MODULAR A-METRIC SPACES

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Abstract. *In this article, we introduce modular A-metric spaces. Also, we give topology induced by this metric and some results obtained from this. As an application, we prove the uniqueness and existence of fixed point of Banach contraction mapping in modular A-metric spaces.*
Keywords: Modular Metric, Modular A-metric, Fixed Point, Contraction Theorem.

1. INTRODUCTION

The notion of modular spaces on linear spaces was initiated by Nakano [5,6] and was intensively developed by Koshi, Shimogaki, Yamamuro [4,8] and others. In 1959, it was redefined by Orlicz [7] as follows:

Definition 1: [7] A modular on a real linear space X is a functional $p : X \to [0,\infty]$ satisfying the following conditions for all $x, y \in X$:

($p1$) $p(0) = 0$,

($p2$) If $p(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$,

($p3$) $p(-x) = p(x)$,

($p4$) $p(\alpha x + \beta y) \le p(x) + p(y)$ for all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

In 2010, the notions of a metric modular on an arbitrary set and the corresponding modular space, more general than a metric space, were introduced and studied by Chistyakov [3] as follows:

Definition 2: [3] The modular metric on X where X is non-empty is defined by a mapping $w : (0,\infty) \times X \times X \to [0,\infty]$ that satisfying following conditions for all $x, y, z \in X$ and all $\lambda, \mu > 0$:

(i) $w_\lambda(x, y) = 0 \iff x = y$,

(ii) $w_\lambda(x, y) = w_\lambda(y, x)$,

(iii) $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(z, y)$.

Behind this new concept, there exists the physical interpretation of the modular. A modular on a set bases on a nonnegative (possibly infinite valued) "field of (generalized) velocities" while a metric on a set stands for non-negative finite distances between any two points of the set. When we come to the explain of generalized velocities, we can say that to

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each "time" absolute value of an average velocity $w_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $w_\lambda(x, y)$.

2. MODULAR A-METRIC SPACES

In this work, as a new perspective in both modular metric space and A-metric space which is introduced by M. Abbas *et al.* [1], we define the notion of modular $A-$ metric spaces and give some basic properties of it. Also, we examine Banach Contraction Theorem on complete modular $A-$ metric spaces.

Definition 3: The modular $A-$ metric on X where X is non-empty is defined by a mapping $A_\lambda : (0,\infty) \times X^n \to [0,\infty]$ that satisfying following conditions for all $x_i, a \in X$, $\lambda_i > 0$, $i = \overline{1,n}$ and $\lambda > 0$:

(A1) $A_\lambda(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0,$

(A2) $A_\lambda(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = ... = x_{n-1} = x_n,$

$(A3) A_{\lambda_1 + \lambda_2 + ... + \lambda_n}(x_1, x_2, x_3, ..., x_{n-1}, x_n) \leq A_{\lambda_1}(x_1, x_1, ..., (x_1)_{n-1}, a)$

$$+ A_{\lambda_2}(x_2, x_2, ..., (x_2)_{n-1}, a)$$

$$+ A_{\lambda_n}(x_n, x_n, ..., (x_n)_{n-1}, a)$$

The pair (X, A) is said to be a modular A-metric space.

Example 1: Let $X = R$. Define a function $A_\lambda : (0,\infty) \times X^n \to [0,\infty]$ by

$$A_\lambda(x_1, x_2, x_3, ..., x_{n-1}, x_n) = \frac{\lambda}{n}[|x_1 - x_2| + |x_1 - x_3| + ... + |x_1 - x_n|$$

$$+ |x_2 - x_3| + |x_2 - x_4| + ... + |x_2 - x_n|$$

$$\vdots$$

$$+ |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n|$$

$$+ |x_{n-1} - x_n|]$$

$$= \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|$$

for all $\lambda > 0$ and $x_1, x_2, ..., x_n \in X$.

Then (X, A) is an usual modular A-metric space on X.

