ORIGINAL PAPER

GEODESICS, LINE OF CURVATURES AND ASYMPTOTIC CURVES VERSUS RELAXED ELASTIC LINES ON AN ORIENTED SURFACE

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Abstract. *Let $\alpha(s)$ be an arc on a connected oriented surface S in E^3, parameterized by arc length s, with curvature κ and length l. The total square curvature K of α is defined by $K=\int_0^l \kappa^2 ds$. The arc α is called a relaxed elastic line if it is an extremal for the variational problem of minimizing the value of K within the family of all arcs of length l on S having the same initial point and initial direction as α. In this study, we show that a geodesic is a relaxed elastic line if and only if it is planar and an asymptotic curve cannot be a relaxed elastic line. Also, we obtain a criterion for a line of curvature to be a relaxed elastic line.*

Keywords: *Relaxed elastic line, geodesic, line of curvature, asymptotic curve.*

1. INTRODUCTION

Let $\alpha(s)$ denote an arc on a connected oriented surface S in E^3, parameterized by arc length $s, 0 \le s \le l,$ with curvature $\kappa(s)$. The total square curvature K of α is defined by

$$K=\int_0^l \kappa^2 ds.$$

An arc is called a relaxed elastic line if it is an extremal for the variational problem of minimizing the value of K within the family of all arcs of length l on S having the same initial point and initial direction as α [1]. In [1] authors derive the intrinsic equations for a relaxed elastic line on an oriented surface. Hilbert and Cohn-Vossen [2] incorrectly suggested a flexible knitting needle, constrained to conform to a surface, as one model for a geodesic on a surface. This model actually gives a relaxed elastic line on the surface, and is not generally a geodesic unless the surface lies in a plane or on a sphere [1]. Physical motivation for study of the problem of elastic lines on surfaces may be found in the nucleosome core particle [3, 4]. There are several papers about this kind of minimization problems [5-8]. Santaló studied the minimization of Frenet first curvature dependent energies for spaces of curves constrained to lie in a surface of the Euclidean 3-space [9]. Garay and Pauley analyze an extension of that problem by considering energy functionals, determined by Lagrangians which depend not only on the curvature but also on the torsion, acting on spaces of surface contained clamped curves of real 3-space forms [10].

In [11] authors handled the problem of minimizing the total square torsion on an oriented surface and defined the relaxed elastic line of second kind. However, they only gave

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Euler-Lagrange equations for this problem. Bayram and Kasap obtained a differential equation and boundary conditions for a relaxed elastic line of second kind on an oriented surface in Euclidean and Minkowski 3-space [12, 13].
In this paper, we show that a geodesic is a relaxed elastic line if and only if it is planar. Also, we observe that an asymptotic curve cannot be a relaxed elastic line and obtain a criterion for a line of curvature to be a relaxed elastic line.

2. RELAXED ELASTIC LINES

Let $\alpha(s)$ denote an arc on a connected oriented surface S in E^3, parameterized by arc length s, $0 \le s \le l$, with curvature $\kappa(s)$. The total square curvature K of α is defined by
$K = \int_0^l \kappa^2 ds.$

Definition 1: The arc α is called a relaxed elastic line if it is an extremal for the variational problem of minimizing the value of K within the family of all arcs of length l on S having the same initial point and initial direction as α [1].

It is obvious that any arc of a straight line $(\kappa \equiv 0)$ on S is a relaxed elastic line. Thus, it is so natural to assume that $\kappa \neq 0,\ \forall s$ on α. On α, let $T(s) = \alpha'(s)$ denote the unit tangent vector field, $n(s)$ denote the unit surface normal vector field to S and $Q(s) = nxT$. Then, $\{T, Q, n\}$ gives an orthonormal basis on α and $\{T, Q\}$ gives a basis for the vectors tangent to S at $\alpha(s)$. The frame $\{T, Q, n\}$ is called Darboux frame. Derivative equations for the Darboux frame is

$$\begin{pmatrix} T' \\ Q' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ n \end{pmatrix},$$

where κ_g, κ_n and τ_g are geodesic curvature, normal curvature and geodesic torsion, respectively [14].

Theorem 2. The intrinsic equations for a relaxed elastic line of length l on a connected oriented surface in E^3 are given by the differential equation

$$(DE) \quad 2\kappa_g'' - 4\tau_g\kappa_n' - 2\kappa_n\tau_g' + \kappa_g\left[\kappa_g^2 + \kappa_n^2 - 2\tau_g^2 + \kappa_n^2(l)\right] = 0$$

together with the boundary conditions

$$(BCI) \quad \kappa_g(l) = 0$$

$$(BCII) \quad \kappa_g'(l) = 2\kappa_n(l)\tau_g(l)$$

at the free end [1].

Theorem 3. On a general surface, an arc of a geodesic is a relaxed elastic line if and only if it satisfies $\kappa_n^2\tau_g = 0$ [1].

Proposition 4: An arc of an asymptotic line on a catenoid cannot be a relaxed elastic line [1].

Theorem 5. A geodesic with nonvanishing curvature on a general surface is a relaxed elastic line if and only if it is planar.

Proof: Let α be a geodesic with $\kappa \neq 0$ on a surface. Since α is a geodesic $\kappa_g \equiv 0$. By the formula $\kappa^2 = \kappa_g^2 + \kappa_n^2$ and $\kappa \neq 0$ we have $\kappa_n \neq 0$ along α. Thus using Theorem 3, α is a relaxed elastic line if and only if $\tau_g \equiv 0$. Since $\tau = \tau_g + \frac{\kappa_g\kappa_n' - \kappa_g'\kappa_n}{\kappa_g^2+\kappa_n^2}$, we get $\tau = \tau_g = 0$, that is α is a plane curve.

Theorem 6. An asymptotic curve with nonvanishing curvature on a general surface cannot be a relaxed elastic line.

Proof: Let α be an asymptotic curve with $\kappa \neq 0$ on a surface. Since α is an asymptotic curve $\kappa_n \equiv 0$. By the formula $\kappa^2 = \kappa_g^2 + \kappa_n^2$ and $\kappa \neq 0$ we have $\kappa_g \neq 0$ along α.. This implies that α does not satisfy boundary condition (BC I) in Theorem 2. Thus α cannot be relaxed elastic line.

Lemma 7. For a planar line of curvature $\frac{\kappa_n}{\kappa_g} = \text{constant}$.

Proof: Let α be a planar line of curvature, that is $\tau = \tau_g = 0$. Since $\tau = \tau_g + \frac{\kappa_g\kappa_n' - \kappa_g'\kappa_n}{\kappa_g^2+\kappa_n^2}$, we get

$$\frac{\kappa_g\kappa_n' - \kappa_g'\kappa_n}{\kappa_g^2+\kappa_n^2} = 0 \Rightarrow \kappa_g\kappa_n' - \kappa_g'\kappa_n = 0 \Rightarrow \left(\frac{\kappa_n}{\kappa_g}\right)' = 0 \Rightarrow \frac{\kappa_n}{\kappa_g} = \text{constant.}$$

Theorem 8. Let α be a planar line of curvature. α is a relaxed elastic line if and only if it satisfies the differential equation

$$(DE) \quad 2\kappa_g'' + \kappa_g^3\left(c^2+1\right) = 0$$

together with the boundary conditions

$$(BCI) \quad \kappa_g(l)=0$$
$$(BCII) \quad \kappa_g'(l)=0,$$

where $c=\frac{\kappa_n}{\kappa_g}=\text{constant}$.

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