SOLUTION OF DIFFERENT TYPES OF FUZZY INTEGRO-DIFFERENTIAL EQUATIONS VIA LAPLACE HOMOTOPY PERTURBATION METHOD

JAMSHAD AHMAD[1], HIRA NOSHER[2]

Abstract. *In this paper we shall consider the various types of linear and non-linear fuzzy integro-differential equations of second kind like fuzzy Volterra integro-differential equation, fuzzy Fredholm integro-differential equation and mixed fuzzy Volterra-Fredholm integro-differential equation and suggested the algorithm of advanced numerical technique like Laplace homotopy perturbation method to find out their solutions. The application of this method on various types of fuzzy integro-differential equations shows the efficiency and reliability of the proposed method.*

Keywords: Laplace homotopy perturbation method, linear fuzzy integro-differential equations, non-linear fuzzy integro-differential equations.

1. INTRODUCTION

Integral equation plays a vital role with in many disciplines of sciences, engineering and mathematics. Using of integral equations with exact parameter within many modeling physical problems is not quite easy or better to say impossible in real problems. To overcome this difficulty one of the most recent approach is to use fuzzy concept. Basic concept of fuzzy was first introduced by professor Zadeh in 1965 after his publication on fuzzy set theory [1, 2]. Thus in 1978 Dubois and Prade introduced the concept of arithmetic operations on fuzzy numbers or can say they presented the fuzzy calculus [3, 4], then as well as time pass many different fields of mathematics use this concept of fuzzy set theory and introduced fuzzy functions, relations, groups, subgroups etc. Recently twenty years ago in Japan a person name M. Sugeno introduced the concept of fuzzy integrals [5, 6], then it's becoming a research oriented topic. Homotopy perturbation method (HPM) is a coupling of perturbation method and homotopy technique was firstly introduced by He JH in 1999 [7, 8], then it was farther developed by him [9, 10]. Laplace homotopy perturbation method (LHPM) was introduced by Amini khan and Hemmatnezhad in 2012 [11-13], to find the solution of non-linear ordinary differential equations. LHPM is combination of Laplace transformation and Homotopy perturbation method. The purpose to use Laplace transform is to overcome the deficiency of other semi-analytical methods such as HPM, VIM and ADM that is mainly caused by unsatisfied conditions.

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[1] University of Gujrat, Faculty of Sciences, Department of Mathematics, 50700 Gujrat, Pakistan.
E-mail: jamshadahmadm@gmail.com.
[2] National College of Business Administration and Economics, Department of Mathematics, Gujrat Campus, Pakistan.

The paper is organized as follows: In 1st Section we discuss the literature of topic and the method. In 2nd section we define fuzzy integro-differential equations. In 3rd section we introduce the method of LHPM to find solution of various types of fuzzy integro-differential equations. In 4th section we find the solution of numerical problems by utilizing LHPM. Finally a brief conclusion is given at the end.

2. DEFINITION

The general nth-order fuzzy integro-differential equation is as follows

$$u^{(n)}(x,\alpha) = f(x,\alpha) + \lambda \int_{a(x)}^{b(x)} k(x,t)u(t,\alpha)dt \tag{1}$$

with $u(0,\alpha) = (a_0, b_0), u'(0,\alpha) = (a_1, b_1), u''(0,\alpha) = (a_2, b_2), ..., u^{(n-1)}(0,\alpha) = (a_{n-1}, b_{n-1})$

Where $(a_i, b_i), i = 0, 1, 2, ..., n-1$ are real constants, $u^{(n)}(x,\alpha)$ is n^{th} order derivative of fuzzy function $u(x,\alpha)$ also $f(x,\alpha)$ is fuzzy function given in advance, α is the fuzzy parameter whose value lies between $[0,1]$ i.e. $0 \le \alpha \le 1$, λ is constant parameter, $k(x,t)$ is known function of two variables x and t called kernel of fuzzy integro-differential equation, $a(x)$ and $b(x)$ are limits of fuzzy integro-differential equation, if both of limits $a(x)$ and $b(x)$ are constant, then integro-differential equation is known as Fredholm fuzzy integro-differential equation, if one of limit can say $a(x)$ is constant and one of limit say $b(x)$ is variable then equation is called fuzzy Volterra integro-differential equation and an equation containing both type of integrals is called mixed fuzzy integro-differential equation.

The parametric representation of Eq. (1) is as follows,

$$\begin{cases} \underline{u}^{(n)}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_{a(x)}^{b(x)} \underline{k(x,t)u(t,\alpha)}dt \\ \overline{u}^{(n)}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_{a(x)}^{b(x)} \overline{k(x,t)u(t,\alpha)}dt \end{cases}$$

For $0 \le \alpha \le 1$
Where $u(x,\alpha) = (\underline{u}(x,\alpha), \overline{u}(x,\alpha))$, $f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ and

$$\begin{cases} \underline{k(x,t)u(t,\alpha)} = k(x,t)\underline{u}(t,\alpha) & k(x,t) \ge 0 \\ \overline{k(x,t)u(t,\alpha)} = k(x,t)\overline{u}(t,\alpha) & k(x,t) \ge 0 \end{cases}$$

3. ANALYSIS OF HPM

To solve Eq. (1) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v}, p, \alpha) = (1-p)\left[\underline{v}^{(n)}(x,\alpha) - \underline{u}_0(x,\alpha)\right] + p\left[\underline{v}^{(n)}(x,\alpha) - \underline{f}(x,\alpha) - \int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt\right] = 0 \\ H(\overline{v}, p, \alpha) = (1-p)\left[\overline{v}^{(n)}(x,\alpha) - \overline{u}_0(x,\alpha)\right] + p\left[\overline{v}^{(n)}(x,\alpha) - \overline{f}(x,\alpha) - \int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt\right] = 0 \end{cases} \tag{2}$$

