ORIGINAL PAPER

BINOMIAL TRANSFORMS OF QUADRAPELL SEQUENCES AND QUADRAPELL MATRIX SEQUENCES

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Abstract. *In this paper, we apply the binomial transform to the Quadrapell sequence. We investigate some interesting properties the so-obtained new sequence. Moreover we define the matrix sequence of the Quadrapell numbers. Then, we give properties of these new matrix sequences. Finally, we apply binomial transform to Quadrapell matrix sequence and give some algebraic properties of the new sequence.*

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1. INTRODUCTION

Special numbers such as Fibonacci, Lucas and Pell numbers have been long interested mathematicians for their intrinsic theory and applications. For rich applications of these numbers in science and nature [1, 2]. In [3], Taşcı defined Quadrapell numbers by the following recurrence relation for $n \geq 4$,

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4} \tag{1.1}$$

with initial values $D_0 = D_1 = D_2 = 1$ and $D_3 = 2$. The author considered the characteristic equation of Quadrapell recurrence relation

$$x^4 - x^2 - 2x - 1$$

and gave the roots of this equation $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\gamma = \frac{-1+\sqrt{3}i}{2}$ and $\delta = \frac{-1-\sqrt{3}i}{2}$ respectively.

Moreover he gave generating function, Binet formula and summation formulas for the Quadrapell numbers. On the other hand, the matrix sequences have taken so much interest for different type of numbers [4-6]. For instance, in [4], authors defined matrix generalization for Fibonacci and Lucas numbers. In [5], the authors gave generalizations for (s, t) Pell and (s, t) Pell-Lucas numbers. In [6], the authors defined the matrix sequences in terms of Padovan and Perrin numbers.

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In addition, some matrix based transforms can be introduced for a given sequences. Binomial transform is one of these transforms, there are also other ones such as rising and falling binomial transforms [7-12]. In [7], the binomial transform B of the integer sequence $A = \{ a_n\}$ which is denoted by $B(A) = \{ b_n\}$ and defined by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.$$

In [10], Falcon and Plaza applied the binomial transform to the k –Fibonacci sequences. In [11], Bhadouria et al. investigated binomial transform of k –Lucas sequences using the similar method to [10]. In [12], Yılmaz and Taşkara studied the binomial transform to Padovan and Perrin matrix sequences.

Motivated by the works referred to above, we shall investigate in the present paper the binomial transform of Quadrapell numbers and define Quadrapell matrix sequences. Moreover we apply binomial transform Quadrapell matrix sequences. The paper is organized as follows. In section 2, we apply binomial transform of Quadrapell sequence and give some properties of them. In section 3, we define Quadrapell matrix sequences and give recurrence relation, Binet formula, generating function. In section 4, we present binomial transform of Quadrapell matrix sequence and give some algebraic relations on it.

2. BIOMIAL TRANSFORM OF QUADRAPELL SEQUENCE

In this section, we focus on binomial transform of Quadrapell sequence to get some important results.

Definition 1. Let D_n be the Quadrapell numbers. The binomial transform of Quadrapell sequence is

$$b_n = \sum_{i=0}^{n} \binom{n}{i} D_i \,.$$

Lemma 1. The binomial transform of the Quadrapell sequence verifies the relation

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (D_i + D_{i+1}). \tag{2.1}$$

Proof. By using the Definition 1 and well known binomial equality

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1},$$

we obtain

$$b_{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} D_i$$

$$= \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] D_i + D_0$$

$$= \sum_{i=0}^{n} \binom{n}{i} D_i + \sum_{i=0}^{n} \binom{n}{i} D_{i+1}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (D_i + D_{i+1}).$$

Note that Equation (2.1) can also be written as

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} D_{i+1}.$$

Theorem 1. For $n > 0$, recurrence relation of sequences $\{ b_n\}$ is

$$b_{n+3} = 4b_{n+2} - 5b_{n+1} + 4b_n - b_{n-1} \tag{2.2}$$

with initial conditions $b_0 = 1$, $b_1 = 2$, $b_2 = 4$, $b_3 = 9$.

