)\right)$$

Consider $P(z)=\sum_{r=0}^n a_r z^r$, then $P\left(-\frac{y^2}{|\sin\theta|^2}\right)\left(h^\theta_{\alpha,\beta}p(y)\right)=h^\theta_{\alpha,\beta}\left(f(y)\right)$.

Therefore $h^\theta_{\alpha,\beta}p(y)=\dfrac{h^\theta_{\alpha,\beta}\left(f(y)\right)}{P\left(-\frac{y^2}{|\sin\theta|^2}\right)}$.

Applying inverse fractional generalized Hankel-Clifford transformation gives the solution

$$p(x)=\left(h^\theta_{\alpha,\beta}\right)^{-1}\left[\frac{h^\theta_{\alpha,\beta}\left(f(y)\right)}{P\left(-\frac{y^2}{|\sin\theta|^2}\right)}\right].$$

Example 2: Using the fractional generalized Hankel-Clifford transformation, investigate the solution of the generalized differential equation

$$\frac{\partial^2 u}{\partial x^2}+(1-\alpha-\beta)ix\cot\theta\frac{\partial u}{\partial x}+\left\{\frac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}+i\cot\theta-x\cot^2\theta\right\}u+\frac{\partial^2 u}{\partial z^2}=0,\ x\geq 0, z\geq 0,$$

with $u(x,z)\to 0$ as $z\to\infty$ and $x\to\infty$,

and $u(x,0)=f(x),\ x\geq 0.$

Applying fractional generalized Hankel-Clifford transformation to equation ( ),

$$y^{-\alpha-\beta}\int_0^\infty\left[\frac{\partial^2 u}{\partial x^2}+(1-\alpha-\beta)ix\cot\theta\frac{\partial u}{\partial x}+\left\{\frac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}+i\cot\theta-x\cot^2\theta\right\}u+\frac{\partial^2 u}{\partial z^2}\right]C^\theta_{\alpha,\beta}(x,y)dx=0.$$

$$y^{-\alpha-\beta}\int_0^\infty\left[\frac{\partial^2 u}{\partial x^2}+(1-\alpha-\beta)ix\cot\theta\frac{\partial u}{\partial x}+\left\{\begin{matrix}\frac{-3\alpha^2-3\beta^2+10\alpha\beta}{4x}\\+i\cot\theta-x\cot^2\theta\end{matrix}\right\}u\right]C^\theta_{\alpha,\beta}(x,y)dx+y^{-\alpha-\beta}\int_0^\infty\frac{\partial^2 u}{\partial z^2}C^\theta_{\alpha,\beta}(x,y)dx=0.$$

The differential equation becomes

$$-\frac{y^2}{|\sin\theta|^2}\tilde{u}+\frac{d^2\tilde{u}}{dz^2}=0,$$

$$\left(D^2-\frac{y^2}{|\sin\theta|^2}\right)\tilde{u}=0$$

where $D=\frac{d}{dz}$.

The solution of equation ( ) is given by

$$\tilde{u}(y,z)=C_1e^{\frac{yz}{|\sin\theta|}}+C_2e^{-\frac{yz}{|\sin\theta|}}.$$

Taking fractional generalized Hankel-Clifford transformation of order zero of ( ), $\tilde{u}(y,z)\to 0$ as $z\to\infty$, if $C_1=0$.

Applying the inversion formula,

$$\tilde{u}(x,z)=y^{-\alpha-\beta}\int_0^\infty \overline{C_{\alpha,\beta}^{\theta}(x,y)}\left(C_1e^{\frac{yz}{|\sin\theta|}}+C_2e^{-\frac{yz}{|\sin\theta|}}\right)dy.$$

3. CONTINUOUS FRACTIONAL GENERALIZED HANKEL-CLIFFORD WAVELET TRANSFORMATION

The continuous fractional generalized Hankel-Clifford wavelet transformation is a generalization of the ordinary continuous generalized Hankel-Clifford wavelet transformation with parameter θ, that is, continuous generalized Hankel-Clifford wavelet transformation is a special case of continuous fractional generalized Hankel-Clifford wavelet transformation with parameter $\theta=\frac{\pi}{2}$.