Thus the initial approximation is taken as

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \end{cases} \tag{3}$$

Substituting Eq. (3) in Eq. (2) reduces to

$$\begin{cases} \underline{v}^{(n)}(x,\alpha) = \underline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt \\ \overline{v}^{(n)}(x,\alpha) = \overline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt \end{cases} \tag{4}$$

Now by applying Laplace transformation on both sides of Eq. (4), we get

$$\begin{cases} \mathcal{L}\{\underline{v}^{(n)}(x,\alpha)\} = \mathcal{L}\{\underline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt\} \\ \mathcal{L}\{\overline{v}^{(n)}(x,\alpha)\} = \mathcal{L}\{\overline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt\} \end{cases} \tag{5}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\frac{1}{s^n}\right)\{s^{n-1}\underline{v}(o,\alpha) + s^{n-2}\underline{v}'(o,\alpha) + ... + \underline{v}^{(n-1)}(o,\alpha) + \mathcal{L}\{\underline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt\}\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\frac{1}{s^n}\right)\{s^{n-1}\overline{v}(o,\alpha) + s^{n-2}\overline{v}'(o,\alpha) + ... + \overline{v}^{(n-1)}(o,\alpha) + \mathcal{L}\{\overline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt\}\} \end{cases} \tag{6}$$

Applying inverse Laplace transformation on both sides of Eq. (6), we obtain

$$\begin{cases} \underline{v}(x,\alpha) = \mathcal{L}^{-1}\{\left(\frac{1}{s^n}\right)\{s^{n-1}\underline{v}(o,\alpha) + s^{n-2}\underline{v}'(o,\alpha) + ... + \underline{v}^{(n-1)}(o,\alpha) + \mathcal{L}\{\underline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt\}\}\} \\ \overline{v}(x,\alpha) = \mathcal{L}^{-1}\{\left(\frac{1}{s^n}\right)\{s^{n-1}\overline{v}(o,\alpha) + s^{n-2}\overline{v}'(o,\alpha) + ... + \overline{v}^{(n-1)}(o,\alpha) + \mathcal{L}\{\overline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt\}\}\} \end{cases} \tag{7}$$

Assume the solution of Eq. (7) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{8}$$

Where $(\underline{v}_i, \overline{v}_i)$ are unknown to determined.

Now by putting Eq. (8) in Eq. (7) and by comparing coefficient like power of p, we get the following iterations are as follows

$$p^0: \begin{cases} \underline{v}_0(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{s^{n-1}\underline{v}(o,\alpha) + s^{n-2}\underline{v}'(o,\alpha) + ... + \underline{v}^{(n-1)}(o,\alpha) + \mathcal{L}\{\underline{f}(x,\alpha)\}\right\}\right\} \\ \overline{v}_0(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{s^{n-1}\overline{v}(o,\alpha) + s^{n-2}\overline{v}'(o,\alpha) + ... + \overline{v}^{(n-1)}(o,\alpha) + \mathcal{L}\{\overline{f}(x,\alpha)\}\right\}\right\} \end{cases} \quad (9)$$

$$p^1: \begin{cases} \underline{v}_1(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{\mathcal{L}\left\{p\int_{a(x)}^{b(x)}k(x,t)\underline{v}_0(t,\alpha)dt\right\}\right\}\right\} \\ \overline{v}_1(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{\mathcal{L}\left\{p\int_{a(x)}^{b(x)}k(x,t)\overline{v}_0(t,\alpha)dt\right\}\right\}\right\} \end{cases} \quad (10)$$

$$p^2: \begin{cases} \underline{v}_2(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{\mathcal{L}\left\{p\int_{a(x)}^{b(x)}k(x,t)\underline{v}_1(t,\alpha)dt\right\}\right\}\right\} \\ \overline{v}_2(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{\mathcal{L}\left\{p\int_{a(x)}^{b(x)}k(x,t)\overline{v}_1(t,\alpha)dt\right\}\right\}\right\} \end{cases} \quad (11)$$

And so on…

Thus the solution of FIDE-2 is given as

$$\begin{cases} \underline{u}(x,\alpha) = \lim_{p\to 1}\underline{v}(x,\alpha) = \sum_{i=0}^{\infty}\underline{v}_i(x,\alpha) \\ \overline{u}(x,\alpha) = \lim_{p\to 1}\overline{v}(x,\alpha) = \sum_{i=0}^{\infty}\overline{v}_i(x,\alpha) \end{cases} \quad (12)$$

4. NUMERICAL EXAMPLES

Example 4.1 Consider the linear fuzzy Volterra integro-differential equation of 2nd kind

$$u'(x,\alpha) = f(x,\alpha) - \int_0^x u(t,\alpha)dt \quad (13)$$

with $u(0,\alpha) = (0,0)$

where $\lambda = 1, \ 0 \le t \le x, 0 \le \alpha \le 1, k(x,t) = 1 \ and \ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)) \ i.e.$
$$f(x,\alpha) = ((\alpha^2 + \alpha),(5-\alpha)).$$

To solve Eq. (13) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,\alpha) = \underline{v}'(x,\alpha) - (\alpha^2 + \alpha) + p\int_0^x \underline{v}(t,\alpha)dt = 0 \\ H(\overline{v},p,\alpha) = \overline{v}'(x,\alpha) - (5-\alpha) + p\int_0^x \overline{v}(t,\alpha)dt = 0 \end{cases}. \quad (14)$$

