ORIGINAL PAPER

GAME CHROMATIC NUMBER OF $W_n \square P_2$

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Abstract. *In this paper we find an exact value for the game chromatic number of the Cartesian product graph $W_n \square P_2$ of two graphs, n-wheel W_n and the path graph P_2. This extends a previous work of Sia on the game chromatic number of certain families of Cartesian product graphs. We prove that the game chromatic number of graph $W_n \square P_2$ is 5, if $n \geq 3$.*

Keywords: *Game chromatic number; wheel graph; Cartesian product of graphs.*
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1. INTRODUCTION

In this study we consider a graph coloring game invented by Bodlaender and Brams independently (see [2, 3]). Let $G = (V, E)$ be a finite simple graph and X be a set of colors. The game chromatic number of G is defined via a two-person finite game. Two players, generally called Alice and Bob, with Alice going first, alternatively color the uncolored vertices of G with a color from a color set X, such that no two adjacent vertices have the same color. Bob wins if at any stage of the game before the G is completely colored, one of the players has no legal move; otherwise, that is, if all the vertices of G are colored properly, Alice wins. Certainly, the result of the game depends on the graph G and the number of colors in X. It is obvious that if $|X| \geq |V|$, then Alice can win, but if $|X| < \chi(G)$, then Bob always wins, where $\chi(G)$ is the chromatic number of G. The *game chromatic number* $\chi_g(G)$ of G is the least number of colors in the color set X for which Alice has a winning strategy. It is clear that the game chromatic number $\chi_g(G)$ satisfies

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1, \tag{1}$$

where $\Delta(G)$ is the largest vertex degree in G.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with a vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. Here, $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of graph G respectively. Although

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\},$$

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there is no similar equality for the game chromatic number of the Cartesian product of arbitrary graphs (see [1]). If $v \in V(H)$, then the subgraph G_v of $G\square H$ obtained by $\{(u,v) : u \in V(G)\}$ is called a G-*fiber*. H-fiber is defined similarly.

The game coloring number of a graph G is defined by modifying the rules of the coloring game as follows. The players fix a positive integer k, and instead of coloring vertices, only mark an unmarked vertex each turn. Bob wins if at some stage, some unmarked vertex has k marked neighbors, while Alice wins if this never happens. The *game coloring number* of a graph G is defined as the least number k for which Alice has a winning strategy on graph G, and it is denoted by the symbol $\mathrm{col}_g(G)$.

The following results provide a useful upper bound for the game chromatic number of the Cartesian product of the graphs in view of the game coloring number.

Proposition 1.1 ([4]) Suppose that $G = (V,E)$ is a graph with $E = E_1 \cup E_2$. Let $G_1 = (V,E_1)$ and $G_2 = (V,E_2)$. Then

$$\chi_g(G) \le \mathrm{col}_g(G) \le \mathrm{col}_g(G_1) + \Delta(G_2)$$

holds.

Corollary 1.2 ([7]) For any Cartesian product graph $G\square H$, we have

$$\chi_g(G\square H) \le \mathrm{col}_g(G\square H) \le \mathrm{col}_g\left(\bigcup_{v\in V(H)} G_v\right) + \Delta(H).$$

Here, G_v denotes the G-fiber in $G\square H$.

In [8], Zhu proposes another useful upper bound for the game chromatic number of the Cartesian product of the graphs by means of acyclic chromatic number of a graph.

2. EXACT VALUE FOR $\chi_g(W_n\square P_2)$

In this section we give an exact value for the game chromatic number of the graph $W_n\square P_2$, which is the Cartesian product of the two graphs n-wheel W_n and path graph P_2. Here, *wheel graph* W_n is a graph with $n+1$ vertices ($n \ge 3$), formed by connecting a single vertex v_0, called a *center*, to all vertices of an n-cycle, called a *rim* of the wheel.

Using Corollary 1 and Inequality (1) we immediately get the following result.

Proposition 2.1 If $n \ge 3$ then $3 \le \chi_g(W_n\square P_2) \le 5$.

