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SOME FRACTIONAL INTEGRAL INEQUALITIES INVOLVING APPELL HYPERGEOMETRIC FUNCTION

SUNIL JOSHI1, EKTA MITTAL1, RUPAKSHI MISHRA PAND EY1

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Abstract. *The aim of the present investigation is to establish certain new fractional integral inequalities involving the Appell hypergeometric function considering the extended Chebyshev functional in the case of synchronous functions. In the conclusions numerous special cases as fractional integral inequalities involving Saigo, Erdelyi-Kober,and Riemann-Liouville type fractional integral operators are presented.Further, we also reflect their significance with other related known results due to Purohit, Raina and Belarbi and Dahmani.*

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1. INTRODUCTION

Fractional integral inequalities have several applications, the most valuable ones are in establishing uniqueness of solutions in fractional boundary value problems and in fractional Partial differential equations, under various assumptions (Chebyshev inequality, Grüss inequality, Hermite- Hadamard inequality, Ostrowski inequality, etc.), inequalities are playing a very significant role in all fields of mathematics, particularly in the theory of approximations [2, 6]. Here we shall use the following definitions with their related details.

Definition 1. Two functions f and g are said to be synchronous on $[a, b]$, if

$$\{(f(x) - f(y))(g(x) - g(y))\} \geq 0, \text{ for any } x, y \in [a, b]. \quad (1)$$

Definition 2. A real-valued function $f(x)$ $(x > 0)$ is said to being the space $\mathbb{C}_\lambda, \lambda \in \mathbb{R}$ if there exists a real number $p > \lambda$ such that $f(x) = x^p \phi(x)$, where $\phi(x) \in \mathbb{C}(0, \infty)$.

[1] Manipal University Jaipur, Department of Mathematics and Statistics, 303007 Rajasthan, India.
E-mail: sunil.joshi@jaipur.manipal.edu.

Definition 3. Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$ *and* $\mathrm{Re}(\gamma)>0$ then a generalized fractional integral,

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f(x)=\frac{x^{-\alpha}}{\Gamma\gamma}\int_0^x (x-t)^{\gamma-1}t^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{t}{x},1-\frac{x}{t}\ )f(t)dt \tag{2}$$

Where the function $F_3(-)$ appearing as a kernel for the operator is the Appell hypergeometric function defined by

$$F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\ \max\left[|x|<1,|y|<1\right] \tag{3}$$

and $(\alpha)_m$ is pochhammer symbol such that,

$$(\alpha)_m=\alpha(\alpha+1)(\alpha+2)...(\alpha+m-1) \qquad (\alpha)_0=1 \tag{4}$$

The purpose of the present investigation is to acquire certain Chebyshev type integral inequalities involving the generalized fractional integral operators [13] which comprises in the kernel, the Appell hypergeometric function .The concluding section gives some special cases of the main results.

Preliminary Lemma

The following lemma are required to establish our main results.

Lemma 1. [7, p. 394, eqn. (4.18)]

Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$

If $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\rho)>$max [0, $\mathrm{Re}(\alpha+\alpha'+\beta-\gamma)$, $\mathrm{Re}(\alpha'-\beta'$], then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\ x^{\rho-1}=x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma\rho\,\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\,\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\,\Gamma(\rho+\gamma-\alpha'-\beta)\ \Gamma(\rho+\beta')} \tag{5}$$

Theorem 1. Let f and g be two synchronous functions on [0, ∞), then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)g(x)\}\geq\frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha-\alpha')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{-\gamma+\alpha+\alpha'}I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f(x)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}g(x)$$

$$\forall\ x>0,\ \beta'>-1,\ 1>\max.\{0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\},(\gamma-\alpha')>\max.(1-\beta\ ,1-\alpha) \tag{6}$$

Proof: Let f and g be two synchronous functions; then using Definition 1, for all τ,

$\rho\in(0,\mathrm{x})$, x ≥ 0, we have $\{(f\ (\tau)-\mathrm{f}\ (\rho))\ (\mathrm{g}\ (\tau)-\mathrm{g}\ (\rho))\}\geq 0$, (7)

Which implies that,

$$(\tau)\ g\ (\tau) + f\ (\rho)\ \mathrm{g}\ (\rho) \geq f\ (\tau)\ \mathrm{g}\ (\rho) + f\ (\rho)\ \mathrm{g}\ (\tau) \tag{8}$$

Now multiply by $\frac{x^{-\alpha}}{\Gamma\gamma}(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\tau}{x},1-\frac{x}{\tau})$ and integrate with respect to τ from 0 to x in equation (8), we get