Now by applying Laplace transformation on both sides of Eq. (14), we get

$$\begin{cases} \mathcal{L}\left\{\underline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{\left(\alpha^2+\alpha\right)-p\int_0^x \underline{v}(t,\alpha)dt\right\} \\ \mathcal{L}\left\{\overline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{(5-\alpha)-p\int_0^x \overline{v}(t,\alpha)dt\right\} \end{cases} \tag{15}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{\left(\alpha^2+\alpha\right)-p\int_0^x \underline{v}(t,\alpha)dt\right\}\right\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(5-\alpha)-p\int_0^x \overline{v}(t,\alpha)dt\right\}\right\} \end{cases} \tag{16}$$

Applying inverse Laplace transformation on both sides of Eq. (16), we obtain

$$\begin{cases} \underline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{\left(\alpha^2+\alpha\right)-p\int_0^x \underline{v}(t,\alpha)dt\right\}\right\}\right\} \\ \overline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(5-\alpha)-p\int_0^x \overline{v}(t,\alpha)dt\right\}\right\}\right\} \end{cases}, \tag{17}$$

Assume the solution of Eq. (17) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases}. \tag{18}$$

Now by putting Eq. (18) in Eq. (17) and by comparing coefficient like power of p, we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = \left(\alpha^2+\alpha\right)x, \\ \overline{v}_0(x,\alpha) = (5-\alpha)x \end{cases}, \tag{19}$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = -\dfrac{x^3}{3!}\left(\alpha^2+\alpha\right) \\ \overline{v}_1(x,\alpha) = -\dfrac{x^3}{3!}(5-\alpha) \end{cases}, \tag{20}$$

$$p^2 : \begin{cases} \underline{v}_2(x,\alpha) = \dfrac{x^5}{5!}\left(\alpha^2+\alpha\right) \\ \overline{v}_2(x,\alpha) = \dfrac{x^5}{5!}(5-\alpha) \end{cases}, \tag{21}$$

$$p^3 : \begin{cases} \underline{v}_3(x,\alpha) = \dfrac{x^7}{7!}\left(\alpha^2 + \alpha\right) \\[2mm] \overline{v}_3(x,\alpha) = \dfrac{x^7}{7!}\left(5 - \alpha\right) \end{cases}, \tag{22}$$

and so on…

As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\[4mm] \overline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{23}$$

Thus by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = x(\alpha^2 + \alpha) - \dfrac{x^3}{3!}(\alpha^2 + \alpha) + \dfrac{x^5}{5!}(\alpha^2 + \alpha) - \dfrac{x^7}{7!}(\alpha^2 + \alpha) + ... \\[3mm] \overline{u}(x,\alpha) = x(5 - \alpha) - \dfrac{x^3}{3!}(5 - \alpha) + \dfrac{x^5}{5!}(5 - \alpha) - \dfrac{x^7}{7!}(5 - \alpha) + ... \end{cases} \tag{24}$$

And the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \sin x(\alpha^2 + \alpha) \\[2mm] \overline{u}(x,\alpha) = \sin x(5 - \alpha) \end{cases}. \tag{25}$$
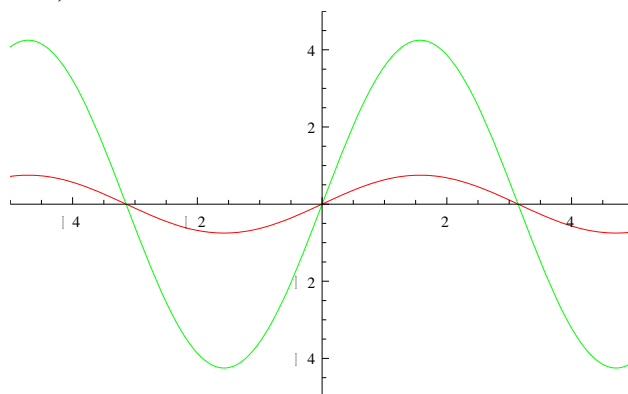

Figure 1.Plot of Solution of Example 1.

Example 4.2 Consider the linear fuzzy Fredholm integro-differential equation of 2nd kind

$$u''(x,\alpha) = f(x,\alpha) + \int_0^\pi tu(t,\alpha)dt \tag{26}$$

with $\quad u(0,\alpha) = (\alpha,(6 - \alpha)),\, u'(0,\alpha) = (0,0),$

where $\lambda = 1,\, 0 \le t \le \pi,\, 0 \le \alpha \le 1,\, k(x,t) = t \quad and \quad f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ *i.e.*

$$f(x,\alpha) = ((2 - \cos x)\alpha, (2 - \cos x)(6 - \alpha)).$$

To solve Eq. (26) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v}, p, \alpha) = \underline{v}''(x,\alpha) - (2 - \cos x)\alpha - p\int_0^\pi t\underline{v}(t,\alpha)dt = 0 \\ H(\overline{v}, p, \alpha) = \overline{v}''(x,\alpha) - (2 - \cos x)(6 - \alpha) - p\int_0^\pi t\overline{v}(t,\alpha)dt = 0 \end{cases} \tag{27}$$