Proof. It is not difficult to see that for $n \ge 3$,

$$\chi(W_n\square P_2) = \begin{cases} 3 & , \quad if \;\; n \;\; is \;\; even \\ 4 & , \quad if \;\; n \;\; is \;\; odd \end{cases}$$

Hence, using inequality (1) we find for any integer $n \ge 3$,

$$3 \le \chi_g(W_n\square P_2).$$

In Corollary 1, for $n \geq 3$, if we set $G = W_n$ and $H = P_2$, then it is clear that $\Delta(P_2) = 1$. Furthermore, since $\mathrm{col}_g(W_n) = 4$, then we get

$$\chi_g(W_n \square P_2) \leq \mathrm{col}_g(W_n \square P_2) \leq 5. \quad (2)$$

This completes the proof.

Indeed, Alice clearly wins with five or more colors. The degree of vertices of the Cartesian product graph $W_n \square P_2$ is 4, except for the centers of the wheel-fibers. The centers of the wheel-fibers are of the degree $n + 1$, so Alice wins if she colors these centers on her first moves.

In [7], the exact value for the game chromatic number of the Cartesian product graph $W_n \square P_2$ is presented as follows:

Proposition 2.2 ([7]) For any integer $n \geq 9$, $\chi_g(W_n \square P_2) = 5$.

We improve the result given in [7], by Theorem 2.3

Theorem 2.3 For $n \geq 3$, we have $\chi_g(W_n \square P_2) = 5$.

Proof. Using Proposition 2, for $n \geq 3$ we have $\chi_g(W_n \square P_2) \leq 5$. Therefore, if we describe a winning strategy for Bob with four colors, we complete the proof.

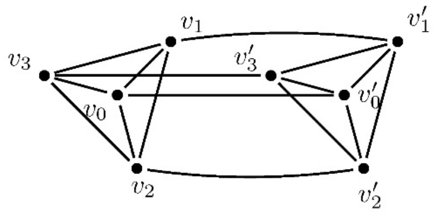

Figure 1. Cartesian product graph $W_3 \square P_2$.

Let $n = 3$ (see Fig. 1). Since $W_3 \cong K_4$, no matter which vertex Alice colors on her first move. Without loss of generality, assume that Alice colors the center of any wheel-fiber with any color, say v_0 with color 1. Then Bob simply responds by coloring v_1' with color 2. Then all possible cases for Alice's second move and Bob's response are as indicated in Table 1. When Bob responds to Alice's move, he gets two critical vertices, which she cannot defend at the same time, or a vertex requiring a fifth color, and he wins the game.

Table 1: All possible cases for Alice's second move and Bob's responses for $n = 3$.

Alice's 2nd Move	Bob's Responses	Critical Vertices
v_3', color 1	v_2, color 3	v_1, v_2'
v_3', color 3	v_2', color 4	v_0'
v_2', color 1	v_3, color 3	v_1, v_3'
v_2', color 3	v_3', color 4	v_0'
v_0', color 3	v_3, color 4	v_1, v_3'
v_1, color 3	v_2', color 4	v_0', v_2
v_2, color 2	v_3', color 3	v_0', v_3
v_2, color 3	v_3, color 4	v_1
v_3, color 2	v_2', color 3	v_2, v_0'
v_3, color 3	v_2, color 4	v_1

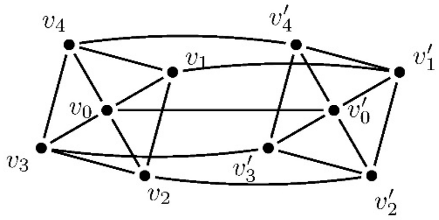

Figure 2. Cartesian product graph $W_4 \square P_2$.

Now, let $n = 4$ (see Fig. 2). Then again there are two cases for Alice's first move:

Case 1. Alice can color a vertex on the rim of the wheel-fibers. Without loss of generality, assume that Alice colors v_1 with color 1. Then Bob colors a vertex on the rim of an uncolored wheel-fiber, say v_3', with color 1. No matter whose turn it is, the remaining uncolored vertices of the graph cannot be colored with color 1. Therefore, Alice has to use a new color. Note that, if Alice colors v_2 with color 1 instead of v_1 then Bob colors v_4' with color 1, or if she colors v_3 or v_4 with color 1 then Bob colors v_1' or v_2' with color 1 respectively. Therefore, in any case, on her second move, she has to use a new color. No matter which vertex Alice colors with her second move, if Bob colors a vertex according to the possible cases given below, Bob wins the game.