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(\tau)g(\tau)\} + f(\rho)g(\rho)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}(1) \geq g(\rho)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f(\tau) + f(\rho)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}g(\tau) \tag{9}$$

Again multiply by $\frac{x^{-\alpha}}{\Gamma\gamma}(x-\rho)^{\gamma-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\rho}{x},1-\frac{x}{\rho})$ and integrate with respect to ρ from 0 to x in equation (9), we get

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(\tau)g(\tau)\}I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}(1) + I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(\rho)g(\rho)\}I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}(1) \geq I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}g(\rho)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f(\tau)$$
$$+ I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f(\rho)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}g(\tau) \tag{10}$$

$$\Rightarrow\ 2I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(\tau)g(\tau)\}I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}(1)\ \geq\ 2I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}g(\tau)I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f(\tau) \tag{11}$$

Using the formula (5) (by putting $\rho = 1$) and we obtained required result.

Theorem 2. Let $(f_i)_{i=1,2,\ldots n}$ be n positive increasing functions on $[0,\infty)$, then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left(\prod_{i=1}^{n} f_i(x)\right) \geq \left[\frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\gamma-\alpha'-\alpha)}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{-\gamma+\alpha+\alpha'}\right]^{n-1}\prod_{i=1}^{n} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_i(x)\}$$

$$\forall\, x > 0,\ \beta' > -1,\ 1 > \max.\{0, \mathrm{Re}(\alpha+\alpha'+\beta'-\gamma), \mathrm{Re}(\alpha'-\beta')\}, (\gamma-\alpha') > \max.(1-\beta, 1-\alpha) \tag{12}$$

Proof: This theorem has proved by induction principle by putting $n = 1$ in equation (12).

We have $I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_1(x)\} \geq I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_1(x)\}$

Next, for n = 2 in (12), we get

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_1(x)f_2(x)\} \geq \left[\frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\gamma-\alpha'-\alpha)}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{-\gamma+\alpha+\alpha'}\right]$$
$$\times[I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_1(x)\}I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_2(x\}] \tag{13}$$

By mathematical induction principal, let us suppose it is true for (n-1), then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left(\prod_{i=1}^{n-1} f_i(x)\right) \geq \left[\frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\gamma-\alpha'-\alpha)}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{-\gamma+\alpha+\alpha'}\right]^{n-2}\prod_{i=1}^{n-1} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f_i(x)\} \tag{14}$$

Now, given that $(f_i)_{i=1,2,...n}$ are increasing function which imply that the function $\prod_{i=1}^{n-1} f_i(x)$ is also an increasing function. Therefore, we apply inequality (6) to the function $\prod_{i=1}^{n-1} f_i(x)=$ g(x) and $f_n = f$ to get the desired result.

Theorem 3. Let f and g be two synchronous function on $[0,\infty)$, then

$$\frac{\Gamma(1+c-a-a'-b)\Gamma(1+b'-a')}{\Gamma(1+b')\,\Gamma(1+c-a'-b)\,\Gamma(1+c-a-a')}\; x^{-a-a'+c}\, I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)g(x)\}$$
$$+\frac{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha-\alpha')} I_{0,x}^{a,a',b,b',c}\{f(x)g(x)\} \geq I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f(x) I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} g(x)$$
$$+ I_{0,x}^{a,a',b,b',c} f(x) I_{0,x}^{a,a',b,b',c} g(x) \qquad (15)$$

$for\,all\, x>0, \gamma, c>0, \beta', b'>-1,\ (\beta'-\alpha')>0,\ (1+\gamma)>\max.\{0,(\alpha+\alpha'),(\alpha'+\beta),(\alpha+\alpha'+\beta)\}$, $(b'-a')>0,\ (1+c)>\max.\{0,(a+a'),(a'+b),(a+a'+b)\}$,

Proof: Multiplying equation (9) by $\frac{x^{-a}}{\Gamma c}(x-\rho)^{c-1}\rho^{-a'}F_3(a,a',b,b',c,1-\frac{\rho}{x},1-\frac{x}{\rho})$ and integrate with respect to ρ both side from 0 to x, we have

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(\tau)g(\tau)\} I_{0,x}^{a,a',b,b',c}(1) + I_{0,x}^{a,a',b,b',c}\{f(\rho)g(\rho)\} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}(1)$$
$$\geq I_{0,x}^{a,a',b,b',c} g(\rho) I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f(\tau) + I_{0,x}^{a,a',b,b',c} f(\rho) I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} g(\tau) \qquad (16)$$

Finally, using equation (5) in above equation we will get the desired result.