A fractional generalized Hankel-Clifford wavelet is a function $\psi\in L^2(\mathbb{R}_+)$ which satisfies the condition

$$A_{\beta,\psi,\theta}=\int_0^\infty x^{-2\alpha-1}\left|\left(h_{\alpha,\beta}^{\theta}\psi\right)(x)\right|^2dx<\infty,\ \alpha-\beta\geq -1/2, \tag{16}$$

where $A_{\beta,\psi,\theta}$ is called the admissibility condition of the fractional generalized Hankel-Clifford wavelet transformation. The fractional generalized Hankel-Clifford wavelets $\psi_{b,a}^{\theta}(x)$ can be generated from one single function $\psi\in L^2(\mathbb{R}_+)$ by dilation and translation with $b\in\mathbb{R}$ and $a>0$ as in [7]. $\psi_{b,a}(x)=\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$, $v(x,t)$ where a is called the scaling parameter which measures the degree of compression or scale and b is a translation parameter which determines the time location of the wavelet as

$\psi_{b,a}^{\theta}(x)=\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)e^{-\frac{i}{2}\left(x^2-b^2\right)\cot\theta}$, for all a,b and θ as above.

$$D_{\alpha,\beta}^{\theta}(x,y,z)$$

$$=c_{\alpha,\beta}^{\theta}e^{\frac{i}{2}\left(x^2+y^2+z^2\right)|\cot\theta|}\int_0^{\infty}\xi^{-\alpha}\left(\frac{x\xi}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{x\xi}{|\sin\theta|}\right)^{1/2}\right)\left(\frac{y\xi}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{y\xi}{|\sin\theta|}\right)^{1/2}\right)$$

$$\times\left(\frac{z\xi}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{z\xi}{|\sin\theta|}\right)^{1/2}\right)d\xi$$

$$=\frac{c_{\alpha,\beta}^{\theta}e^{\frac{i}{2}\left(x^2+y^2+z^2\right)|\cot\theta|}\nabla^{-2\beta}2^{-\beta-\frac{1}{2}}}{(xyz)^{-\beta}\sqrt{\pi}\,\Gamma(\alpha)},$$

where $\nabla(x,y,z)$ denotes the area of a triangle with sides x,y,z of such a triangle exists and zero otherwise. Clearly $\left|D_{\alpha,\beta}^{\theta}(x,y,z)\right|\geq 0$ and is symmetric in x,y,z. Setting $\xi=0,$ as in [8] ,

$$\int_0^{\infty}\left|D_{\alpha,\beta}^{\theta}(x,y,z)\right|z^{\alpha}dz\leq\frac{(xy)^{\alpha}}{2^{(\alpha-\beta)/2}\left|(\sin\theta)^{\alpha}\right|\Gamma\left(\alpha+\frac{1}{2}\right)} \tag{17}$$

for $x,y,z\in\mathbb{R}_+$.

Applying inverse fractional generalized Hankel-Clifford wavelet transformation of $D_{\alpha,\beta}^{\theta}(x,y,z)$ we obtain

$$\overline{c_{\alpha,\beta}^{\theta}}\int_0^{\infty}e^{-\frac{i}{2}\left(z^2+\xi^2\right)|\cot\theta|}\left(\frac{z\xi}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{z\xi}{|\sin\theta|}\right)^{1/2}\right)D_{\alpha,\beta}^{\theta}(x,y,z)\,dz$$

$$=e^{\frac{i}{2}\left(x^2+y^2-\xi^2\right)|\cot\theta|}\xi^{-\alpha}\left(\frac{x\xi}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{x\xi}{|\sin\theta|}\right)^{1/2}\right)\left(\frac{y\xi}{|\sin\theta|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{y\xi}{|\sin\theta|}\right)^{1/2}\right).$$

Lemma 4.1 If $\psi\in L^2(\mathbb{R}_+)$, then $\left\|\psi_{b,a}^{\theta}\right\|_{L^2}\leq\|\psi\|_{L^2}$.

