ORIGINAL PAPER

SOLVING TWO POINT BOUNDARY VALUE PROBLEMS BY NUMEROV TYPE METHOD WITH AN INTERNAL BOUNDARY CONDITION

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Manuscript received: 06.06.2016; Accepted paper: 04.10.2016; Published online: 30.03.2017.

Abstract. *The objective of this article is to describe numerical method for the solution of two-point boundary value problems with an internal boundary condition. We present the derivation of the propose Numerov type method. The order of the propose method is four. In numerical section of the article we shall consider some linear and non linear model problem and solve those problems by the propose method to illustrate the efficiency and the accuracy.*
Keywords: *Boundary value problems, Finite difference method, Fourth order method, Internal boundary condition, Second order differential equation.*
2010 for the AMS Subject Classification: *65L05, 65L12.*

1. INTRODUCTION

In this work, we consider second order boundary value problems of the form

$$u''(x) = f(x, u), \qquad a < x < \frac{a+b}{2} \text{ and } \frac{a+b}{2} < x \le b \tag{1}$$

with the boundary condition

$$y(a) = \alpha\,, \;\; y\left(\frac{a+b}{2}\right) = \beta \tag{2}$$

or

$$y\left(\frac{a+b}{2}\right) = \alpha\,, \;\; y(b) = \beta \tag{3}$$

where $\alpha,\ \beta$ are real finite constants. We assume that f is continuous function in $[a, b] \times \mathbb{R}$.

In these problems we are unable to integrate from one side of the domain to the other. So it will difficult to solve these problems numerically by using Numerov method. It is possible to solve the problem away from the internal point at which value is prescribed if we have smooth derivative and solution in the neighborhood of this internal point. To solve problem (1) with boundary conditions (2) or (3) a fitting point method can be applied [1].

Ordinary differential equations are used to understand different kind of physical problems in nature. To understand how nature works, we need to solve these differential

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equations using some analytical methods which satisfy the certain conditions. In most cases it is impossible to solve these modeled problems under realistic conditions analytically. So we prefer other mathematical techniques to solve these problems. One such technique is numerical approximation / solution and a literature regarding the numerical solution of the two-point boundary value problem is given in [2-5]. The existence and uniqueness of the solution for the problem (1) is assumed. The specific assumption on $f(x, y)$ to ensure existence and uniqueness will not be considered in this article however I may refer literature in [3, 6-8]. The specific problems (1), some literary work in [1] but to the best of our knowledge, no method similar to proposed method for the numerical solution have been discussed in the literature to date.

In this article we shall develop a Numerov Type finite difference method for solving problems (1) numerically. The order and accuracy of the proposed method under certain conditions is four. A numerical experiment performed to demonstrate the effectiveness of the method.

We have presented our work in this article as follows. In the next section we will discuss the finite difference method and in Section 3 we will derive our propose method. In Section 4, we have discussed convergence under appropriate condition. The applications of the proposed method to the model problems and illustrative numerical results have been produced to show the efficiency in Section 5. Discussion and conclusion on the performance of the method are presented in Section 6.

2. DEVELOPMENT OF THE FINITE DIFFERENCE METHOD

We define N-1 an odd and finite numbers of nodal points in the domain (a, b), in which the solution of the problem (1) is desired, as $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ using uniform step length h such that $x_i = a + i.h, i = 0\ (1)\ N$ and $x_{\frac{N}{2}} = \frac{a+b}{2}$, as shown below in the Fig. 1.

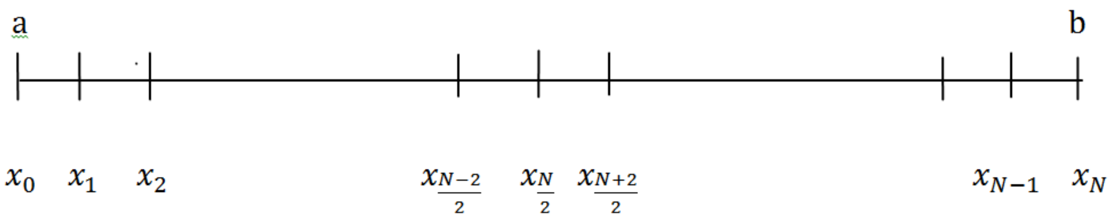

Figure 1.