Case 1.1. Alice, on her second move, colors v_2 with color 2. Then Bob colors v_4' with color 2. In this case, the remaining uncolored vertices of the graph cannot be colored by color 2. All possible alternatives for Alice's third move and Bob's responses to Alice are listed in Table 2. In all of the alternatives, Bob wins.

Table 2: Case 1.1: All possible cases for Alice's third move and Bob's responses for $n = 4$.

Alice's 3rd Move	Bob's Responses	Critical Vertex
v_0, color 3	v_4, color 4	v_3
v_3, color 3	v_4, color 4	v_0
v_4, color 3	v_3, color 4	v_0

Case 1.2. Alice, on her second turn, colors v_3 with color 2. Bob then plays color 2 in v_1'. In this case, Alice, on her third move, has to use a third color. In Table 3, a winning strategy for Bob is given for all of Alice's possible moves.

Table 3: Case 1.2: All possible cases for Alice's third move and Bob's responses for $n = 4$.

Alice's 3rd Move	Bob's Responses	Critical Vertex
v_0, color 3	v_2', color 4	v_2
v_2, color 3	v_4, color 4	v_0
v_4, color 3	v_2, color 4	v_0

Case 1.3. If Alice, on her second move, colors v_4 with color 2, then Bob colors v_2' with color 2. Then Alice, on her next move, must use color 3. All possible alternatives for Alice's third move and Bob's responses to Alice are presented in Table 4. According to Table 4, Bob wins.

Table 4: Case 1.3: All possible cases for Alice's third move and Bob's responses for $n = 4$.

Alice's 3rd Move	Bob's Responses	Critical Vertex
v_0, color 3	v_1', color 4	v_0'
v_2, color 3	v_3, color 4	v_0

v_3, color 3	v_2, color 4	v_0

Case 1.4. Alice, on her second move, can color v_0 with color 2. Then Bob colors v_1' with color 2. Alice, again on her next move, must use color 3. All possible moves for Alice's third move and Bob's responses to Alice are given in Table 5. In all situations, Bob wins.

Table 5: Case 1.4: All possible cases for Alice's third move and Bob's responses for $n = 4$.

Alice's 3rd Move	Bob's Responses	Critical Vertex
v_2, color 3	v_4, color 4	v_3
v_3, color 3	v_4', color 4	v_4
v_4, color 3	v_2, color 4	v_3

All possible sequences of moves in Case 1. are given in Fig. 3.

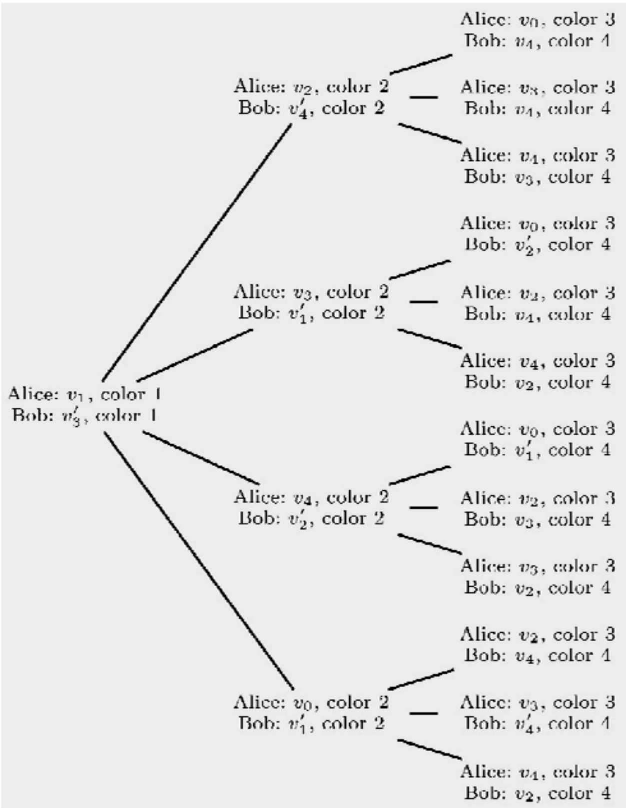

Figure 3. For $n = 4$ all possible sequences of moves in Case 1.