Proof. Let be

$$b_{n+3} = Ab_{n+2} + Bb_{n+1} + Cb_n + Db_{n-1}$$

If we take $n = 1, 2, 3$ and 4, we take the system,

$$\begin{cases} b_4 = Ab_3 + Bb_2 + Cb_1 + Db_0 \\ b_5 = Ab_4 + Bb_3 + Cb_2 + Db_1 \\ b_6 = Ab_5 + Bb_4 + Cb_3 + Db_2 \\ b_7 = Ab_6 + Bb_5 + Cb_4 + Db_3 \end{cases}$$

By considering Definition 1 and Cramer rule for the system, we obtain

$$A = 4,\ B = -5,\ C = 4,\ D = -1$$

which is completed the proof.

Theorem 2. The generating function of the binomial transform for $\{D_n\}$ is

$$b(x) = \frac{1 - 2x + x^2 - x^3}{1 - 4x + 5x^2 - 4x^3 + x^4}.$$

Proof. Assume that

$$b(x) = \sum_{i=0}^{\infty} b_i x^i$$

is the generating function of the binomial transform for $\{D_n\}$. Then

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$4xb(x) = 4b_0 x + 4b_1 x^2 + 4b_2 x^3 + \cdots$$

$$5x^2 b(x) = 5b_0 x^2 + 5b_1 x^3 + 5b_2 x^4 + \cdots$$

$$4x^3 b(x) = 4b_0 x^3 + 4b_1 x^4 + 4b_2 x^5 + \cdots$$

$$x^4 b(x) = b_0 x^4 + b_1 x^5 + b_2 x^6 + \cdots$$

Since from Equation (2.2), we obtain

$$(1 - 4x + 5x^2 - 4x^3 + x^4)\, b(x) = 1 - 2x + x^2 - x^3,$$

and hence the generating function for the binomial transform of the $\{b_n\}_{n=0}^{\infty}$ is

$$b(x) = \frac{1 - 2x + x^2 - x^3}{1 - 4x + 5x^2 - 4x^3 + x^4} \quad .$$

We note that, $b(x)$ may be obtained from the generating function of the Quadrapell sequence,

$$g(x) = \frac{1 + x - x^3}{1 - x^2 - 2x^3 - x^4} \quad .$$

It is seen by using the following result proved by Prodinger [9]:

$$b(x) = \frac{1}{1 - x}\, g\left(\frac{x}{1 - x}\right).$$

Theorem 3. The Binet formula for b_n is

$$b_n = \frac{3}{2}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)-\left(\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\right)+\frac{(-1)^{n+1}}{2}\left(\frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta}\right).$$

Proof. Note that the generating function is

$$b(x) = \frac{1-2x+x^2-x^3}{1-4x+5x^2-4x^3+x^4} \quad .$$

It is easily seen that

$$b(x) = \frac{\frac{3}{2}x}{x^2-3x+1}-\frac{1}{x^2-3x+1}-\frac{\frac{1}{2}x}{x^2-x+1}$$

$$= \sum_{n=0}^{\infty}\left[\frac{3}{2}\left(\frac{(\alpha+1)^n-(\beta+1)^n}{\alpha-\beta}\right)-\left(\frac{(\alpha+1)^{n+1}-(\beta+1)^{n+1}}{\alpha-\beta}\right)-\frac{1}{2}\left(\frac{(\gamma+1)^n-(\delta+1)^n}{\gamma-\delta}\right)\right]x^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{3}{2}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)-\left(\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\right)+\frac{(-1)^{n+1}}{2}\left(\frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta}\right)\right]x^n$$

Thus, by the equality of generating function, we get

$$b_n = \frac{3}{2}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)-\left(\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\right)+\frac{(-1)^{n+1}}{2}\left(\frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta}\right).$$

Corollary 1. For $n > 0$, one has

$$b_n = \frac{3}{2}F_{2n}-F_{2n+2}+\frac{(-1)^{n+1}}{2}c_{2n}$$

where F_n denoted Fibonacci sequences and $c_n = \frac{\gamma^n-\delta^n}{\gamma-\delta}$.

3. THE MATRIX SEQUENCE OF QUADRAPELL NUMBERS

In this section, we will define matrix sequence of Quadrapell numbers. Moreover we will also present recurrence relation, Binet formula and generating function. Firstly we will define the Quadrapell matrix sequence.