Now by applying Laplace transformation on both sides of Eq. (27), we get

$$\begin{cases} \mathcal{L}\{\underline{v}''(x,\alpha)\} = \mathcal{L}\left\{(2 - \cos x)\alpha + p\int_0^\pi t\underline{v}(t,\alpha)dt\right\} \\ \mathcal{L}\{\overline{v}''(x,\alpha)\} = \mathcal{L}\left\{(2 - \cos x)(6 - \alpha) + p\int_0^\pi t\overline{v}(t,\alpha)dt\right\} \end{cases} \tag{28}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\frac{1}{s^2}\right)\left\{\alpha s + \mathcal{L}\left\{(2 - \cos x)\alpha + p\int_0^\pi t\underline{v}(t,\alpha)dt\right\}\right\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\frac{1}{s^2}\right)\left\{(6 - \alpha)s + \mathcal{L}\left\{(2 - \cos x)(6 - \alpha) + p\int_0^\pi t\overline{v}(t,\alpha)dt\right\}\right\} \end{cases} \tag{29}$$

Applying inverse Laplace transformation on both sides of Eq. (29), we obtain

$$\begin{cases} \underline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2}\right)\left\{\alpha s + \mathcal{L}\left\{(2 - \cos x)\alpha + p\int_0^\pi t\underline{v}(t,\alpha)dt\right\}\right\}\right\} \\ \overline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2}\right)\left\{(6 - \alpha)s + \mathcal{L}\left\{(2 - \cos x)(6 - \alpha) + p\int_0^\pi t\overline{v}(t,\alpha)dt\right\}\right\}\right\} \end{cases} \tag{30}$$

Assume the solution of Eq. (30) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{31}$$

Now by putting Eq. (31) in Eq. (30) and by comparing coefficient like power of p, we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = (x^2 + \cos x)\alpha \\ \overline{v}_0(x,\alpha) = (x^2 + \cos x)(6 - \alpha) \end{cases}, \tag{32}$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = \dfrac{\pi^4}{8}x^2\alpha - x^2\alpha \\[2mm] \overline{v}_1(x,\alpha) = \dfrac{\pi^4}{8}x^2(6-\alpha) - x^2(6-\alpha) \end{cases}, \tag{33}$$

$$p^2 : \begin{cases} \underline{v}_2(x,\alpha) = \dfrac{\pi^8}{64}x^2\alpha - \dfrac{\pi^4}{8}x^2\alpha \\[2mm] \overline{v}_2(x,\alpha) = \dfrac{\pi^8}{64}x^2(6-\alpha) - \dfrac{\pi^4}{8}x^2(6-\alpha) \end{cases}, \tag{34}$$

and so on….

As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\[3mm] \overline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{35}$$

Thus by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \cos x\alpha + x^2\alpha - x^2\alpha + \dfrac{\pi^4}{8}x^2\alpha - \dfrac{\pi^4}{8}x^2\alpha + \dfrac{\pi^8}{64}x^2\alpha + ... \\[3mm] \overline{u}(x,\alpha) = \cos x(6-\alpha) + x^2(6-\alpha) - x^2(6-\alpha) + \dfrac{\pi^4}{8}x^2(6-\alpha) - \dfrac{\pi^4}{8}x^2(6-\alpha) + \dfrac{\pi^8}{64}x^2(6-\alpha) + ... \end{cases} \tag{36}$$

And the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \cos x\alpha \\[2mm] \overline{u}(x,\alpha) = \cos x(6-\alpha) \end{cases}. \tag{37}$$
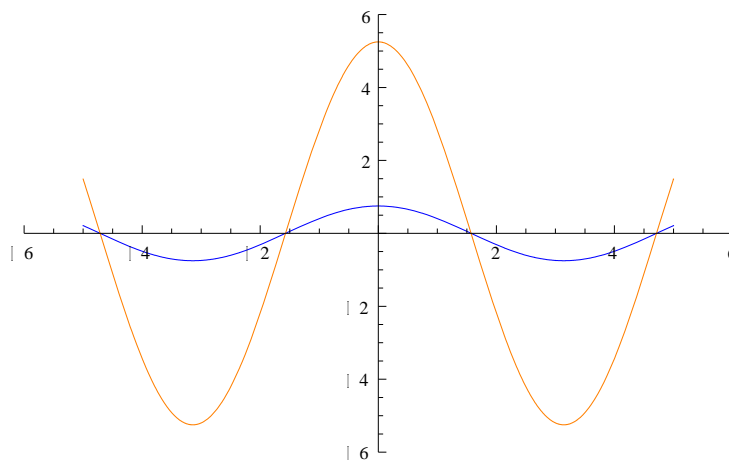

Figure 2. Plot of Solution of Example 2.

Example 4.3 Consider the linear fuzzy Volterra-Fredholm integro-differential equation of 2nd kind

$$u'(x,\alpha) = f(x,\alpha) + \int_0^x u(t,\alpha)dt - \int_0^1 e^{x-t}u(t,\alpha)dt \tag{38}$$

with $u(o,\alpha) = (2\alpha, 2(3-\alpha))$,
where

$$0 \le t \le 1, 0 \le t \le x, \ 0 \le \alpha \le 1, \ k_1(x,t) = 1, k_2(x,t) = e^{x-t} and \ \ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)) \ \ i.e.$$
$$f(x,\alpha) = ((e^x+1)\alpha, \ (e^x+1)(3-\alpha)).$$

To solve Eq. (38) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,\alpha) = \underline{v}'(x,\alpha) - (e^x+1)\alpha - p\int_0^x \underline{v}(t,\alpha)dt + p\int_0^1 e^{x-t}\underline{v}(t,\alpha)dt = 0 \\ H(\overline{v},p,\alpha) = \overline{v}'(x,\alpha) - (e^x+1)(3-\alpha) - p\int_0^x \overline{v}(t,\alpha)dt + p\int_0^1 e^{x-t}\overline{v}(t,\alpha)dt = 0 \end{cases} \tag{39}$$