Proof: We have

$$\psi_{b,a}^{\theta}(x)=\frac{1}{\sqrt{a}}e^{-\frac{i}{2}\left(\frac{b^2}{a^2}+\frac{x^2}{a^2}\right)\cot\theta}\int_0^{\infty}D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\psi(z)\,dz.$$

Now,

$$\left|\psi_{b,a}^{\theta}(x)\right| \le \frac{1}{\sqrt{a}}\int_0^\infty \left|\psi(z)\right| z^{-\alpha/2}\left|\left\{D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\right\}^{1/2}\right| z^{\alpha/2}\times\left|\left\{D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\right\}^{1/2}\right| dz.$$

$$\le \frac{1}{\sqrt{a}}\left(\int_0^\infty z^{-\alpha}\left|\psi(z)\right|^2\left|\left\{D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\right\}\right| dz\right)^{1/2}\left(\int_0^\infty \left|\left\{D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\right\}^{1/2}\right| z^{\alpha} dz\right)^{1/2}$$

$$\le \frac{1}{\sqrt{a}}\left(\frac{(bx)^{\alpha}}{a^{2\alpha}\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\right)^{1/2}\left(\int_0^\infty z^{-\alpha}\left|\psi(z)\right|^2\left|\left\{D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\right\}\right| dz\right)^{1/2}.$$

Therefore

$$\int_0^\infty \left|\psi_{b,a}^{\theta}(x)\right|^2 dx \le \left(\frac{(b)^{\alpha}}{a^{2\alpha+1}\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\right)\int_0^\infty z^{-\alpha}\left|\psi(z)\right|^2 dz\int_0^\infty \left|D_{\alpha,\beta}^{\theta}(b/a,x/a,z)\right| x^{\alpha} dx.$$

Thus

$$\int_0^\infty \left|\psi_{b,a}^{\theta}(x)\right|^2 dx \le \frac{a^{-2\alpha}b^{2\alpha}}{\left(\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)\right)^2}\int_0^\infty \left|\psi(z)\right|^2 dz$$

$$\left\|\psi_{b,a}^{\theta}(x)\right\|_{L^2} \le \frac{a^{-\alpha}b^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left\|\psi\right\|_{L^2}.$$

Theorem 4.1 Let $f,\psi \in L^2(\mathbb{R}_+)$. Then the continuous fractional generalized Hankel-Clifford wavelet $H_{\alpha,\beta,\psi}^{\theta}$ is defined on f by

$$\left(H_{\alpha,\beta,\psi}^{\theta} f\right)(b,a) = \overline{c_{\alpha,\beta}^{\theta}}\sin\theta a^{-(\alpha-\beta)/2}\int_0^\infty x^{-\alpha}\left(h_{\alpha,\beta}^{\theta} e^{-\frac{i}{2}(\bullet)^2\cot\theta} f(x)\right)$$
$$\times e^{-(ax)^2\left|\cot\theta\right|\frac{i}{2}\left(\frac{1}{a^2}-1\right)}\left(\frac{bx}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{bx}{\left|\sin\theta\right|}\right)^{1/2}\right)\left(\overline{h_{\alpha,\beta}^{\theta}\psi}\right)(ax)\,dx.$$

Proof: We have

$$\left(H_{\alpha,\beta,\psi}^{\theta} f\right)(b,a) = \left\langle f,\psi_{b,a}^{\theta}\right\rangle$$
$$= \int_0^\infty f(t)\overline{\psi_{b,a}^{\theta}(t)}\,dt$$
$$= \int_0^\infty f(t)\overline{\left(\frac{1}{\sqrt{a}} e^{-\frac{i}{2}\left(\frac{b^2}{a^2}+\frac{t^2}{a^2}\right)\cot\theta}\int_0^\infty D_{\alpha,\beta}^{\theta}(b/a,t/a,z)\psi(z)\,dz\right)}dt.$$