Remark 1. It may be noted that inequalities (6) and (15) are reversed if the functions are asynchronous on [0, ∞); that is,

$$\{(f(x) - f(y))\,(g(x) - g(y))\} \leq 0, \text{ for any } x, y \in [0, \infty).$$

Remark 2. For $a=\alpha,\ a'=\alpha', b=\beta,\ b'=\beta',\ c=\gamma$, Theorem 4 immediately reduces to theorem 1.

Theorem 4. Let f and g be two functions defined on $[0,\infty)$ such that f is increasing, g is differentiable, and there exists a real number $m=\inf_{x\geq 0} g'(x)$, then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)g(x)\} \geq \frac{\Gamma(1+\beta')\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha'-\alpha)}{\Gamma(1+\gamma-\alpha'-\alpha-\beta)\Gamma(1+\beta'-\alpha')} x^{-\gamma+\alpha+\alpha'} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)\} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{g(x)\}$$
$$-m\frac{(1+\gamma-\alpha-\alpha'-\beta)(1+\beta'-\alpha')x}{(1+\beta')(1+\gamma-\alpha'-\beta)(1+\gamma-\alpha-\alpha')} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)\} + m I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{g(x)\} \quad (17)$$

$for\,all\, x>0, \gamma>0, \beta'>-1,\ (\beta'-\alpha')>0,\ (1+\gamma)>\max.\{0,(\alpha+\alpha'),(\alpha'+\beta),(\alpha+\alpha'+\beta)\}$

Proof: Let s(x) = g(x) – mx, it is clear that it is differentiable and it is increasing on $[0,\infty)$, therefore by theorem (1) [eq. (6)], we have

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}[f(x)\{g(x)-mx\}] \geq \frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha-\alpha')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')} x^{-\gamma+\alpha+\alpha'}$$
$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f(x) I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{g(x)-mx\} \quad (18)$$

$$\geq \frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha-\alpha')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')} x^{-\gamma+\alpha+\alpha'} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f(x) I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} g(x)$$
$$-m\frac{\Gamma(1+\beta')\,\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha-\alpha')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')} x^{-\gamma+\alpha+\alpha'} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f(x) I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} g(x) \quad (19)$$

By using equation (5) and putting $\rho = 2$ we acquire a desired result.

Theorem 5. Let f and g be two functions defined on [0, ∞), such that f is increasing, g is differentiable, and there exists a real number $M = \sup_{t\geq 0} g'(t)$, then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)g(x)\} \geq \frac{\Gamma(1+\beta')\Gamma(1+\gamma-\alpha'-\beta)\,\Gamma(1+\gamma-\alpha'-\alpha)}{\Gamma(1+\gamma-\alpha'-\alpha-\beta)\Gamma(1+\beta'-\alpha')} x^{-\gamma+\alpha+\alpha'} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)\} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{g(x)\}$$
$$-M\frac{(1+\gamma-\alpha-\alpha'-\beta)(1+\beta'-\alpha')x}{(1+\beta')(1+\gamma-\alpha'-\beta)(1+\gamma-\alpha-\alpha')} I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{f(x)\} + M I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\{g(x)\} \quad (20)$$

$for\,all\, x>0, \gamma>0, \beta'>-1,\ (\beta'-\alpha')>0,\ (1+\gamma)>\max.\{0,(\alpha+\alpha'),(\alpha'+\beta),(\alpha+\alpha'+\beta)\}$

Proof: Let s(x) = g(x) – Mx, it is clear that it is differentiable and increasing on $[0,\infty)$.By applying the similar procedure as of in theorem 3, one can easily establish the above theorem.

CONCLUDING REMARKS

We consider some special consequences of the result derive in the previous section. Following Saigo and Maeda [7], the operator (2) would reduce immediately to the extensively investigated Saigo, Erdelyi-Kober and Riemann-Liouville type fractional integral operators respectively given the following relationships [4, 5].

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x) = \frac{x^{-\alpha}}{\Gamma\gamma}\int_0^x (x-t)^{\gamma-1} {}_2F_1(\alpha,\beta,\gamma,1-\frac{t}{x}) f(t)dt \quad (\gamma>0,\alpha,\beta\in R) \tag{21}$$

$$I_{0,x}^{\gamma,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,-\beta} f(x) = \frac{x^{-\gamma+\beta}}{\Gamma\gamma}\int_0^x (x-t)^{\gamma-1} t^{-\beta} f(t)dt \quad (\gamma>0,\beta\in R) \tag{22}$$

$$I_{0,x}^{0,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma} f(x) = \frac{1}{\Gamma\gamma}\int_0^x (x-t)^{\gamma-1} f(t)dt \quad (\gamma>0) \tag{23}$$

Now if we put $\alpha'=0$ and make theorem (1) to (3) and remark (1), (2) in terms of relation (21), the known fractional integral inequalities due to Purohit and Raina [2].
Again, for $\alpha'=0$, theorem 6 and 7 provide, respectively. The following inequalities involving Saigo fractional integral operators.