Now by applying Laplace transformation on both sides of Eq. (39), we get

$$\begin{cases} \mathcal{L}\left\{\underline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{(e^x+1)\alpha + p\int_0^x \underline{v}(t,\alpha)dt - p\int_0^1 e^{x-t}\underline{v}(t,\alpha)dt\right\} \\ \mathcal{L}\left\{\overline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{(e^x+1)(3-\alpha) + p\int_0^x \overline{v}(t,\alpha)dt - p\int_0^1 e^{x-t}\overline{v}(t,\alpha)dt\right\} \end{cases} \tag{40}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\frac{1}{s}\right)\left\{\mathcal{L}\left\{2\alpha + (e^x+1)\alpha + p\int_0^x \underline{v}(t,\alpha)dt - p\int_0^1 e^{x-t}\underline{v}(t,\alpha)dt\right\}\right\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\frac{1}{s}\right)\left\{\mathcal{L}\left\{2(3-\alpha) + (e^x+1)(3-\alpha) + p\int_0^x \overline{v}(t,\alpha)dt - p\int_0^1 e^{x-t}\overline{v}(t,\alpha)dt\right\}\right\} \end{cases} \tag{41}$$

Applying inverse Laplace transformation on both sides of Eq. (41), we obtain

$$\begin{cases} \underline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)\left\{\mathcal{L}\left\{2\alpha + (e^x+1)\alpha + p\int_0^x \underline{v}(t,\alpha)dt - p\int_0^1 e^{x-t}\underline{v}(t,\alpha)dt\right\}\right\}\right\} \\ \overline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)\left\{\mathcal{L}\left\{2(3-\alpha) + (e^x+1)(3-\alpha) + p\int_0^x \overline{v}(t,\alpha)dt - p\int_0^1 e^{x-t}\overline{v}(t,\alpha)dt\right\}\right\}\right\} \end{cases} \tag{42}$$

Assume the solution of Eq. (42) can be written as power series in

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \, \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \, \overline{v}_i(x,\alpha) \end{cases} \tag{43}$$

Now by putting Eq. (43) in Eq. (42) and by comparing coefficient like power of p, we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = (e^x + x + 1)\alpha \\ \overline{v}_0(x,\alpha) = (e^x + x + 1)(3-\alpha) \end{cases} \tag{44}$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = (-x - 1 + \dfrac{1}{2}x^2 + ...)\alpha \\ \overline{v}_1(x,\alpha) = (-x - 1 + \dfrac{1}{2}x^2 + ...)(3-\alpha) \end{cases} \tag{45}$$

and so on…

As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\ \overline{u}(x,\alpha) = \sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{46}$$

Thus by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = e^x \alpha + x\alpha + \alpha - x\alpha - \alpha + \dfrac{1}{2}x^2\alpha + ... \\ \overline{u}(x,\alpha) = e^x(3-\alpha) + x(3-\alpha) + (3-\alpha) - x(3-\alpha) - (3-\alpha) + \dfrac{1}{2}x^2(3-\alpha) + ... \end{cases} \tag{47}$$

And the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = e^x \alpha \\ \overline{u}(x,\alpha) = e^x(3-\alpha) \end{cases} \tag{48}$$

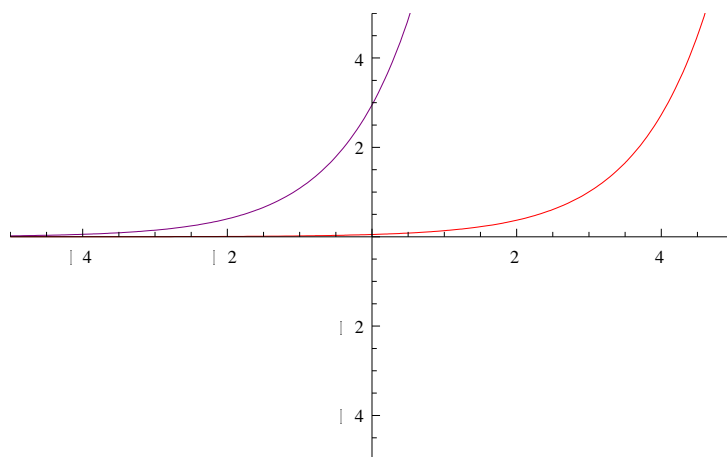**Figure 3. Plot of Solution of Example 3.**

Example 4.4 Consider the non-linear fuzzy Volterra integro-differential equation of 2nd kind

$$u'(x,\alpha) = f(x,\alpha) + \int_0^x u^2(t,\alpha)dt \tag{49}$$

with $\quad u(0,\alpha) = (0,0),$

where $\lambda = 1, \; 0 \le t \le x, \; 0 \le \alpha \le 1, \; k(x,t) = 1 \;$ and $\; f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)) \;$ i.e.
$f(x,\alpha) = ((\alpha),(7-\alpha)).$

To solve Eq. (49) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v}, p, \alpha) = \underline{v}'(x,\alpha) - \alpha - p\int_0^x \underline{v}^2(t,\alpha)dt = 0 \\ H(\overline{v}, p, \alpha) = \overline{v}'(x,\alpha) - (7-\alpha) - p\int_0^x \overline{v}^2(t,\alpha)dt = 0 \end{cases} \tag{50}$$