$$= \frac{1}{\sqrt{a} c_{\alpha,\beta}^{\theta}} \int_0^\infty c_{\alpha,\beta}^{\theta} e^{|\cot\theta| \frac{i}{2}\left(t^2 + \frac{\xi^2}{a^2}\right)} \left(\frac{t}{|\sin\theta|}\frac{\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{t}{|\sin\theta|}\frac{\xi}{a}\right)^{1/2}\right) e^{-i\left(\frac{t^2}{2}\right)\cot\theta} f(t)\,dt$$

$$\times e^{-\xi^2|\cot\theta| \frac{i}{2}\left(\frac{1}{a^2}-1\right)} \xi^{-\alpha} \left(\frac{b}{|\sin\theta|}\frac{\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{b}{|\sin\theta|}\frac{\xi}{a}\right)^{1/2}\right) \left(\overline{h_{\alpha,\beta}^{\theta} \psi(z)}\right)(\xi)\,d\xi$$

$$= \frac{1}{c_{\alpha,\beta}^{\theta}\sqrt{a}} \int_0^\infty \xi^{-\alpha} \left(h_{\alpha,\beta}^{\theta} e^{-\frac{i}{2}(\bullet)^2 \cot\theta} f\right)$$

$$\times e^{-\xi^2|\cot\theta| \frac{i}{2}\left(\frac{1}{a^2}-1\right)} \left(\frac{b}{|\sin\theta|}\frac{\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{b}{|\sin\theta|}\frac{\xi}{a}\right)^{1/2}\right) \left(\overline{h_{\alpha,\beta}^{\theta} \psi(z)}\right)(\xi)\,d\xi$$

Substituting $\frac{\xi}{a} = x,$ then the continuous fractional generalized Hankel-Clifford wavelet can be written as

$$\left(H_{\alpha,\beta,\psi}^{\theta} f\right)(b,a) = \overline{c_{\alpha,\beta}^{\theta}} \sin\theta a^{-(\alpha-\beta)/2} \int_0^\infty x^{-\alpha} \left(h_{\alpha,\beta}^{\theta} e^{-\frac{i}{2}(\bullet)^2 \cot\theta} f(x)\right)$$

$$\times e^{-(ax)^2|\cot\theta| \frac{i}{2}\left(\frac{1}{a^2}-1\right)} \left(\frac{bx}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{bx}{|\sin\theta|}\right)^{1/2}\right) \left(\overline{h_{\alpha,\beta}^{\theta} \psi}\right)(ax)\,dx.$$

$$h_{\alpha,\beta}^{\theta}\left(e^{-\frac{ib^2}{2}\cot\theta} \left(H_{\alpha,\beta,\psi}^{\theta} f\right)(b,a)\right) = \sin\theta a^{-(\alpha-\beta)/2} \left(x^{-\alpha} e^{\frac{i}{2}(ax)^2|\cot\theta|} h_{\alpha,\beta}^{\theta} e^{-\frac{i}{2}(\bullet)^2\cot\theta} f(x)\right) \left(\overline{h_{\alpha,\beta}^{\theta} \psi}\right)(ax).$$

Theorem 4.2 If ψ_1 and ψ_2 are two wavelets and $\left(H_{\alpha,\beta,\psi_1}^{\theta} f\right)(b,a)$ and $\left(H_{\alpha,\beta,\psi_2}^{\theta} g\right)(b,a)$ denote the continuous fractional generalized Hankel-Clifford wavelet transformations of $f, g \in L^2(\mathbb{R}_+)$ respectively, then

$$\int_0^\infty \int_0^\infty \left(H_{\alpha,\beta,\psi_1}^{\theta} f\right)(b,a) \left(H_{\alpha,\beta,\psi_2}^{\theta} g\right)(b,a) \frac{dbda}{a^2} = \sin^2\theta C_{\alpha,\beta,\psi_1,\psi_2,\theta} \langle f, g\rangle, \tag{17}$$

where

$$C_{\alpha,\beta,\psi_1,\psi_2,\theta} = \int_0^\infty a^{-(\alpha-\beta)-2} \left(\overline{h_{\alpha,\beta}^{\theta} \psi_1}\right)(a) \left(\overline{h_{\alpha,\beta}^{\theta} \psi_2}\right)(a)\,da < \infty.$$