Actually, first two conditions can be easily verified for this example. Now, we will show whether the final condition can be verified or not:

$$A_\lambda\left(x_1, x_2, x_3, ..., x_{n-1}, x_n\right) = \frac{\lambda}{n}[|x_1 - x_2| + |x_1 - x_3| + ... + |x_1 - x_n|$$
$$+ |x_2 - x_3| + |x_2 - x_4| + ... + |x_2 - x_n|$$
$$\vdots$$
$$+ |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n|$$
$$+ |x_{n-1} - x_n|]$$

$$A_\lambda\left(x_1, x_2, x_3, ..., x_{n-1}, x_n\right)$$
$$\leq \frac{\lambda}{n}[|x_1 - a| + |a - x_2| + |x_1 - a| + |a - x_3| ... + |x_1 - a| + |a - x_n|$$
$$+ |x_2 - a| + |a - x_3| + |x_2 - a| + |a - x_4| ... + |x_2 - a| + |a - x_n|$$
fi
$$+ |x_{n-2} - a| + |a - x_{n-1}| + |x_{n-2} - a| + |a - x_n|$$
$$+ |x_{n-1} - a| + |a - x_n|]$$
$$= \frac{\lambda}{n}[(n-1)|x_1 - a| + (n-1)|x_2 - a| + ... + (n-1)|x_n - a|]$$
$$= \frac{\lambda}{n}[(|x_1 - x_1| + |x_1 - x_1| + ... + |x_1 - a|) + (|x_1 - x_1| + ... + |x_1 - a|)$$
$$+ ... + (|x_1 - x_1| + |x_1 - a|) + (|x_1 - a|)]$$
$$+ \frac{\lambda}{n}[(|x_2 - x_2| + |x_2 - x_2| + ... + |x_2 - a|)$$
$$+ ... + (|x_2 - x_2| + |x_2 - a|) + (|x_2 - a|)]$$
fi
$$+ \frac{\lambda}{n}[(|x_n - x_n| + |x_n - x_n| + ... + |x_n - a|) +$$
$$+ ... + (|x_n - x_n| + |x_n - a|) + (|x_n - a|)]$$
$$= A_{\frac{\lambda}{n}}\left(x_1, x_1, x_1, ..., x_1, a\right) + A_{\frac{\lambda}{n}}\left(x_2, x_2, x_2, ..., x_2, a\right) + ... + A_{\frac{\lambda}{n}}\left(x_n, x_n, x_n, ..., x_n, a\right)$$

Thus, (X, A) is an usual modular A-metric space.

Lemma 1: Let the function $\lambda \to A_\lambda\left(x, x, x, ..., x, y\right)$ be continuous on $(0, \infty)$ for all $x, y \in X$. Then, $A_\lambda\left(x, x, x, ..., x, y\right) = A_\lambda\left(y, y, y, ..., y, x\right)$ is obtained.

Proof: From $(A3)$ condition of modular A-metric space, there exists $\varepsilon > 0$ such that

$$A_\lambda(x,x,x,...,x,y) \leq A_\varepsilon(x,x,x,...,x,x)$$
$$+A_\varepsilon(x,x,x,...,x,x)$$
$$+A_\varepsilon(x,x,x,...,x,x)$$
$$\vdots$$
$$+A_\varepsilon(x,x,x,...,x,x)$$
$$+A_{\lambda-(n-1)\varepsilon}(y,y,y,...,y,x)$$
$$= A_{\lambda-(n-1)\varepsilon}(y,y,y,...,y,x) \qquad (2.2.1)$$

By taking the limit of inequality (2.2.1) as $\varepsilon \to 0$, we get

$$A_\lambda(x,x,x,...,x,y) \leq A_\lambda(y,y,y,...,y,x) \qquad (2.2.2)$$

Similarly,

$$A_\lambda(y,y,y,...,y,x) \leq A_\varepsilon(y,y,y,...,y,y)$$
$$+A_\varepsilon(y,y,y,...,y,y)$$
$$+A_\varepsilon(y,y,y,...,y,y)$$
$$\vdots$$
$$+A_\varepsilon(y,y,y,...,y,y)$$
$$+A_{\lambda-(n-1)\varepsilon}(x,x,x,...,x,y)$$
$$= A_{\lambda-(n-1)\varepsilon}(x,x,x,...,x,y). \qquad (2.2.3)$$

When the limit of inequality (2.2.3) is taken as $\varepsilon \to 0$, the desired result is obtained

$$A_\lambda(y,y,y,...,y,x) \leq A_\lambda(x,x,x,...,x,y) \qquad (2.2.4)$$

The result follows from (2.2.2) and (2.2.4).