Definition 2. Let n be a natural number. The Quadrapell matrix sequence is defined by

$$\mathfrak{D}_n = \mathfrak{D}_{n-2} + 2\mathfrak{D}_{n-3} + \mathfrak{D}_{n-4}, \tag{3.1}$$

with initial conditions

$$\mathfrak{D}_0 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{pmatrix}, \mathfrak{D}_1 = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{pmatrix},$$

$$\mathfrak{D}_2 = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \mathfrak{D}_3 = \begin{pmatrix} 2 & 4 & 5 & 9 \\ 1 & 2 & 4 & 5 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

The following theorem, we present n^{th} general term of the sequence in (3.1) via Quadrapell numbers.

Theorem 4. Let $\mathfrak{D}_n$ be the matrix sequence of the Quadrapell numbers. For $n \geq 0$,

$$\mathfrak{D}_n = \begin{pmatrix} D_n & D_{n+1} & D_{n+2} & D_{n+3} \\ D_{n-1} & D_n & D_{n+1} & D_{n+2} \\ D_{n-2} & D_{n-1} & D_n & D_{n+1} \\ D_{n-3} & D_{n-2} & D_{n-1} & D_n \end{pmatrix}. \tag{3.2}$$

Proof. We prove this by induction on n. First of all, let us consider (1.1) and then $D_{-3} = -2$, $D_{-2} = 2$, $D_{-1} = -1$, $D_0 = 1$, $D_1 = 1$, $D_2 = 1$, $D_3 = 2$. These equalities which gives the following first step of the induction:

$$\mathfrak{D}_0 = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 \\ D_{-1} & D_0 & D_1 & D_2 \\ D_{-2} & D_{-1} & D_0 & D_1 \\ D_{-3} & D_{-2} & D_{-1} & D_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{pmatrix}.$$

Assuming the equation in (3.2) holds for all positive integer $n = k$, we will prove it for $n = k + 1$. Thus,

$$\mathfrak{D}_{k-1} + 2\mathfrak{D}_{k-2} + \mathfrak{D}_{k-3} = \begin{pmatrix} D_{k-1} & D_k & D_{k+1} & D_{k+2} \\ D_{k-2} & D_{k-1} & D_k & D_{k+1} \\ D_{k-3} & D_{k-2} & D_{k-1} & D_k \\ D_{k-4} & D_{k-3} & D_{k-2} & D_{k-1} \end{pmatrix} + 2\begin{pmatrix} D_{k-2} & D_{k-1} & D_k & D_{k+1} \\ D_{k-3} & D_{k-2} & D_{k-1} & D_k \\ D_{k-4} & D_{k-3} & D_{k-2} & D_{k-1} \\ D_{k-5} & D_{k-4} & D_{k-3} & D_{k-2} \end{pmatrix}$$

$$+\begin{pmatrix} D_{k-3} & D_{k-2} & D_{k-1} & D_k \\ D_{k-4} & D_{k-3} & D_{k-2} & D_{k-1} \\ D_{k-5} & D_{k-4} & D_{k-3} & D_{k-2} \\ D_{k-6} & D_{k-5} & D_{k-4} & D_{k-3} \end{pmatrix}$$

$$= \begin{pmatrix} D_{k+1} & D_{k+2} & D_{k+3} & D_{k+4} \\ D_k & D_{k+1} & D_{k+2} & D_{k+3} \\ D_{k-1} & D_k & D_{k+1} & D_{k+2} \\ D_{k-2} & D_{k-1} & D_k & D_{k+1} \end{pmatrix}$$

$$= \mathfrak{D}_{k+1}$$

Hence the proof is completed.

Theorem 5. Let n be a natural number. Binet formula for the Quadrapell matrix sequence as the form

$$\mathfrak{D}_n = Ax^n + By^n + Cz^n + Dt^n$$

where

$$A = \frac{\mathfrak{D}_3 + (\mathfrak{D}_2 + \mathfrak{D}_1)x - \mathfrak{D}_0 y}{(x-t)(x-y)(x-z)},$$

$$B = \frac{\mathfrak{D}_3 + (\mathfrak{D}_2 + \mathfrak{D}_1)y - \mathfrak{D}_0 x}{(t-y)(x-y)(y-z)},$$

$$C = \frac{\mathfrak{D}_3 + \mathfrak{D}_2 z + \mathfrak{D}_1(t-1) + \mathfrak{D}_0 t}{(z-t)(x-z)(y-z)},$$

$$D = \frac{\mathfrak{D}_3 + \mathfrak{D}_2 t + \mathfrak{D}_1(z-1) + \mathfrak{D}_0 z}{(t-x)(t-y)(t-z)},$$

such that x, y, z and t are roots of characteristic equation of (3.1).