Proof: We have

$$\left(H_{\alpha,\beta,\psi_1}^{\theta} f\right)(b,a) = \overline{c_{\alpha,\beta}^{\theta}} \sin\theta a^{-(\alpha-\beta)/2} \int_0^\infty x^{-\alpha} \left(h_{\alpha,\beta}^{\theta} e^{-\frac{i}{2}(\bullet)^2 \cot\theta} f(x)\right)$$

$$\times e^{-(ax)^2|\cot\theta| \frac{i}{2}\left(\frac{1}{a^2}-1\right)} \left(\frac{bx}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{bx}{|\sin\theta|}\right)^{1/2}\right) \left(\overline{h_{\alpha,\beta}^{\theta} \psi_1}\right)(ax)\,dx.$$

Now

$$\int_0^\infty\int_0^\infty \left(H^\theta_{\alpha,\beta,\psi_1} f\right)(b,a)\overline{\left(H^\theta_{\alpha,\beta,\psi_2} g\right)}(b,a)\frac{dbda}{a^2}$$

$$=\int_0^\infty\int_0^\infty \left[\begin{array}{l} \overline{c^\theta_{\alpha,\beta}}\sin\theta a^{-(\alpha-\beta)/2}\int_0^\infty x^{-\alpha}\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} f(x)\right) \\ \times e^{-(ax)^2|\cot\theta|\frac{i}{2}\left(\frac{1}{a^2}-1\right)}\left(\frac{bx}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{bx}{|\sin\theta|}\right)^{1/2}\right)\left(\overline{h^\theta_{\alpha,\beta}\psi_1}\right)(ax)\,dx \end{array}\right]$$

$$\times\left[\begin{array}{l} \overline{\overline{c^\theta_{\alpha,\beta}}\sin\theta a^{-(\alpha-\beta)/2}\int_0^\infty y^{-\alpha}\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} g(y)\right)} \\ \times e^{-(ay)^2|\cot\theta|\frac{i}{2}\left(\frac{1}{a^2}-1\right)}\left(\frac{by}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{by}{|\sin\theta|}\right)^{1/2}\right)\left(\overline{h^\theta_{\alpha,\beta}\psi_2}\right)(ay)\,dy \end{array}\right] db\frac{da}{a^2}$$

$$=\int_0^\infty\int_0^\infty a^{-(\alpha-\beta)-2}x^{-\alpha}\sin^3\theta e^{-\cot\theta\frac{i}{2}\left(2-a^2\right)x^2}\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} f(x)\right)\left(\overline{h^\theta_{\alpha,\beta}\psi_1}\right)(ax)$$

$$\left\{\begin{array}{l} c^\theta_{\alpha,\beta}\int_0^\infty e^{\cot\theta\frac{i}{2}\left(b^2+x^2\right)}\left(\frac{bx}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{bx}{|\sin\theta|}\right)^{1/2}\right) \\ \times\left(\begin{array}{l} \overline{c^\theta_{\alpha,\beta}}\int_0^\infty e^{-\frac{i}{2}\left(b^2+y^2\right)\cot\theta}\left(\frac{by}{|\sin\theta|}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\left(\frac{by}{|\sin\theta|}\right)^{1/2}\right) \\ \times e^{\frac{i}{2}\left(2-a^2\right)y^2\cot\theta}\, y^{-\alpha}\left(\overline{h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} g}\right)(y)\left(h^\theta_{\alpha,\beta}\psi_2\right)(ay)\,dy \end{array}\right) db \end{array}\right\} dxda$$

$$=\sin^3\theta\int_0^\infty\int_0^\infty a^{-(\alpha-\beta)-2}x^{-\alpha}\left(\overline{h^\theta_{\alpha,\beta}\psi_1}\right)(ax)\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} f\right)(x)\overline{\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} g\right)}(x)\left(h^\theta_{\alpha,\beta}\psi_2\right)(ax)\,dxda$$