Now by applying Laplace transformation on both sides of Eq. (50), we get

$$\begin{cases} \mathcal{L}\left\{\underline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{\alpha + p\int_0^x \underline{v}^2(t,\alpha)dt\right\} \\ \mathcal{L}\left\{\overline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{(7-\alpha) + p\int_0^x \overline{v}^2(t,\alpha)dt\right\} \end{cases} \tag{51}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{\alpha + p\int_0^x \underline{v}^2(t,\alpha)dt\right\}\right\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(7-\alpha) + p\int_0^x \overline{v}^2(t,\alpha)dt\right\}\right\} \end{cases} \tag{52}$$

Applying inverse Laplace transformation on both sides of Eq. (52), we obtain

$$\begin{cases} \underline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{\alpha + p\displaystyle\int_0^x \underline{v}^2(t,\alpha)dt\right\}\right\}\right\} \\[4mm] \overline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(7-\alpha) + p\displaystyle\int_0^x \overline{v}^2(t,\alpha)dt\right\}\right\}\right\} \end{cases} \qquad (53)$$

Assume the solution of Eq. (53) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\[4mm] \overline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \qquad (54)$$

Now by putting Eq. (54) in Eq. (53) and by comparing coefficient like power of p, we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = \alpha\, x \\[2mm] \overline{v}_0(x,\alpha) = (7-\alpha)x \end{cases}, \qquad (55)$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = \alpha^2\, \dfrac{x^4}{12} \\[4mm] \overline{v}_1(x,\alpha) = (5-\alpha)\dfrac{x^4}{12} \end{cases}, \qquad (56)$$

$$p^2 : \begin{cases} \underline{v}_2(x,\alpha) = \alpha^3\, \dfrac{x^7}{252} \\[4mm] \overline{v}_2(x,\alpha) = (5-\alpha)^3\, \dfrac{x^7}{252} \end{cases}, \qquad (57)$$

And so on…

As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\[4mm] \overline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases}. \qquad (58)$$

Thus by utilizing above iterative results the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \alpha\, x + \alpha^2\, \dfrac{x^4}{12} + \alpha^3\, \dfrac{x^7}{252} + ... \\[4mm] \overline{u}(x,\alpha) = (7-\alpha)\, x + (7-\alpha)^2\, \dfrac{x^4}{12} + (7-\alpha)^3\, \dfrac{x^7}{252} + ... \end{cases} \qquad (59)$$
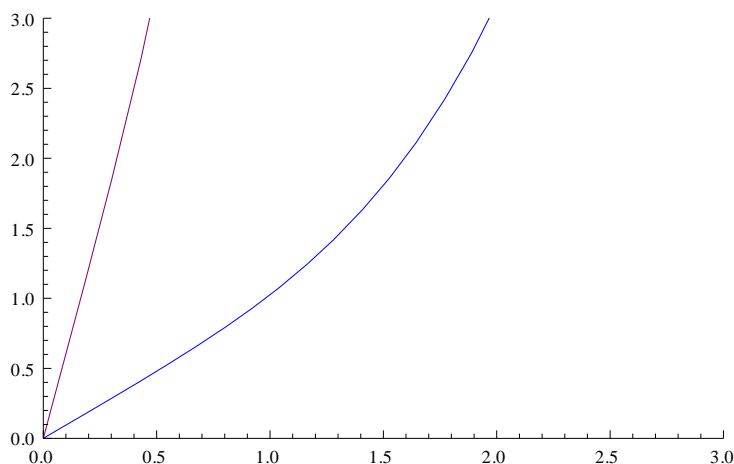
Figure 4 Plot of Solution of Example 4.

Example 4.5 Consider the non-linear fuzzy Fredholm integro-differential equation of 2nd kind

$$u'(x,\alpha) = f(x,\alpha) + \int_0^1 \frac{x^2}{10} tu^2(t,\alpha)dt, \tag{60}$$

with $u(0,\alpha) = (0,0)$,

where $\lambda = 1$, $0 \le t \le 1$, $0 \le \alpha \le 1$, $k(x,t) = \frac{x^2}{10}t$ and $f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ i.e.

$$f(x,\alpha) = ((\alpha - \frac{x^2}{40}\alpha^2), ((2-\alpha) - \frac{x^2}{40}(2-\alpha)^2)).$$

To solve Eq. (60) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v}, p, \alpha) = \underline{v}'(x,\alpha) - (\alpha - \frac{x^2}{40}\alpha^2) - p\int_0^1 \frac{x^2}{10} t\underline{v}^2(t,\alpha)dt = 0 \\ H(\overline{v}, p, \alpha) = \overline{v}'(x,\alpha) - ((2-\alpha) - \frac{x^2}{40}(2-\alpha)^2) - p\int_0^1 \frac{x^2}{10} t\overline{v}^2(t,\alpha)dt = 0 \end{cases} \tag{61}$$

Now by applying Laplace transformation on both sides of Eq. (61), we get

$$\begin{cases} \mathcal{L}\{\underline{v}'(x,\alpha)\} = \mathcal{L}\left\{ (\alpha - \frac{x^2}{40}\alpha^2) + p\int_0^1 \frac{x^2}{10} t\underline{v}^2(t,\alpha)dt \right\} \\ \mathcal{L}\{\overline{v}'(x,\alpha)\} = \mathcal{L}\left\{ ((2-\alpha) - \frac{x^2}{40}(2-\alpha)^2) + p\int_0^1 \frac{x^2}{10} t\overline{v}^2(t,\alpha)dt \right\} \end{cases} \tag{62}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\frac{1}{s}\right)\left\{ \mathcal{L}\left\{ (\alpha - \frac{x^2}{40}\alpha^2) + p\int_0^1 \frac{x^2}{10} t\underline{v}^2(t,\alpha)dt \right\} \right\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\frac{1}{s}\right)\left\{ \mathcal{L}\left\{ ((2-\alpha) - \frac{x^2}{40}(2-\alpha)^2) + p\int_0^1 \frac{x^2}{10} t\overline{v}^2(t,\alpha)dt \right\} \right\} \end{cases} \tag{63}$$