Proof. Let us consider equation (3.1). The roots of the characteristic equation of (3.1) are x, y, z, t. Thus the its general solution of it is given by

$$\mathfrak{D}_n = Ax^n + By^n + Cz^n + Dt^n.$$

Using initial conditions in Definition 1 and applying linear algebra operations, we get the A, B, C and D.

In [3], the author obtained the Binet formula for the Quadrapell numbers. Now as a different approximation in the following corollary, we give the Binet formula by means of the matrix sequence.

Corollary 2. The Binet formula for Quadrapell numbers in terms of matrix sequence is given by

$$D_n = \frac{x^n(2x-y+2)}{(x-t)(x-y)(x-z)} + \frac{y^n(2y-x+2)}{(t-y)(x-y)(y-z)} + \frac{z^n(2t+z+1)}{(z-t)(x-z)(y-z)} + \frac{t^n(2z+t+1)}{(t-x)(t-y)(t-z)} \; .$$

Proof. If we use the Theorem 5, we can write

$$\mathfrak{D}_n = Ax^n + By^n + Cz^n + Dt^n$$

$$= \frac{x^n}{(x-t)(x-y)(x-z)}\begin{pmatrix} 2x-y+2 & 3x-y+4 & 6x-y+5 & 9x-2y+9 \\ 2x+y+1 & 2x-y+2 & 3x-y+4 & 6x-y+5 \\ 1-2y & 2x+y+1 & 2x-y+2 & 3x-y+4 \\ x+2y+1 & 1-2y & 2x+y+1 & 2x-y+2 \end{pmatrix}$$

$$+\frac{y^n}{(t-y)(x-y)(y-z)}\begin{pmatrix} 2y-x+2 & 3y-x+4 & 6y-x+5 & 9y-2x+9 \\ x+2y+1 & 2y-x+2 & 3y-x+4 & 6y-x+5 \\ 1-2x & x+2y+1 & 2y-x+2 & 3y-x+4 \\ 2x+y+1 & 1-2x & x+2y+1 & 2y-x+2 \end{pmatrix}$$

$$+\frac{z^n}{(z-t)(x-z)(y-z)}\begin{pmatrix} 2t+z+1 & 2t+2z+3 & 3t+4z+3 & 6t+5z+5 \\ z & 2t+z+1 & 2t+2z+3 & 3t+4z+3 \\ t+z+2 & z & 2t+z+1 & 2t+2z+3 \\ -z-1 & t+z+2 & z & 2t+z+1 \end{pmatrix}$$

$$+\frac{t^n}{(t-x)(t-y)(t-z)}\begin{pmatrix} t+2z+1 & 2t+2z+3 & 4t+3z+3 & 5t+6z+5 \\ t & t+2z+1 & 2t+2z+3 & 4t+3z+3 \\ t+z+2 & t & t+2z+1 & 2t+2z+3 \\ -t-1 & t+z+2 & t & t+2z+1 \end{pmatrix}.$$

Also, by using Theorem 5, and if we compare the 1^{st} row and 1^{st} column entries, we obtain

$$D_n = \frac{x^n(2x-y+2)}{(x-t)(x-y)(x-z)} + \frac{y^n(2y-x+2)}{(t-y)(x-y)(y-z)} + \frac{z^n(2t+z+1)}{(z-t)(x-z)(y-z)} + \frac{t^n(2z+t+1)}{(t-x)(t-y)(t-z)} \quad .$$

Theorem 6. The generating function for the Quadrapell matrix sequence is

$$\frac{1}{1-x^2-2x^3-x^4}\begin{pmatrix} -x^3+x+1 & x^3+x^2+x+1 & x^3+3x^2+2x+1 & x^3+3x^2+4x+2 \\ 2x^3+2x^2+x-1 & -x^3+x+1 & x^3+x^2+x+1 & x^3+3x^2+2x+1 \\ -2x^3-x^2-x+2 & 2x^3+2x^2+x-1 & -x^3+x+1 & x^3+x^2+x+1 \\ 3x^3+x^2+2x-2 & -2x^3-x^2-x+2 & 2x^3+2x^2+x-1 & -x^3+x+1 \end{pmatrix} \; .$$