$$=\sin^3\theta\int_0^\infty\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} f\right)(x)\overline{\left(h^\theta_{\alpha,\beta}e^{-\frac{i}{2}(\bullet)^2\cot\theta} g\right)}(x)\left(\int_0^\infty (ax)^{-(\alpha-\beta)-2}\left(\overline{h^\theta_{\alpha,\beta}\psi_1}\right)(ax)\left(h^\theta_{\alpha,\beta}\psi_2\right)(ax)\,x^{2-\beta}da\right)dx$$

$$=\sin^3\theta C_{\alpha,\beta,\psi_1,\psi_2,\theta}\left\langle h^\theta_{\alpha,\beta}f, h^\theta_{\alpha,\beta}g\right\rangle$$

$$=\sin^2\theta C_{\alpha,\beta,\psi_1,\psi_2,\theta}\left\langle f, g\right\rangle.$$

Remark 4.1 If $f=g$ and $\psi_1=\psi_2=\psi$ in the theorem 4.2, then

$$\int_0^\infty\int_0^\infty\left|\left(H^\theta_{\alpha,\beta,\psi} f\right)(b,a)\right|^2\frac{dbda}{a^2}=\sin^2\theta C_{\alpha,\beta,\psi,\theta}\left\langle f, g\right\rangle.$$

Theorem 4.3 If $f,\psi\in L^2(\mathbb{R}_+)$ and $\left(H^\theta_{\alpha,\beta,\psi} f\right)(b,a)$ is continuous fractional generalized Hankel-Clifford wavelet transformation, then

(i). $\left(H^\theta_{\alpha,\beta,\psi} f\right)(b,a)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+$, (ii). $\left\|\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)\right\|_{L^{\infty}} \leq \dfrac{a^{-\alpha}b^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\|f\|_{L^2}\|\psi\|_{L^2}.$

Proof (i) Let (b_0, a_0) be an arbitrary but fixed point in $\mathbb{R}_+ \times \mathbb{R}_+$. Then by Hölder inequality ,

$$\left|\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)-\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b_0,a_0)\right|$$
$$\leq \frac{1}{\sqrt{a}}\int_0^{\infty}\int_0^{\infty}\left| f(x)\psi(z)\left[D^{\theta}_{\alpha,\beta}(b/a,x/a,z)-D^{\theta}_{\alpha,\beta}(b_0/a_0,x/a_0,z)\right]\right|dxdz$$
$$=\frac{1}{\sqrt{a}}\left(\int_0^{\infty}x^{-\alpha}\left|f(x)\right|^2dx\int_0^{\infty}z^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a,x/a,z)-D^{\theta}_{\alpha,\beta}(b_0/a_0,x/a_0,z)\right|dz\right)^{1/2}$$
$$\times\left(\int_0^{\infty}z^{-\alpha}\left|\psi(z)\right|^2dz\int_0^{\infty}x^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a,x/a,z)-D^{\theta}_{\alpha,\beta}(b_0/a_0,x/a_0,z)\right|dx\right)^{1/2}.$$

Now it is known that

$$\int_0^{\infty}z^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a,x/a,z)-D^{\theta}_{\alpha,\beta}(b_0/a_0,x/a_0,z)\right|dz\leq\frac{z^{\alpha}}{\left|(\sin\theta)^{\alpha}\right|2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left[\frac{b^{\alpha}}{a^{2\alpha}}-\frac{b_0^{\alpha}}{a_0^{2\alpha}}\right],$$

and also

$$\int_0^{\infty}x^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a,x/a,z)-D^{\theta}_{\alpha,\beta}(b_0/a_0,x/a_0,z)\right|dx\leq\frac{x^{\alpha}}{\left|(\sin\theta)^{\alpha}\right|2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left[ab^{\alpha}-a_0b_0^{\alpha}\right]$$

By dominated convergence theorem and the continuity of $D^{\theta}_{\alpha,\beta}(b/a,x/a,z)$ in the variable b and a,

$$\lim_{b\to b_0}\lim_{a\to a_0}\left|\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)-\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b_0,a_0)\right|=0.$$

Thus the theorem is proved.