Applying inverse Laplace transformation on both sides of Eq. (63), we obtain

$$
\begin{cases}
\underline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)\left\{\mathcal{L}\left\{(\alpha - \frac{x^2}{40}\alpha^2) + p\int_0^1 \frac{x^2}{10}t\underline{v}^2(t,\alpha)dt\right\}\right\}\right\} \\
\overline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)\left\{\mathcal{L}\left\{((2-\alpha) - \frac{x^2}{40}(2-\alpha)^2) + p\int_0^1 \frac{x^2}{10}t\overline{v}^2(t,\alpha)dt\right\}\right\}\right\}
\end{cases}
\tag{64}
$$

Assume the solution of Eq. (64) can be written as power series in p

$$
\begin{cases}
\underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\
\overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha)
\end{cases}
\tag{65}
$$

Now by putting Eq. (65) in Eq. (64) and by comparing coefficient like power of p, we get

$$
p^0 : \begin{cases}
\underline{v}_0(x,\alpha) = \alpha\,x - \alpha^2\frac{x^3}{120} \\
\overline{v}_0(x,\alpha) = (2-\alpha)x - (2-\alpha)^2\frac{x^3}{120}
\end{cases}
\tag{66}
$$

$$
p^1 : \begin{cases}
\underline{v}_1(x,\alpha) = \alpha^2\frac{x^3}{120} - \alpha^3\frac{x^3}{10800} + \alpha^4\frac{x^3}{3456000} \\
\overline{v}_1(x,\alpha) = (2-\alpha)^2\frac{x^3}{120} - (2-\alpha)^3\frac{x^3}{10800} + (2-\alpha)^4\frac{x^3}{3456000}
\end{cases}
\tag{67}
$$

$$
p^2 : \begin{cases}
\underline{v}_2(x,\alpha) = \alpha^3\frac{x^3}{10800} - \alpha^4\frac{x^3}{3456000} + ... \\
\overline{v}_2(x,\alpha) = (2-\alpha)^3\frac{x^3}{10800} - (2-\alpha)^4\frac{x^3}{3456000} + ...
\end{cases}
\tag{68}
$$

and so on…

As we know the solution is given as

$$
\begin{cases}
\underline{u}(x,\alpha) = \sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\
\overline{u}(x,\alpha) = \sum_{i=0}^{\infty} \overline{v}_i(x,\alpha)
\end{cases}
\tag{69}
$$

Thus by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \alpha\, x - \alpha^2\, \dfrac{x^3}{120} + \alpha^2\, \dfrac{x^3}{120} - \alpha^3\, \dfrac{x^3}{10800} + \alpha^4\, \dfrac{x^3}{3456000} + ... \\[3mm] \overline{u}(x,\alpha) = (2-\alpha)\, x - (2-\alpha)^2\, \dfrac{x^3}{120} + (2-\alpha)^2\, \dfrac{x^3}{120} - (2-\alpha)^3\, \dfrac{x^3}{10800} + (2-\alpha)^4\, \dfrac{x^3}{3456000} + ... \end{cases} \tag{70}$$

And the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = x\alpha \\ \overline{u}(x,\alpha) = x(2-\alpha) \end{cases}. \tag{71}$$

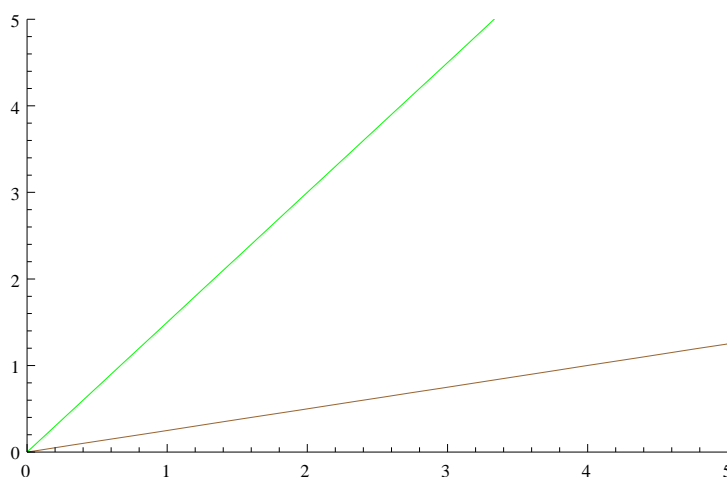**Figure 5. Plot of Solution of Example 5.**

Example 4.6 Consider the non-linear mixed fuzzy Volterra-Fredholm integro differential equation of 2nd kind

$$u'(x,\alpha) = f(x,\alpha) + \int_0^x \int_0^1 u(t,\alpha)\,dt\,dr, \tag{72}$$

with $u(0,\alpha) = (0,0)$,

where $0 \le t \le 1, 0 \le r \le x, 0 \le \alpha \le 1$, $k(x,r,t) = 1$ *and* $f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ *i.e.*

$f(x,\alpha) = ((\alpha^2 + 2\alpha),(3-\alpha))$.