Let denote the exact solution $u(x)$ at $x = x_i$ by u_i. Also let us denote f_i the approximation of the theoretical value of the force function $f(x_i, \mathrm{u}(x_i))$ at node $x = x_i$, $i = 1\ (1)\ N$. Similarly we can define other notations used in this article i.e. $f_{i\pm1}$ and $u_{i\pm1}$ etc.. Using these notations we can redefine problem (1) at node x = x_i and may be written as,

$$u_i^{''} = f_i \quad , \quad i = 0,1,..,N. \tag{4}$$

Suppose we have to determine u_i, which is an approximation to the numerical value of the theoretical solution $u(x)$ of the problem (1) at the nodal point x $= x_i, i = 0,1, \frac{\mathrm{N}}{2} - 1, \frac{\mathrm{N}}{2} + 1,..,N-1$. Following the ideas in [8, 9] we discretize problem (4) at nodes in [a, b],

$$u_i - 2u_{i+1} + u_{i+2} = \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}), \qquad 0 \le i \le \frac{N}{2} - 3,$$

$$u_i - 2u_{i+1} = -u_{i+2} + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}), \qquad i = \frac{N}{2} - 2,$$

$$u_i + u_{i+2} = 2u_{i+1} + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}), \qquad i = \frac{N}{2} - 1,$$

$$-2u_{i+1} + u_{i+2} = -u_i + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}), \qquad i = \frac{N}{2}, \tag{5}$$

$$u_i - 2u_{i+1} + u_{i+2} = \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}), \qquad \frac{N}{2} + 1 \le i \le N - 3,$$

$$u_i - 2u_{i+1} = -u_{i+2} + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}), \qquad i = N - 2.$$

3. DERIVATION OF THE FINITE DIFFERENCE METHOD

In this section we discuss the derivation of the proposed finite difference method (5). Consider following linear combination of $u_i'', u_{i+1}'', u_{i+2}'', u_i, u_{i+1}$ and u_{i+2},

$$a_i u_i + a_{i+1}u_{i+1} + a_{i+2}u_{i+2} + h^2\left(c_i u_i'' + c_{i+1}u_{i+1}'' + c_{i+2}u_{i+2}''\right) = 0 \tag{6}$$

where $a_i's$ and $c_i's$ are constant. Let us assume that $u^{(5)}(x)$ is continuous in [a,b] and $u^{(6)}(x)$ exist in (a,b). To determine the constants $a_i's$ and $c_i's$, let us write expansion of each term of (6) in Taylor series about point x_i. Compare the coefficients of $h^p, p = 01,2,..,5$ in the expansion of (6). We find a system of linear equations in $a_i's$ and $c_i's$. Solving the system of equation so obtained, we find that

$$(a_i, a_{i+1}, a_{i+2}, c_i, c_{i+1}, c_{i+2}) = \frac{1}{12}(12, -24, 12, -1, -10, -1) \tag{7}$$

Thus,

$$u_i - 2u_{i+1} + u_{i+2} = \frac{h^2}{12}\left(u_i'' + 10u_{i+1}'' + u_{i+2}''\right) + \frac{h^6}{240}u^{(6)}(x_i + \theta_i h) \tag{8}$$

and $0 \le \theta_i \le 1$.

Using (4) in (8), we have

$$u_i - 2u_{i+1} + u_{i+2} = \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + \frac{h^6}{240}u^{(6)}(x_i + \theta_i h) \tag{9}$$

and $1 \leq\ i < \frac{N}{2}$. Similarly we can derive other equation by writing Taylor series expansion about the point $x_{i+1},\ \frac{N}{2} \leq\ i \leq N-2$.

Thus we have obtained

$$
\begin{aligned}
u_i - 2u_{i+1} + u_{i+2} &= \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + t_i\,, && 0 \leq i \leq \frac{N}{2} - 3,\\
u_i - 2u_{i+1} &= -u_{i+2} + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + t_i\,, && i = \frac{N}{2} - 2,\\
u_i + u_{i+2} &= 2u_{i+1} + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + t_i\,, && i = \frac{N}{2} - 1,\\
-2u_{i+1} + u_{i+2} &= -u_i + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + t_i\,, && i = \frac{N}{2}, \qquad (10)\\
u_i - 2u_{i+1} + u_{i+2} &= \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + t_i\,, && \frac{N}{2} + 1 \leq i \leq N - 3,\\
u_i - 2u_{i+1} &= -u_{i+2} + \frac{h^2}{12}(f_i + 10f_{i+1} + f_{i+2}) + t_i\,, && i = N - 2.
\end{aligned}
$$

where,

$$
t_i = \frac{h^6}{480}\begin{cases} 2u^{(6)}(x_i + \theta_i h), & 0 \leq \theta_i \leq 1, \quad 1 \leq\ i < \dfrac{N}{2} \\ u^{(6)}(x_{i+1} + \theta_{i+1} h), & 0 \leq \theta_{i+1} \leq 1, \quad \dfrac{N}{2} \leq\ i \leq N-2 \end{cases}
$$

Thus we will obtain our proposed difference method (5) for the numerical solution of the problem (1) by truncating the terms t_i in (10). Thus, the method consists in solving the system of $(N-1) \times (N-1)$ equations (5) in $u_i, i = 0,1,..,\frac{N}{2} - 1, \frac{N}{2} + 1, \dots, N-1$ i.e. u_i an approximation of the theoretical solution $u(x_i)$ of the problem (1) .