Proof. Suppose that $F(x)$ is the generating function for the sequence $\{\mathfrak{D}_n\}$. Then we obtain

$$F(x) = \sum_{i=0}^{\infty} \mathfrak{D}_i x^i$$

$$= \mathfrak{D}_0 + \mathfrak{D}_1 x + \mathfrak{D}_2 x^2 + \mathfrak{D}_3 x^3 + \sum_{i=4}^{\infty} \mathfrak{D}_i x^i$$

$$= \mathfrak{D}_0 + \mathfrak{D}_1 x + \mathfrak{D}_2 x^2 + \mathfrak{D}_3 x^3 + \sum_{i=4}^{\infty} (\mathfrak{D}_{i-2} + 2\mathfrak{D}_{i-3} + \mathfrak{D}_{i-4}) x^i$$

$$= \mathfrak{D}_0 + \mathfrak{D}_1 x + \mathfrak{D}_2 x^2 + \mathfrak{D}_3 x^3 - \mathfrak{D}_0 x^2 - \mathfrak{D}_1 x^3 - 2\mathfrak{D}_0 x^3 + x^2 F(x) + 2x^3 F(x) + x^4 F(x)$$

Now, we rearrangement of the above equation, we have

$$F(x) = \frac{\mathfrak{D}_0 + \mathfrak{D}_1 x + (\mathfrak{D}_2 - \mathfrak{D}_0)x^2 + (\mathfrak{D}_3 - \mathfrak{D}_1 - 2\mathfrak{D}_0)x^3}{1 - x^2 - 2x^3 - x^4}.$$

Thus the proof is completed.

Note that, we will compare the 1^{st} row and 1^{st} column entries with the matrix in Theorem 6. Thus we obtained the generating function for Quadrapell numbers by using the matrix sequence method.

4. BINOMIAL TRANSFORM OF QUADRAPELL MATRIX SEQUENCE

In this section, we will focus on binomial transform of Quadrapell matrix sequences. Then, we will also present recurrence relation, Binet formula and generating function.

Definition 3. Let $\{\mathfrak{D}_n\}$ be the Quadrapell matrix sequence. The binomial transform of this matrix sequence can be expressed as follows:

$$\mathbb{D}_n = \sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_i \quad (4.1)$$

Lemma 2. *For* $n \geq 0$, the following equality are held.

$$\mathbb{D}_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (\mathfrak{D}_i + \mathfrak{D}_{i+1}) \quad (4.2)$$

Proof. By using the Definition 3 and binomial relation, we obtain

$$\mathbb{D}_{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} \mathfrak{D}_i$$

$$= \sum_{i=1}^{n+1} \binom{n}{i} \mathfrak{D}_i + \sum_{i=1}^{n+1} \binom{n}{i-1} \mathfrak{D}_i + \mathfrak{D}_0$$

$$= \sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_i + \sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_{i+1}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (\mathfrak{D}_i + \mathfrak{D}_{i+1}).$$

We note that equation (4.2) can be expressed as

$$\mathbb{D}_{n+1} = \mathbb{D}_n + \sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_{i+1}.$$

Theorem 7. For $n > 0$, recurrence relation of sequence $\{\mathbb{D}_n\}$ is

$$\mathbb{D}_{n+3} = 4\mathbb{D}_{n+2} - 5\mathbb{D}_{n+1} + 4\mathbb{D}_n - \mathbb{D}_{n-1} \tag{4.3}$$

with initial conditions

$$\mathbb{D}_0 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{pmatrix}, \ \mathbb{D}_1 = \begin{pmatrix} 2 & 2 & 3 & 6 \\ 0 & 2 & 2 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix},$$

$$\mathbb{D}_2 = \begin{pmatrix} 4 & 5 & 9 & 15 \\ 2 & 4 & 5 & 9 \\ 1 & 2 & 4 & 5 \\ 1 & 1 & 2 & 4 \end{pmatrix}, \ \mathbb{D}_3 = \begin{pmatrix} 9 & 14 & 24 & 38 \\ 6 & 9 & 14 & 24 \\ 3 & 6 & 9 & 14 \\ 2 & 3 & 6 & 9 \end{pmatrix}.$$