Corollary 1. Let f and g be two functions defined on $[0,\infty)$ such that f is increasing and g is differentiable, there exists a real number $m=\inf_{x\geq 0} g'(x)$, then

$$\begin{aligned} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{f(x)g(x)\} \geq & \frac{\Gamma(1+\gamma-\alpha)\,\Gamma(1+\gamma-\beta)}{\Gamma(1+\gamma-\alpha-\beta)} x^{-\gamma+\alpha} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{f(x)\} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{g(x)\} \\ & -m\frac{(1+\gamma-\alpha-\beta)x}{(1+\gamma-\beta)(1+\gamma-\alpha)} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{f(x)\} + mI_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{g(x)\} \end{aligned} \tag{24}$$

for all $x>0, \gamma>\max.\{0,\alpha+\beta-1,\alpha-1,\beta-1\}$.

Corollary 2. Let f and g be two functions defined on [0, ∞), such that f is increasing, g is differentiable, and there exists a real number $M = \sup_{t\geq 0} g'(t)$, then

$$\begin{aligned} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{f(x)g(x)\} \geq & \frac{\Gamma(1+\gamma-\alpha)\,\Gamma(1+\gamma-\beta)}{\Gamma(1+\gamma-\alpha-\beta)} x^{-\gamma+\alpha} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{f(x)\} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{g(x)\} \\ & -M\frac{(1+\gamma-\alpha-\beta)x}{(1+\gamma-\beta)(1+\gamma-\alpha)} I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{f(x)\} + MI_{0,x}^{\gamma,\alpha-\gamma,-\beta}\{g(x)\} \end{aligned} \tag{25}$$

for all $x>0, \gamma>\max.\{0,\alpha+\beta-1,\alpha-1,\beta-1\}$.

Indeed, by suitably specializing the values of parameters α' and α the results presented in this paper would find further fractional inequalities involving the Erdelyi-Kober and Riemann-Liouville type fractional integral operator, on taking relations (22) and (23) into account. For example, if we set $\alpha'=0$ and $\alpha=\gamma$, Theorems 6 and 7 lead to the following results involving Erdélyi-Kober fractional integral operator.

Corollary 3. Let f and g be two functions defined on $[0,\infty)$ such that f is increasing, g is differentiable, and there exists a real number $m=\inf_{x\geq 0} g'(x)$, then

$$I_{0,x}^{\gamma,-\beta}\{f(x)g(x)\} \geq \frac{\Gamma(1+\gamma-\beta)}{\Gamma(1-\beta)} x^{-\gamma+\alpha} I_{0,x}^{\gamma,-\beta}\{f(x)\} I_{0,x}^{\gamma,-\beta}\{g(x)\} - m\frac{(1+\gamma-\alpha-\beta)x}{(1+\gamma-\beta)(1+\gamma-\alpha)} I_{0,x}^{\gamma,-\beta}\{f(x)\} + mI_{0,x}^{\gamma,-\beta}\{g(x)\} \tag{26}$$

for all $x>0, \gamma > \max.\{0, \beta-1\}$.

Corollary 4. Let f and g be two functions defined on [0, ∞), such that f is increasing, g is differentiable, and there exists a real number $M = \sup_{t\geq 0} g'(t)$, then

$$I_{0,x}^{\gamma,-\beta}\{f(x)g(x)\} \geq \frac{\Gamma(1+\gamma-\beta)}{\Gamma(1-\beta)} x^{-\gamma+\alpha} I_{0,x}^{\gamma,-\beta}\{f(x)\} I_{0,x}^{\gamma,-\beta}\{g(x)\} - M\frac{(1+\gamma-\alpha-\beta)x}{(1+\gamma-\beta)(1+\gamma-\alpha)} I_{0,x}^{\gamma,-\beta}\{f(x)\} + MI_{0,x}^{\gamma,-\beta}\{g(x)\} \tag{27}$$

for all $x>0, \gamma > \max.\{0, \beta-1\}$.

Finally, if we take $\alpha'=0$ and $\alpha=0$ ($c'=0$ and $c=0$ additionally for Theorem 3), then Theorems 1 to 5, yield the known result due to Belarbi and Dahmani [1].We conclude with the remark that the results derived in this paper are general in character and give some contributions to the theory of integral inequalities and fractionalcalculus. Moreover, they are expected to find some applications for establishing uniqueness of solutions in fractional boundary value problems and in fractional partial differentialequations.

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