ORIGINAL PAPER

ZERO CONSTANT MEAN CURVATURE SURFACE FAMILY IN MINKOWSKI SPACE

SEDAT KAHYAOĞLU[1], EMIN KASAP[2]

Manuscript received: 09.02.2016; Accepted paper: 14.03.2016;
Published online: 30.03.2016.

Abstract. *We study zero constant mean curvature surface family prescribed by a given curve in Minkowski 3-space. We derive necessary and sufficient condition for zero constant mean curvature surface family prescribed a timelike or spacelike curve. We indicate the condition of a given curve is a geodesic curve or asymptotic curve on zero constant mean curvature surfaces.*
Keywords: *Constant mean curvature surface, Spacelike maximal surface, Timelike minimal surface, Frenet frame.*

1. INTRODUCTION

One of the oldest problems in geometry is the isoperimetric problem which find surfaces of least area enclosing a prescribed volume. Constant mean curvature surfaces are the local solutions of this problem. Therefore they are play an substantial role in differential geometry. These surfaces are of great interest jointly for mathematicians, physicists and engineers since they provide mathematical models a physical system which has minimum energy. Constant mean curvature surfaces in Minkowski space is important in general relativity. The mean curvature can be used as a global time coordinate and makes a time gauge, which it is important in the study of singularities, the positivity of mass and gravitational radiation[2,4].

In Minkowski 3-space R_1^3, a spacelike surface with zero constant mean curvature is called spacelike maximal surface and a timelike surface with zero constant mean curvature is called timelike minimal surface. Weierstrass type representation formula for spacelike maximal surface was derived by Kobayashi [11], for timelike minimal surface was derived by Minor [13]. Using the Frenet frame of a given curve, parametric representation of minimal surface family was derived in Euclidean space[5]. In Minkowski space, spacelike maximal surface family prescribed by a spacelike curve and timelike minimal surface family prescribed by a spacelike curve was studied in [6] and [7].

In this paper, we investigate general conditions to zero constant mean curvature surface family prescribed a given spacelike or timelike curve. We derive a diferential equation system as necessary and sufficient conditions for zero constant mean curvature surface family prescribed a given spacelike or timelike curve. We use the Frenet frame of a given curve for deriving necessary and sufficient conditions. We indicate conditions of a given curve is a

[1] Department of Mechatronics, Yeşilyurt Demir.Çelik. V. S., Ondokuz Mayis University, 55300 Samsun, Turkey. E-mail: sedatkahyaoglu@hotmail.com.
[2] Department of Mathematics, Art and Science Faculty, Ondokuz Mayis University, 55300 Samsun, Turkey. E-mail: kasape@omu.edu.tr.

geodesic curve or asymptotic curve. We derive some timelike minimal surface family for timelike curves as examples.

2. PRELIMINARIES

In this section we give some principal properties of Minkowski 3-space.
Let R_1^3 be Minkowski 3-space with metric $\langle , \rangle$. In terms of natural coordinates, the metric $\langle , \rangle$ is defined as

$$\langle , \rangle = de_1^2 + de_2^2 - de_3^2 \ .$$

A vector X is called

$$\begin{cases} \text{spacelike if } \langle X, X \rangle > 0 \\ \text{timelike if } \langle X, X \rangle > 0 \\ \text{null if } \langle X, X \rangle > 0 \end{cases} .$$

The vector product of the vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ is

$$X \times Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} .$$

The Frenet frame of the unit speed curve $r = r(u)$ is an othonormal basis $\{T, N, B\}$ where

$$\begin{cases} T = \dfrac{dr}{du} = r', \\ N = \dfrac{r''}{\|r''\|}, \\ B = T \times N \end{cases}$$

they are called tangent, normal and binormal of the curve r, respectively. The curve r is called spacelike (timelike, null) if its tangent vector T is spacelike (timelike, null) in everywhere. Frenet equations are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \eta\kappa & 0 \\ -\varepsilon\kappa & 0 & -\varepsilon\eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $\varepsilon = \langle T, T \rangle$, $\eta = \langle N, N \rangle$.The curvature of the curve r is $\kappa = \langle T', N \rangle$, the torsion of the curve r is $\tau = \langle N', B \rangle$. Full details can be found in [12,14].