Proof. By considering the right hand side of equality in (4.3) and Pascal's identity

$$4\sum_{i=0}^{n+2} \binom{n+2}{i} \mathfrak{D}_i - 5\sum_{i=0}^{n+1} \binom{n+1}{i} \mathfrak{D}_i + 4\sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_i - \sum_{i=0}^{n-1} \binom{n-1}{i} \mathfrak{D}_i$$

$$= \sum_{i=0}^{n+2} \binom{n+2}{i} (\mathfrak{D}_i + \mathfrak{D}_{i+1}) - \sum_{i=0}^{n+2} \binom{n+1}{i-1} \mathfrak{D}_{i+1} + 2\sum_{i=0}^{n+2} \binom{n+1}{i-1} \mathfrak{D}_i - 2\sum_{i=0}^{n+1} \binom{n+1}{i} \mathfrak{D}_i$$

$$+4\sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_i - \sum_{i=0}^{n-1} \binom{n-1}{i} \mathfrak{D}_i$$

From the Lemma 2 and properties of binomial sum, we have

$$\mathbb{D}_{n+3} - \sum_{i=0}^{n+2} \binom{n+1}{i-1} \mathfrak{D}_{i+1} + 2\sum_{i=0}^{n+2} \binom{n+1}{i-1} \mathfrak{D}_i - 2\sum_{i=0}^{n+1} \binom{n+1}{i} \mathfrak{D}_i + 4\sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}_i - \sum_{i=0}^{n-1} \binom{n-1}{i} \mathfrak{D}_i$$

$$= \mathbb{D}_{n+3} - \sum_{i=0}^{n-1} \binom{n-1}{i} (\mathfrak{D}_{i+4} - \mathfrak{D}_{i+2} - 2\mathfrak{D}_{i+1} - \mathfrak{D}_i)$$

$$= \mathbb{D}_{n+3}$$

Note that the Binet formula for $\{\mathbb{D}_n\}$ is

$$\mathbb{D}_n = A_1\lambda_1^n + A_2\lambda_2^n + A_3\lambda_3^n + A_4\lambda_4^n$$

where

$$A_1 = \frac{\mathbb{D}_3 - (\lambda_2+1)\mathbb{D}_2 + (\lambda_2\lambda_3 + \lambda_2\lambda_4 + 1)\mathbb{D}_1 - \lambda_2\mathbb{D}_0}{(\lambda_1-\lambda_4)(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)},$$

$$A_2 = \frac{\mathbb{D}_3 - (\lambda_1+1)\mathbb{D}_2 + (\lambda_1\lambda_3 + \lambda_1\lambda_4 + 1)\mathbb{D}_1 - \lambda_1\mathbb{D}_0}{(\lambda_4-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)},$$

$$A_3 = \frac{\mathbb{D}_3 - (\lambda_4+3)\mathbb{D}_2 + (\lambda_1\lambda_4 + \lambda_2\lambda_4 + 1)\mathbb{D}_1 - \lambda_4\mathbb{D}_0}{(\lambda_3-\lambda_4)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)},$$

$$A_4 = \frac{\mathbb{D}_3 - (\lambda_3+3)\mathbb{D}_2 + (\lambda_1\lambda_3 + \lambda_2\lambda_3 + 1)\mathbb{D}_1 - \lambda_3\mathbb{D}_0}{(\lambda_4-\lambda_3)(\lambda_2-\lambda_4)(\lambda_1-\lambda_4)},$$

$\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the roots of the $x^4 - 4x^3 + 5x^2 - 4x + 1$.