Remark 1: Consider the function $\lambda \to A_\lambda(x_1,x_2,...,x_n)$ where it is a modular A-metric on a set X for any $x_1,x_2,...,x_n \in X$. Here $\lambda \to A_\lambda(x_1,x_2,...,x_n)$ is non-increasing on $(0,\infty)$. Remark 1 can be easily verified as the following way:
If $0 < \lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_{n-1} < \lambda_n$, then

$$A_{\lambda_n}(x,x,x,...,x,y) \leq A_{\lambda_n-\lambda_{n-1}}(x,x,x,...,x,x)$$
$$+A_{\lambda_{n-1}-\lambda_{n-2}}(x,x,x,...,x,x)$$
$$\text{fi}$$
$$+A_{\lambda_2-\lambda_1}(x,x,x,...,x,x)$$
$$+A_{\lambda_1}(y,y,y,...,y,x)$$
$$= A_{\lambda_1}(y,y,y,...,y,x).$$

Hence, from $(A2)$ condition of modular A -metric space,

$$A_{\lambda_n}\left(x, x, x, ..., x, y\right) \le A_{\lambda_1}\left(y, y, ..., y, x\right)$$

is gotten. Then, by using Lemma 1, the following is obtained which is the desired result:

$$A_{\lambda_n}\left(x, x, x, ..., x, y\right) \le A_{\lambda_1}\left(x, x, ..., x, y\right).$$

Lemma 2: Let (X, A) be a modular A -metric space. Then, for all $x, y, z \in X$, we have

$$A_{\lambda}\left(x, x, x, ..., x, z\right) \le (n-1)A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right) + A_{\frac{\lambda}{n}}\left(z, z, z, ..., z, y\right)$$

and

$$A_{\lambda}\left(x, x, x, ..., x, z\right) \le (n-1)A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right) + A_{\frac{\lambda}{n}}\left(y, y, y, ..., y, z\right).$$

Proof: From $(A3)$ condition of modular A -metric space, following inequality is satisfied:

$$A_{\lambda}\left(x, x, x, ..., x, z\right) \le \underbrace{A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right) + ... + A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right)}_{(n-1)} + A_{\frac{\lambda}{n}}\left(z, z, z, ..., z, y\right)$$

$$\le (n-1)A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right) + A_{\frac{\lambda}{n}}\left(z, z, z, ..., z, y\right).$$

Similarly, other inequality is satisfied:

$$A_{\lambda}\left(x, x, x, ..., x, z\right) \le \underbrace{A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right) + ... + A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right)}_{(n-1)} + A_{\frac{\lambda}{n}}\left(y, y, y, ..., y, z\right)$$

$$\le (n-1)A_{\frac{\lambda}{n}}\left(x, x, x, ..., x, y\right) + A_{\frac{\lambda}{n}}\left(y, y, y, ...y, z\right).$$

The desired result is obtained.

Definition 4: Let A be a modular A-metric on X. The set

$$B_{A_{\lambda}}(x_0, r) = \{ y \in X \; : \; A_{\lambda}(y, y, y, ..., y, x_0) < r \}$$

is said to be an open ball with center x_0 and radius r for $x_0 \in X$, $r > 0$.

The set

$$\overline{B_{A_{\lambda}}(x_0, r)} = \{ y \in X \; : \; A_{\lambda}(y, y, y, ..., y, x_0) \le r \}$$

is said to be a closed ball with center x_0 and radius r for $x_0 \in X$, $r > 0$.

Definition 5: Let A be a modular A-metric on X and $Y \subset X$.
$Y \subset X$ is said to be an open subset if for all $\lambda > 0$ and all $x \in X$, there exists a $r > 0$

such that $B_{A_\lambda}(x, r) \subset Y$.

    Let

$$\tau := \{Y \subset X \; : \; x \in Y \text{ iff there exists a } r > 0 \text{ such that } B_{A_\lambda}(x, r) \subset Y\}.$$

τ is said to be a topology in modular A -metric space.