4. THE CONVERGENCE ANALYSIS OF THE METHOD

In this section we analysis method (5) for the purpose of its convergence. So we will consider following linear problem,

$$u''(x) = f(x) \tag{11}$$

subject to the boundary conditions $u\left(\frac{a+b}{2}\right) = \alpha$ and $u(b) = \beta$. We will solve the problem (11) by proposed method (10) and we write in the matrix form. Let $\boldsymbol{U}$ be the exact solution of the system of equations (10), thus we have

$$\boldsymbol{A} \cdot \boldsymbol{U} = \boldsymbol{C_0} \cdot \boldsymbol{U}^* + \frac{h^2}{12}\boldsymbol{C_1} \cdot \boldsymbol{f} + \boldsymbol{T} \tag{12}$$

where

$$\boldsymbol{A}=\begin{pmatrix}1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & ..\\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0\\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & ..\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2\end{pmatrix}_{N-1\times N-1}$$

$$\boldsymbol{C_0}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ .. & .. & .. & .. & .. & .. & .. & .. & ..\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ .. & .. & .. & .. & 0 & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & .. & .. & .. & 1\end{pmatrix}_{N-1\times N+1},$$

$$\boldsymbol{C_1}=\begin{pmatrix}1 & 10 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 10 & 1 & 0 & 0 & 0\\ .. & ... & .. & .. & .. & .. & ..\\ 0 & 0 & 0 & 1 & 10 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 10 & 1\end{pmatrix}_{N-1\times N+1}$$

$$\boldsymbol{U}=\begin{pmatrix}U_0 & U_1 & U_2 & .. & U_{\frac{N}{2}-1} & U_{\frac{N}{2}+1} & .. & U_{N-2} & U_{N-1}\end{pmatrix}^T_{1\times N-1},$$

$$\boldsymbol{U}^*=\begin{pmatrix}U_0 & .. & U_{\frac{N}{2}} & .. & U_N\end{pmatrix}^T_{1\times N+1},\ \boldsymbol{f}=\begin{pmatrix}f_0 & .. & f_{\frac{N}{2}} & .. & f_N\end{pmatrix}^T_{1\times N+1},$$

$$\boldsymbol{T}=(t_0 \quad .. \quad .. \quad .. \quad t_{N-2})^T_{1\times N+1}.$$

Let $\boldsymbol{u}$ be the approximate solution of the system of equations (10), thus we have

$$\boldsymbol{A}\cdot\boldsymbol{u}=\boldsymbol{C_0}\cdot\boldsymbol{u}^*+\frac{h^2}{12}\boldsymbol{C_1}\cdot\boldsymbol{f} \tag{13}$$

where

$$\boldsymbol{u}=\begin{pmatrix}u_0 & u_1 & u_2 & .. & u_{\frac{N}{2}-1} & u_{\frac{N}{2}+1} & .. & u_{N-2} & u_{N-1}\end{pmatrix}^T_{1\times N-1},$$

$\boldsymbol{u}^*=\begin{pmatrix}u_0 & .. & u_{\frac{N}{2}} & .. & u\end{pmatrix}^T_{1\times N+1}$.

Let us define an error in exact and approximate solution of (10), $e_i=U_i-u_i$ at node x_i, so we have $\boldsymbol{e}=\begin{pmatrix}e_0 & e_1 & e_2 & .. & e_{\frac{N}{2}-1} & e_{\frac{N}{2}+1} & .. & e_{N-2} & e_{N-1}\end{pmatrix}^T_{1\times N-1}$. Thus from (12) and (13), we have,

$$\boldsymbol{Ae}=\boldsymbol{T} \tag{14}$$

Thus from (14), we conclude that the convergence of the propose difference method (5) depends on the invertibility of the matrix $\boldsymbol{A}$. To simplify we determine $\boldsymbol{A^{-1}}$=$(b_{l,m})_{N-1\times N-1}$ explicitly where,

$$b_{l,m} = \begin{cases} (l-1)b_{2,m} - (l-2)b_{1,m}\ , & l \le m \le \dfrac{N}{2}, l \le \dfrac{N}{2} \le m \\ \dfrac{(N-2l)}{4}\Big((N-2)b_{2,m} - (N-4)b_{1,m}\Big)\ , & \dfrac{N}{2} < l \le m \\ (N-l)b_{N-1,m}\ , & l > m \end{cases}$$

and

$$b_{1,m} = \begin{cases} m\ , & m \le \dfrac{N}{2} \\ N-m\ , & m > \dfrac{N}{2} \end{cases}, \qquad b_{2,m} = \begin{cases} m-1\ , & m \le \dfrac{N}{2} \\ \dfrac{(N-m)(N-2)}{N} & , m > \dfrac{N}{2} \end{cases},$$

$$b_{N-1,m} = \begin{cases} 0\ , & m \le \dfrac{N}{2} \\ \dfrac{N-2m}{N} & ,\ m > \dfrac{N}{2} \end{cases}.$$