Theorem 8. The generating function of binomial transform for $\{\mathbb{D}_n\}$ is

$$\frac{1}{1 - 4x + 5x^2 - 4x^3 + x^4}$$

$$\times \begin{pmatrix} 3x^3 + x^2 - 2x + 1 & 4x^3 + 2x^2 - 2x + 1 & 3x^3 + 2x^2 - x + 1 & 8x^3 + x^2 - 2x + 2 \\ -2x^3 - 3x^2 + 4x - 1 & 3x^3 + x^2 - 2x + 1 & 4x^3 + 2x^2 - 2x + 1 & 3x^3 + 2x^2 - x + 1 \\ 4x^3 + 7x^2 - 7x + 2 & -2x^3 - 3x^2 + 4x - 1 & 3x^3 + x^2 - 2x + 1 & 4x^3 + 2x^2 - 2x + 1 \\ -2x^3 - 9x^2 + 8x - 2 & 4x^3 + 7x^2 - 7x + 2 & -2x^3 - 3x^2 + 4x - 1 & 3x^3 + x^2 - 2x + 1 \end{pmatrix}.$$

Proof. Assume that $G(x)$ is the generating function of the binomial transform for $\{\mathbb{D}_n\}$. Then we have

$$G(x) = \sum_{i=0}^{\infty} \mathbb{D}_i x^i$$

$$= \mathbb{D}_0 + \mathbb{D}_1 x + \mathbb{D}_2 x^2 + \mathbb{D}_3 x^3 + \sum_{i=4}^{\infty} \mathbb{D}_i x^i$$

$$= \mathbb{D}_0 + \mathbb{D}_1 x + \mathbb{D}_2 x^2 + \mathbb{D}_3 x^3 + \sum_{i=4}^{\infty} (4\mathbb{D}_{i-1} - 5\mathbb{D}_{i-2} + 4\mathbb{D}_{i-3} - \mathbb{D}_{i-4}) x^i$$

$$= \mathbb{D}_0 + (\mathbb{D}_1 - 4\mathbb{D}_0)x + (\mathbb{D}_2 - 4\mathbb{D}_1 + 5\mathbb{D}_0)x^2 + (\mathbb{D}_3 - 4\mathbb{D}_2 + 5\mathbb{D}_1)x^3$$

$$+4xG(x) - 5x^2G(x) + 4x^3G(x) - x^4G(x).$$

We rearrangement of the equation implies that

$$G(x) = \frac{\mathbb{D}_0 + (\mathbb{D}_1 - 4\mathbb{D}_0)x + (\mathbb{D}_2 - 4\mathbb{D}_1 + 5\mathbb{D}_0)x^2 + (\mathbb{D}_3 - 4\mathbb{D}_2 + 5\mathbb{D}_1)x^3}{1 - 4x + 5x^2 - 4x^3 + x^4}$$

Thus, the proof is completed.

5. CONCLUSION

In this paper, we presented binomial transform of Quadrapell sequence and defined Quadrapell matrix sequence. Moreover we studied binomial transform of Quadrapell matrix sequences. On the other hand in [13], author introduced Quadra Fibona-Pell sequence. They are defined by the recurrence relation for $n \geq 4$

$$W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$$

with initial values $W_0 = W_1 = 0$, $W_2 = 1$, $W_3 = 3$. It would be interesting study the binomial transforms of Quadra Fibona-Pell sequence, Quadra Fibona-Pell matrix sequence and research their properties.

REFERENCES

[1] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, 2001.
[2] Vajda, S., Fibonacci & Lucas numbers, and the golden section, *Ellis Horwood Limited*, 1989.
[3] Taşçı, D., *Hacet. J. Math. Stat.*, **38**(3), 265, 2009.
[4] Civciv, H., Türkmen, R., *Ars Combin.*, **87**, 161, 2008.
[5] Gulec, H.H., Taskara, N., *Appl. Math. Lett.*, **25**, 1554 2012.
[6] Yilmaz, N., Taskara, N., *J. Appl. Math.*, Article ID 941673, 2013.
[7] Spivey, M., Steil, L., *J. Integer Seq.*, **9**, Article 06.1.1, 2006.
[8] Chen, K.-W., *J. Number Theory*, **124**(1), 142, 2007.
[9] Prodinger, H., *Fibonacci Quart.*, **32**(5), 412, 1994.
[10] Falcon, S., Plaza, A., *Int. J. Nonlinear Sci. Numer. Simul.*, **10**(11-12), 1527, 2009.
[11] Bhadouria, P., Jhala, D., Singh, B., *J. Math. Computer Sci.*, **8**, 81, 2014.
[12] Yilmaz, N., Taskara, N., *Abstr. Appl. Anal.*, Article ID 497418, 2013.
[13] Özkoç, A., *Adv. Difference Equ.*, **148**, DOI 10.1186/s13662-015-0486-7, 2015.