Case 2. Alice can color the center of the one of the wheel-fibers with any color. Without loss of generality, assume that Alice colors v_0 with color 1. Then Bob colors v_0' with color 2. Now, Alice, on her second move, can color the vertices on the rim of the wheel-fiber centered with v_0, with colors 2, 3 or 4. Similarly, Alice, on her second move, can color the vertices on the rim of the other wheel-fiber with colors 1, 3 or 4. If we label the new unused color as color 3, it is not necessary to study possible moves for color 4. Without loss of generality, suppose that Alice colors the vertices on the rim of the wheel-fiber centered with v_0. Therefore, we only need to consider the following cases:

Case 2.1. Alice can color any vertex on the rim of the wheel-fiber, centered with v_0, with color 2, say v_4. Then Bob colors v_4' with color 1 (if Alice colors v_i with color 2, then Bob colors v_i' with color 1 for $i = 1,2,3,4$). Now there are only four possible cases for Alice's third move. All these moves and Bob's responses to Alice are listed in Table 6.

Table 6: Case 2.1: All possible cases for Alice's third move and Bob's responses for $n = 4$.

Alice's 3rd Move	Bob's Responses	Critical Vertex
v_1, color 3	v_2', color 4	v_1'
v_2, color 2	v_2', color 3	v_1', v_3'
v_2, color 3	v_3', color 4	v_3
v_3, color 3	v_2', color 4	v_3'

Case 2.2. Alice, on her second move, can color a vertex on the rim of the wheel-fiber, centered with v_0, with color 3. In this case, possible options are as indicated in Table 7.

Table 7: Case 2.2: All possible cases for Alice's second move and Bob's responses for $n = 4$.

Alice's 2nd Move	Bob's Responses	Critical Vertex
v_1, color 3	v_4', color 4	v_1', v_4
v_2, color 3	v_3', color 4	v_3, v_2'
v_3, color 3	v_2', color 4	v_2, v_3'
v_4, color 3	v_1', color 4	v_1, v_4'

All possible sequences of moves in Case 2 are given in Fig. 4.

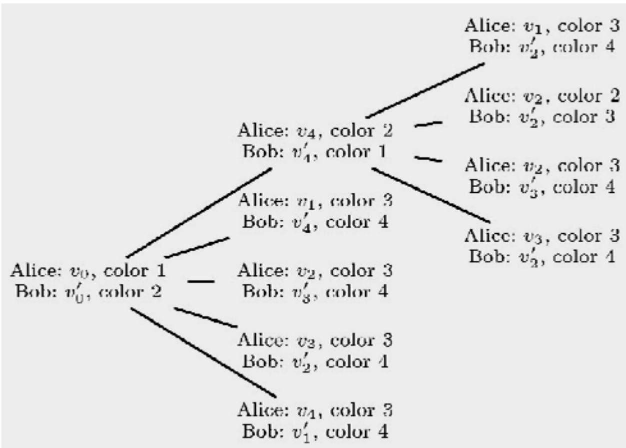

Figure 4. For $n = 4$ all possible sequences of moves in Case 2.

Hence, if Alice colors the center of any wheel-fiber, then Bob colors an uncolored center with a different color and wins the game by 4 colors.

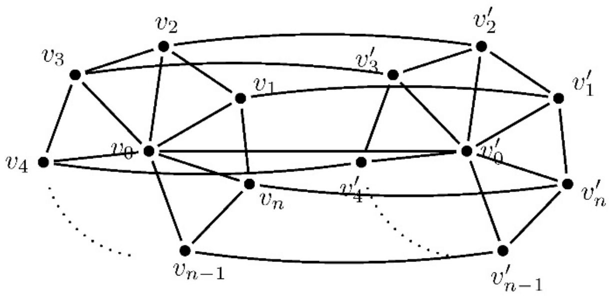

Figure 5. Cartesian product graph $W_n \square P_2$, $n \geq 5$.