To solve Eq. (60) by LHPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v}, p, \alpha) = \underline{v}'(x,\alpha) - (\alpha^2 + 2\alpha) - p\int_0^x \int_0^1 \underline{v}^2(t,\alpha)\,dt\,dr = 0 \\[3mm] H(\overline{v}, p, \alpha) = \overline{v}'(x,\alpha) - (3-\alpha) - p\int_0^x \int_0^1 \overline{v}^2(t,\alpha)\,dt\,dr = 0 \end{cases} \tag{73}$$

Now by applying Laplace transformation on both sides of Eq. (73), we get

$$\begin{cases} \mathcal{L}\left\{\underline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{(\alpha^2 + 2\alpha) + p\int_0^x \int_0^1 \underline{v}^2(t,\alpha)dtdr\right\} \\ \mathcal{L}\left\{\overline{v}'(x,\alpha)\right\} = \mathcal{L}\left\{(3-\alpha) - p\int_0^x \int_0^{1-2} \overline{v}(t,\alpha)dtdr\right\} \end{cases} \tag{74}$$

Using the differential property of LT, we have

$$\begin{cases} \mathcal{L}\{\underline{v}(x,\alpha)\} = \left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(\alpha^2 + 2\alpha) + p\int_0^x \int_0^1 \underline{v}^2(t,\alpha)dtdr\right\}\right\} \\ \mathcal{L}\{\overline{v}(x,\alpha)\} = \left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(3-\alpha) - p\int_0^x \int_0^{1-2} \overline{v}(t,\alpha)dtdr\right\}\right\} \end{cases} \tag{75}$$

Applying inverse Laplace transformation on both sides of Eq. (75), we obtain

$$\begin{cases} \underline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(\alpha^2 + 2\alpha) + p\int_0^x \int_0^1 \underline{v}^2(t,\alpha)dtdr\right\}\right\}\right\} \\ \overline{v}(x,\alpha) = \mathcal{L}^{-1}\left\{\left(\dfrac{1}{s}\right)\left\{\mathcal{L}\left\{(3-\alpha) - p\int_0^x \int_0^{1-2} \overline{v}(t,\alpha)dtdr\right\}\right\}\right\} \end{cases} \tag{76}$$

Assume the solution of Eq. (76) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{77}$$

Now by putting Eq. (77) in Eq. (76) and by comparing coefficient like power of p, we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = (\alpha^2 + 2\alpha)x \\ \overline{v}_0(x,\alpha) = (3-\alpha)x \end{cases}, \tag{78}$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = (\alpha^2 + 2\alpha)^2 \dfrac{x^2}{6} \\ \overline{v}_1(x,\alpha) = (3-\alpha)^2 \dfrac{x^2}{6} \end{cases}, \tag{79}$$

$$p^2 : \begin{cases} \underline{v}_2(x,\alpha) = (\alpha^2 + 2\alpha)^3 \dfrac{x^2}{24} \\ \overline{v}_2(x,\alpha) = (3-\alpha)^3 \dfrac{x^2}{24} \end{cases}, \tag{80}$$

and so on…
As we know the solution is given as

$$
\begin{cases}
\underline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\[4mm]
\overline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \overline{v}_i(x,\alpha)
\end{cases}, \tag{81}
$$

Thus by utilizing above iterative results the solution is given as

$$
\begin{cases}
\underline{u}(x,\alpha) = (\alpha^2 + 2\alpha)\,x + (\alpha^2 + 2\alpha)^2\,\dfrac{x^2}{6} + (\alpha^2 + 2\alpha)^3\,\dfrac{x^2}{24} + \dots \\[4mm]
\overline{u}(x,\alpha) = (3-\alpha)\,x + (3-\alpha)^2\,\dfrac{x^2}{6} + (3-\alpha)^3\,\dfrac{x^2}{24} + \dots
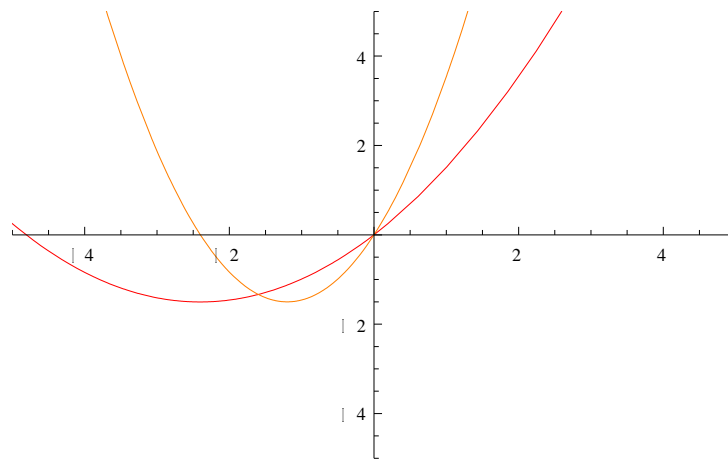\end{cases} \tag{82}
$$

Figure 6. Plot of Solution of Example 6.

5. CONCLUSION

Usually it's difficult to solve fuzzy integro-differential equations analytically. Most probably it's required to obtain the approximate solutions. In this paper we developed a numerical technique like Laplace homotopy perturbation method for finding the solution of linear and non-linear fuzzy integro-differential equations. This technique proved really reliable and affective from achieved results. It gives fast convergence because by utilized less number of iterations we get approximate as well as exact solution.

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