We express parametrization of the surface $x \subset R_1^3$ as

$$x(u,v) = r(u) + f(u,v)T(u) + g(u,v)N(u) + h(u,v)B(u).$$

For $v = v_0$,

$$x(u,v_0) = r(u)$$

is a parameter curve on x and is called isoparametric curve on the surface x. If we calculate partial derivatives of $x(u,v)$ as

$$x_u = \frac{\partial x}{\partial u},$$

$$x_v = \frac{\partial x}{\partial v}$$

then we have,

$$\begin{aligned} x_u &= (1 + f_u - \varepsilon\kappa g)T + (g_u + \eta(\kappa f - \tau h))N + (h_u - \varepsilon\eta\tau g)B \\ x_v &= f_v T + g_v N + h_v B \\ x_{vv} &= f_{vv} T + g_{vv} N + h_{vv} B \end{aligned}$$

$$\begin{aligned} x_{uu} = {} & [(1 + f_u - \varepsilon\kappa g)_u - \varepsilon\kappa(g_u + \eta(\kappa f - \tau h))]T \\ & +[(g_u + \eta(\kappa f - \tau h))_u + \eta\kappa(1 + f_u - \varepsilon\kappa g) - \eta\tau(h_u - \varepsilon\eta\tau g)]N . \\ & +[(h_u - \varepsilon\eta\tau g)_u - \varepsilon\eta\tau(g_u + \eta(\kappa f - \tau h))]B \end{aligned}$$

The unit normal $n = n(u,v)$ of the surface x is defined by

$$n(u,v) = \frac{x_u \times x_v}{\left\| x_u \times x_v \right\|}$$

where

$$x_u \times x_v = \varphi_1 T + \varphi_2 N + \varphi_3 B$$

and

$$\begin{aligned} \varphi_1 &= \varepsilon((g_u + \eta(\kappa f - \tau h))h_v - (h_u - \varepsilon\eta\tau g)g_v) \\ \varphi_2 &= \eta((h_u - \varepsilon\eta\tau g)f_v - (1 + f_u - \varepsilon\kappa g)h_u) \\ \varphi_3 &= \varepsilon\eta((g_u + \eta(\kappa f - \tau h))f_v - (1 + f_u - \varepsilon\kappa g)g_v). \end{aligned}$$

The conditions of the curve r is a geodesic curve on the surface x was derived by Kasap and Akyildiz [8] as

$$\begin{aligned} \varphi_1(u,v_0) &= 0, \\ \varphi_2(u,v_0) &\neq 0, \\ \varphi_3(u,v_0) &= 0. \end{aligned} \tag{1}$$

The condition of the curve r is an asymptotic curve on the surface x was derived by Saffak et al. [15] as

$$h_v(u,v_0) = 0. \tag{2}$$

The mean curvature H of the surface x is

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} \tag{3}$$

where $e = \langle x_{uu}, n \rangle$, $f = \langle x_{uv}, n \rangle$, $g = \langle x_{vv}, n \rangle$, $E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$, $G = \langle x_v, x_v \rangle$ are the coefficient of the first and second fundemental forms.

3. ZERO CONSTANT MEAN CURVATURE SURFACE FAMILY

In this section we derive necessary and sufficient condition of zero constant mean curvature surfaces prescribed a curve.

Let $x \subset R_1^3$ be a spacelike or timelike surface. We know that there exist a conformal parameter $x(u,v)$ on the surface x [9]. We reconsider the conformal parameter as

$$\langle x_u, x_u \rangle = \sigma \langle x_v, x_v \rangle \text{ and } \langle x_u, x_v \rangle = 0 \tag{4}$$

where

$$\sigma = \begin{cases} -1 & x \text{ is timelike,} \\ 1 & x \text{ is spacelike.} \end{cases}$$

Thus, the mean curvature of x vanishes if and only if

$$x_{uu} + \sigma x_{vv} = 0. \tag{5}$$

Hence, we obtain the following theorem.