Finally, let $n \geq 5$ (see Fig. 5). In this case, Bob has a winning strategy again. Bob's strategy is as follows:

Alice, on her first move, has to choose one of two alternatives:

Case 1. Alice can color the center of any wheel-fiber with any color, say v_0 with color 1. Then Bob colors v_1' with color 2. Now Alice, on her second move, can color a vertex from one of the two wheel-fibers:

Case 1.1. If Alice colors a vertex from the wheel-fiber, centered with v_0', then there are two alternatives:

Case 1.1.1. If Alice, on her second move, colors v_0' with color 3, then Bob colors v_2 with color 4. Since Alice cannot defend both v_1 and v_2' at the same time, Bob wins.

Case 1.1.2. If Alice, on her second move, colors any vertex, different from v_0', she can choose one of three alternatives:

Case 1.1.2.1. Alice can color any vertex, but v_0', on the wheel-fiber centered with v_0' with color 2. Then Bob, on his second move, colors v_2' with color 3. In this case, Alice has to defend v_0', that is, she has to color v_0' with color 4. Now if Bob colors a proper vertex from another wheel-fiber, he gets two critical vertices, which Alice cannot defend at the same time. Bob wins.

Case 1.1.2.2. Alice can color any vertex, except v_0', on the wheel-fiber centered with v_0' with color 1, say v_2'. Then Bob, on his second move, colors a vertex with a distance 2 from v_2', and on the same rim with v_2', with color 3, say v_4'. Now Alice, on her third move must defend vertex v_0', i.e., she colors v_0' with color 4. Then Bob colors v_3 with color 2. Since one more color is necessary for the vertex v_3', Bob wins.

Case 1.1.2.3. Alice can color any vertex, except v_0', on the wheel-fiber centered with v_0' with color 3. Then Bob colors any vertex, except v_0', from the wheel-fiber centered with v_0' with color 4. In this case, one more color is necessary for the vertex v_0', and Bob wins.

Case 1.2. Alice, on her second move, can color any vertex on the rim of the wheel-fiber centered with v_0. Then Bob colors the neighbor of the vertex, which is a neighbor of the last two colored vertices by Alice and is on the wheel-fiber centered at v_0', with an unused color. In this case, Bob gets two vertices such that Alice cannot defend both at the same time. Bob wins.

Therefore, when Alice colors the center vertex of any wheel-fiber with any color, Bob always wins by coloring a vertex with a different color on the rim of the uncolored wheel-fiber.

Case 2. Alice can color a vertex on the rim of any wheel-fiber. Then Bob colors the center of the uncolored wheel-fibers with an unused color. Then it is Alice's turn. If Bob uses the strategy as in Case 1, he wins the game.

Therefore, if $n \geq 3$, Bob wins the game with four colors. This completes the proof.

3. CONCLUSION

In the literature, a few exact values are known for the game chromatic number of Cartesian product graphs. For example, the game chromatic number of grid graph $P_m \square P_n$ is still not known, except for certain special values of m and n. Similarly, the game chromatic number of the toroidal graph $C_m \square C_n$ is also not known for arbitrary m and n (see [6]).

In [5], it is proved that the game coloring number of a graph G is at most $c + 4$, if every edge of G belongs to at most c different cycles. Since $c = 0$ for trees, using Corollary 1 we get

$$\chi_g(P_2 \square T) \leq 5,$$

$$\chi_g(P_n \square T) \leq 6, \quad (n \geq 3)$$

$$\chi_g(C_n \square T) \leq 6, \quad (n \geq 3)$$

for any tree T.

In this work, even if we calculate the exact value for the game chromatic number of $W_n \square P_2$, it is still an open question as to what is the game chromatic number of $W_n \square P_m$ for any values of n and m.

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