(ii) As $\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)=\int_0^{\infty}f(t)\left(\frac{1}{\sqrt{a}}e^{-\frac{i}{2}\left(\frac{b^2}{a^2}+\frac{t^2}{a^2}\right)\cot\theta}\int_0^{\infty}D^{\theta}_{\alpha,\beta}(b/a,t/a,z)\psi(z)dz\right)dt,$ by Hölder inequality

$$\left|\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)\right|$$

$$\leq \frac{1}{\sqrt{a}}\left(\int_0^\infty x^{-\alpha}\left|f(x)\right|^2 dx \int_0^\infty z^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a, x/a, z)\right| dz\right)^{1/2}$$

$$\times\left(\int_0^\infty z^{-\alpha}\left|\psi(z)\right|^2 dx \int_0^\infty x^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a, x/a, z)\right| dx\right)^{1/2}$$

$$\leq \frac{1}{\sqrt{a}}\left(\int_0^\infty x^{-\alpha}\left|f(x)\right|^2 dx \frac{x^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left[\frac{b^{\alpha}}{a^{2\alpha}}\right]\right)^{1/2}$$

$$\times\left(\int_0^\infty z^{-\alpha}\left|\psi(z)\right|^2 dz \frac{z^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left[ab^{\alpha}\right]\right)^{1/2}.$$

Since $\displaystyle\int_0^\infty z^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a, x/a, z)\right| dz \leq \frac{z^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left[\frac{b^{\alpha}}{a^{2\alpha}}\right]$,

and also

$$\int_0^\infty x^{\alpha}\left|D^{\theta}_{\alpha,\beta}(b/a, x/a, z)\right| dx \leq \frac{x^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left[ab^{\alpha}\right].$$

Thus

$$\left|\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)\right| \leq \frac{b^{\alpha}a^{-\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(\int_0^\infty \left|f(x)\right|^2 dx\right)^{1/2}\left(\int_0^\infty \left|\psi(z)\right|^2 dz\right)^{1/2}.$$

Thus

$$\left\|\left(H^{\theta}_{\alpha,\beta,\psi} f\right)(b,a)\right\|_{L^\infty} \leq \frac{a^{-\alpha}b^{\alpha}}{\left|(\sin\theta)^{\alpha}\right| 2^{(\alpha-\beta)/2}\Gamma\left(\alpha+\frac{1}{2}\right)}\left\|f\right\|_{L^2}\left\|\psi\right\|_{L^2}.$$

Hence the proof.

Acknowledgement: The study was supported by a grant from National Board of Higher Mathematics Ref No. 2/48(11) 2013/ NBHM/ (R.P.) / R& D II/979

REFERENCES

[1] Akhilesh P., et all., *Boundary Value Problems*, **2013**(1), 40, 2013.
[2] Pathak, R. S., Dixit, M. M., *J. Comput. Appl. Math*., **160**(1-2), 241, 2003.
[3] Upadhyay, S. K., et all., *Integral Transforms Spec. Funct.*, **23**(5), 315, 2012.
[4] Namias, V., *J. Inst. Math. Appl*., **26**, 187, 1980.
[5] Sheppard, C. J. R., Larkin, K.G., *Opt. Commun.*, **154**, 173, 1998.
[6] Zhang, Y., et all., *Opt. Commun.*, **176**, 71, 2000.
[7] Kerr, F. H., *J. Math. Anal. Appl.*, **166**, 65, 1992.
[8] Debnath, L., *Wavelet Transforms and Their Applications*, Birkhäuser, Boston, 2002.
[9] Shi, J., et all., *Sci. China Inf. Sci*., **55**(6), 1270, 2012.
[10] Lakshmi Gorty V. R., *Int.J.Comp.Sci.Engin.(IJCSE)*, **1**(4), 1, 2013.
[11] Malgonde S. P., Bandewar, S. R., *Proc. IndianAcad. Sci*., **110**(3), 293, 2000.
[12] Prasad, A., Singh, V. K., *Ann. Univ. Ferrara*, **59**(1), 141, 2012.
[13] Zemanian, AH., *Generalized Integral Transformations*, Interscience, New York, 1968.