Theorem 1. Let the spacelike or timelike curve $r \subset R_1^3$ be an isoparametric curve on the surface $x \subset R_1^3$, then the mean curvature of x is constant zero if and only if there exist the functions f, g, h which are satisfiying the following conditions

$$\varepsilon(1 + f_u - \varepsilon\kappa g)^2 + \eta(g_u + \eta(\kappa f - \tau h))^2 - \varepsilon\eta(h_u - \varepsilon\eta\tau g))^2 = \sigma(\varepsilon f_v^2 + \eta g_v^2 - \varepsilon\eta h_v^2) \tag{6}$$

$$\varepsilon(1 + f_u - \varepsilon\kappa g)f_v + \eta(g_u + \eta(\kappa f - \tau h))g_v - \varepsilon\eta(h_u - \varepsilon\eta\tau g)h_v = 0 \tag{7}$$

$$(1 + f_u - \varepsilon\kappa g)_u - \kappa(g_u + \eta(\kappa f - \tau h)) = -\sigma f_{vv} \tag{8}$$

$$(g_u + \eta(\kappa f - \tau h))_u + \eta\kappa(1 + f_u - \varepsilon\kappa g) - \eta\tau(h_u - \varepsilon\eta\tau g) = -\sigma g_{vv} \quad (9)$$

$$(h_u - \varepsilon\eta\tau g)_u - \eta\tau(g_u + \eta(\kappa f - \tau h)) = -\sigma h_{vv} \quad (10)$$

Proof. Let the mean curvature of x is constant zero. From (4) we have (6) and (7). Also, from (5) we have (8)-(10).
Conversely, let the equations (6)-(10) are satisfied. Thus, we have $H = 0$. □

Note that, for $\varepsilon = 1, \sigma = 1$ the surface x is a spacelike maximal surface, for $\varepsilon = \pm 1, \sigma = -1$ the surface x is a timelike minimal surface.

From the following proposition, if we solve the equations (8)-(10) then the equations (6) and (7) give us constants which are derived from (8)-(10).

Proposition 1. Let (5) is satisfied. Then $\langle x_u, x_u \rangle = \sigma \langle x_v, x_v \rangle + c_1$ if and only if $\langle x_u, x_v \rangle = c_2$ where c_1, c_2 are constant.

Proof. Let $x_{uu} = -\sigma x_{vv}$. Then

$$\begin{aligned} \langle x_u, x_v \rangle_u &= \langle x_{uv}, x_u \rangle - \sigma \langle x_{vv}, x_v \rangle \\ &= \frac{1}{2}(\langle x_u, x_u \rangle_v - \sigma \langle x_v, x_v \rangle_v), \\ \langle x_u, x_v \rangle_v &= \langle x_{uv}, x_v \rangle - \sigma \langle x_{uu}, x_u \rangle \\ &= \frac{1}{2}(\langle x_v, x_v \rangle_u - \sigma \langle x_u, x_u \rangle_u). \end{aligned}$$

Thus proof is completed. □

The following corollaries give us conditions to the curve r be geodesic curve or asymptotic curve on the zero constant mean curvature surface x.

Corollary 1. Let the curve r be an isoparametric curve on the zero constant mean curvature surface x. Then the curve r is a geodesic curve on the surface x if and only if

$$h_v(u, v_0) = \pm 1, \quad (11)$$

$$\langle N, N \rangle = -\varepsilon\sigma. \quad (12)$$

Proof. From isoparametric condition, we have

$$f(u, v_0) = g(u, v_0) = h(u, v_0) = 0$$

and

$$f_u(u, v_0) = g_u(u, v_0) = h_u(u, v_0) = 0.$$

Let the conditions (11) and (12) are satisfied. From (6) and (7) we have $g_v(u, v_0) = 0$. From (1) the curve r be a geodesic curve on surface x.