The row sum of $\boldsymbol{A^{-1}}$ is

$$\sum_{m=1}^{N-1} b_{l,m} = \frac{N^2 - 3Nl + 2l^2}{4}$$

and

$$\max_{1\le l\le N-1} \sum_{m=1}^{N-1} |b_{l,m}| = b_{1,m} = \frac{N^2 - 3N + 2}{4} \tag{15}$$

Hence from (15) we obtain

$$\|\boldsymbol{A^{-1}}\| = \max_{1\le l\le N-1} \sum_{m=1}^{N-1} |b_{l,m}| \le \frac{1}{4}\frac{(b-a)^2}{h^2} \tag{16}$$

Thus from equation (14) and (16), we have

$$\|\boldsymbol{e}\| \le \frac{1}{4}\frac{(b-a)^2}{h^2}\|\boldsymbol{T}\| \tag{17}$$

Let $M = \max_{a\le x\le b} |u^{(6)}(x)|$, then from (10) and (17) we have

$$\|\boldsymbol{e}\| \le \frac{Mh^4}{960}(b-a)^2 = O(h^4) \tag{18}$$

Thus we conclude from equation (18) that $\|\boldsymbol{e}\| \to 0$ as $h \to 0$. Thus we have proved theoretically that propose method is convergent and order of convergence is at least four.

5. NUMERICAL EXPERIMENTS

In this section, we have applied the proposed method (5) to solve numerically three different linear and nonlinear model problems. We have used Gauss-Seidel and Newton Raphson method respectively to solve the system of linear and nonlinear equations arises from equation (5). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. Let u_i , the numerical value calculated by formulae (3), an approximate value of the theoretical solution $u(x)$ at the grid point $x = x_i$. We have used following formula in calculation of MAE, the maximum absolute error

$$MAE(\mathrm{u}) = \max_{0\le i \le N-1} |u(x_i) - u_i|$$

are shown in Tables 1-3, for different value of N. Also we have shown number of iterations performed to reach the result.

Example 1. Consider the following linear two-point problem

$$u''(x) = \frac{1}{x^2}u(x) - \frac{1}{x} \ , \quad 1 \le x < \frac{3}{2} \cup \frac{3}{2} < x < 2$$

with the boundary condition $y(2) = 0$ and $y\left(\frac{3}{2}\right) = \frac{-27}{152}$. The maximum absolute error presented in exact solution $u(x) = \frac{1}{2}x - \frac{5}{38}x^2 - \frac{18}{19}\frac{1}{x}$ in Tables 1.

Example 2. Consider the following non- linear two-point problem

$$u''(x) = \frac{1}{2}(1.0 + u(x) + x)^3 \ , \qquad 0 \le x < \frac{1}{2} \cup \frac{1}{2} < x < 1$$

with the boundary condition $y(1) = 0$ and $y\left(\frac{1}{2}\right) = \frac{1}{3}$. The maximum absolute error presented in exact solution $u(x) = \frac{2}{2-x} - x - 1$ in Table 2.

Example 3. Consider the following non- linear two-point problem

$$u''(x) = \frac{3}{2}u^2(x) \ , \qquad 0 \le x < \frac{1}{2} \cup \frac{1}{2} < x < 1$$

with the boundary condition $y(1) = 2$ and $y\left(\frac{1}{2}\right) = \frac{16}{5}$. The maximum absolute error presented in exact solution $u(x) = \frac{4}{1+x^2}$ in Table 3.

Table 1. Maximum absolute error $|u(x_i) - u_i|$ in example 1.

	N		
	8	16	32
MAE	.16030512(-4)	.10101419(-5)	.36076494(-6)
Ite.	4	8	15

Table 2. Maximum absolute error $|u(x_i) - u_i|$ in example 2 .

	N		
	8	16	32
MAE	.21871843(-4)	.98513237(-6)	.10369144(-6)
Ite.	14	24	18

Table 3. Maximum absolute error $|u(x_i) - u_i|$ in example 3 .

	N		
	8	16	32
MAE	.41484833(-3)	.28610229(-4)	.95542600(-6)
Ite.	8	9	16

6. CONCLUSION

A finite difference method is developed and discussed for the numerical solution of system of two-point one internal and one at boundary value problems. We conclude from derivation and discussion that the method (5) is at least of fourth order. Numerical results are in good agreement to the theoretical results.

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