Conversely, let the curve r is a geodesic curve on the surface x. From (1) we have $g_v(u, v_0) = 0$. Thus, from (6) and (7) we have (11) and (12). □

Corollary 2. Let the curve r is an isoparametric curve on the zero constant mean curvature surface x. Then the curve r is an asymptotic curve on the surface x if and only if

$$g_v(u, v_0) = \pm 1, \tag{13}$$

$$\left\langle N, N \right\rangle = \varepsilon\sigma . \tag{14}$$

Proof. From (6),(7) and (2) asymptotic we derive (13) and (14)

4. EXAMPLES

We consider $v_0 = 0$ in the all examples

Example 1. Let $r_1(u) = (a\cos u, a\sin u, bu)$ be a timelike curve where $a^2 - b^2 = -1$ and $\sigma = -1$. The Frenet vectors are

$$\begin{aligned} T(u) &= (-a\sin u, a\cos u, b), \\ N(u) &= (-\cos u, -\sin u, 0), \\ B(u) &= (b\sin u, -b\cos u, -a). \end{aligned}$$

Thus we have $\eta = 1$, $\kappa = a$ and $\tau = b$. If we choose $f(u,v) = f(v)$, $g(u,v) = g(v)$, $h(u,v) = h(v)$ then from (9) we have $g(v) = c_1 \sin v - a\cos v + a$, from (8) and (10), we have $f(v) = c_2 \sin v - abcv$ and $h(v) = c_2 \frac{b}{a} \sin v - a^2 cv$ where c, c_1 and c_2 are constant. From (7) and (6) we have $c_2 = abc$ and $c_1 = \pm\sqrt{1 - c^2}$. Thus we have a timelike minimal surface family prescribed the timelike curve r_1 as

$$\begin{aligned} x_1(u, v; c) = {} & (a\cos u, a\sin u, bu) \\ & + abc(v - \sin v)(-a\sin u, a\cos u, b) \\ & + (\pm(\sqrt{1 - c^2})(\sin v) + a(1 - \cos v)(-\cos u, -\sin u, 0) \\ & + c(b^2 \sin v - a^2 v)(b\sin u, -b\cos u, -a). \end{aligned}$$

For $c = \pm 1$, the curve r_1 is a geodesic curve on the timelike minimal surface $x_1(u,v;1)$. For $c = 0$, the curve r_1 is an asymptotic curve on the timelike minimal surface $x_1(u,v;0)$. For $a = 1, b = \sqrt{2}$ we show some members of the timelike minimal surface family $x_1(u,v;c)$ in Fig. 1.

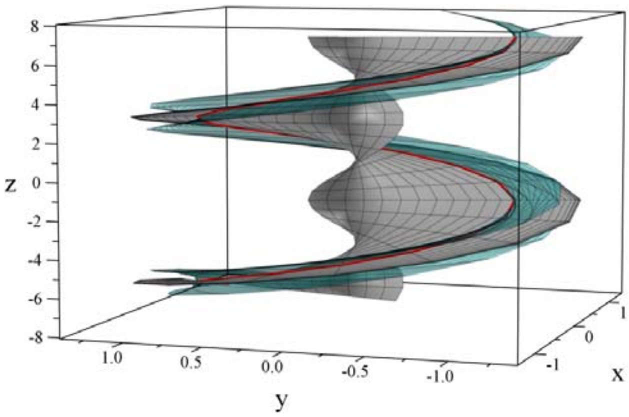

Fig 1. Timelike minimal surfaces $x_1(u,v;0)$, $x_1(u,v;1)$ and the timelike curve r_1.

Example 2. Let $r_2(u) = (0, a\cosh\frac{u}{a}, a\sinh\frac{u}{a})$ be timelike curve and $\sigma = -1$. The Frenet apparatus are

$$T(u) = (0, \sinh\frac{u}{a}, \cosh\frac{u}{a}),$$
$$N(u) = (0, \cosh\frac{u}{a}, \sinh\frac{u}{a}),$$
$$B(u) = (1,0,0),$$

$\tau = 0$, $\kappa = \dfrac{1}{a}$ and $\eta = 1$.

From Theorem 1, we have $g(v) = a(\frac{1\pm\sqrt{1-c^2}}{2}e^{\frac{v}{a}} + \frac{1\mp\sqrt{1-c^2}}{2}e^{\frac{-v}{a}} - 1)$ and $h(u,v) = cv$. Thus we derive a timelike minimal surface family as

$$x_2(u,v;|c|) = (cv, a(\frac{1\pm\sqrt{1-c^2}}{2}e^{\frac{v}{a}} + \frac{1\mp\sqrt{1-c^2}}{2}e^{\frac{-v}{a}})(\cosh\frac{u}{a}, \sinh\frac{u}{a})).$$

For $c = \pm 1$, the curve r_2 is a geodesic curve on the timelike minimal surface $x_2(u,v;1)$. For $c = 0$, the curve r_2 is an asymptotic curve on the timelike plane $x_2(u,v;0)$. For a=4, we show some members of the timelike minimal surface family $x_2(u,v;|c|)$ in Fig. 2.

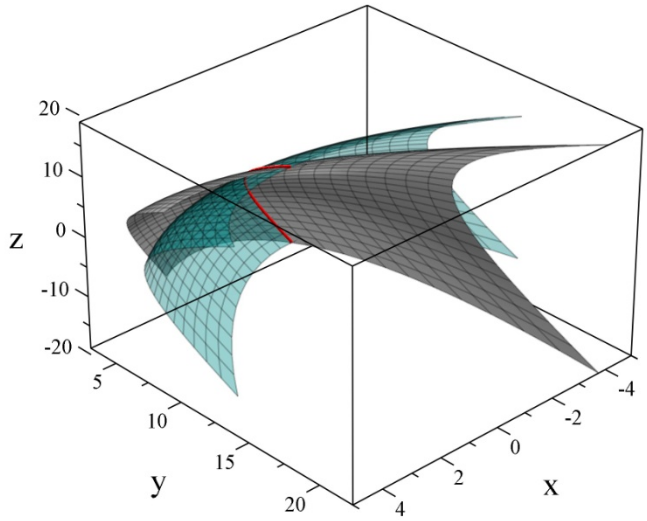

Fig 2. Timelike minimal surfaces $x_2(u,v;0)$, $x_2(u,v;1)$ and the timelike curve r_2.

CONCLUSION

We derive necessary and sufficient condition for zero constantmean curvature surface family prescribed by a given curve without the complex representation formula. We indicate condition of a given curve is a geodesic curve or asymptotic curve on zero constant mean curvature surface.

REFERENCES

[1] Alias, L.J., Chaves, R.M.B., and Mira, P., *Proc. Cambridge Philos. Soc.*, **134**, 289, 2003.
[2] Bartnik, R., *Miniconference on geometry/partial differential equations Centre for Mathematical Analysis*, **2**, 24, 1987.
[3] Chaves, R.M.B., Dussan, M.P., Magi, M., *J. Math. Anal. Appl.*, **377**, 481, 2011.
[4] Choquet B.Y. and York J., *The Cauchy Problem in General Relativity and Gravitation*, Plenum, New York, 1980.
[5] Kahyaoglu, S., Kasap, E., *Int. J. Contemp. Math. Sci.*, **10**, 223, 2015.
[6] Kahyaoglu, S., Kasap, E., *Int. J. Contemp. Math. Sci.*, **11**, 131, 2016.
[7] Kahyaoglu, S., Kasap, E., *Int. Electron. J. Pure Appl. Math.*, **10**, 83, 2016.
[8] Kasap, E., Akyildiz, F. T., *Appl. Math. Comput.*,. **177**, 260, 2006.
[9] Kim, Y.W., Koh, S.-E., Shin, H., Yang, S.D., *J. Korean Math. Soc*,. **48**, 1083, 2011.
[10] Kim, Y.W., Yang, S.D., *J. Geom. Phys.*, **57**, 2167, 2007.
[11] Kobayashi, O., *Tokyo J. Math.*, **6**, 297, 1983.
[12] Kühnel, W., *Differential geometry: curves-surfaces-manifolds*, AMS Student Mathematical Library, 2005.
[13] Milnor T.K., *Michigan Math. J.*, **30**, 309, 1983.
[14] O'Neill, B., *Semi-riemannian geometry with application to general relativity*, Academic Press, 1983.
[15] Saffak, G., Bayram, E., Kasap, E., *http://arxiv.org/abs/1406.0618*, 2013.