       Theorem 1: Let A be a modular A-metric on X. In this case, (X, τ) is a Hausdorff space.

Proof: Assume that $x \neq y$ and $c = A_\lambda(x, x, ..., x, y)$ for $x, y \in X$. Now, consider $U = B_{A_{\frac{\lambda}{n}}}(x, \frac{c}{2(n-1)})$ and $V = B_{A_{\frac{\lambda}{n}}}(y, \frac{c}{2})$. Then, $x \in U$ and $y \in V$. We claim that $U \cap V = \phi$.

Assume that $z \in U \cap V \neq \phi$. Thus, $z \in B_{A_{\frac{\lambda}{n}}}(x, \dfrac{c}{2(n-1)})$ and $z \in B_{A_{\frac{\lambda}{n}}}(y, \dfrac{c}{2})$.

     Therefore,

$$A_{\frac{\lambda}{n}}(z, z, z, ..., z, x) < \frac{c}{2(n-1)},$$

$$A_{\frac{\lambda}{n}}(z, z, z, ..., z, y) < \frac{c}{2}.$$

     By using Lemma 2, we have

$$
\begin{aligned}
c &= A_\lambda(x, x, x, ..., x, y) \\
&\leq (n-1) A_{\frac{\lambda}{n}}(z, z, z, ..., z, x) + A_{\frac{\lambda}{n}}(z, z, z, ..., z, y) \\
&< (n-1) \frac{c}{2(n-1)} + \frac{c}{2} \\
&= c
\end{aligned}
$$

i.e. $c < c$ is a contradiction. So $U \cap V = \phi$ and (X, τ) is a Hausdorff space.

       Definition 6: Let A be a modular A-metric on X, $\{x_n\}_{n \in N} \subset X$ and $x \in X$.

(1) $\{x_n\}$ converges to x if $A_\lambda(x_n, x_n, x_n, ..., x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

In other words, for each $\varepsilon > 0$, there exists a natural number n_0 such that for all $n \geq n_0$, $A_\lambda(x_n, x_n, x_n, ..., x_n, x) \leq \varepsilon$.

(2) $\{x_n\}$ is said to be a Cauchy sequence if $A_\lambda(x_n, x_n, x_n, ..., x_n, x_m) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$.

In other words, for each $\varepsilon > 0$, there exists a natural number n_0 such that for all $n, m \geq n_0$, $A_\lambda(x_n, x_n, x_n, ..., x_n, x_m) \leq \varepsilon$.

(3) (X, A) is said to be complete modular A-metric space if every Cauchy sequence in X is convergent.

Lemma 3: Let A be a modular A-metric on X. If the sequence $\{x_n\}_{n\in N} \in X$ converges to x in X, in this case x is unique.

Proof: Assume that $\{x_n\}_{n\in N}$ converge to x and y. Then, for each $\varepsilon > 0$, there exist $N_1, N_2 \in N$ such that for all $n \geq N_1$, we obtain

$$A_{\lambda_i}(x_n, x_n, x_n, ..., x_n, x) < \frac{\varepsilon}{2(n-1)}, \ i = \overline{1, (n-1)}$$

and for all $n \geq N_2$,

$$A_{\lambda_n}(x_n, x_n, x_n, ..., x_n, y) < \frac{\varepsilon}{2}$$

for all $\lambda_i > 0$, $i = \overline{1, n}$. Take $N = \max\{N_1, N_2\}$. Thus, we obtain

$$A_{\lambda_1 + \lambda_2 + ... + \lambda_n}(x, x, ..., y) \leq A_{\lambda_1}(x, x, ..., x, x_n) + A_{\lambda_2}(x, x, ..., x, x_n) + ... + A_{\lambda_n}(y, y, ...y, , x_n)$$

$$= A_{\lambda_1}(x_n, x_n, ..., x_n, x) + A_{\lambda_2}(x_n, x_n, ..., x_n, x) + ... + A_{\lambda_n}(x_n, x_n, ..., x_n, y)$$

$$< \frac{\varepsilon}{2(n-1)} + \frac{\varepsilon}{2(n-1)} + ... + \frac{\varepsilon}{2}$$

$$= (n-1)\frac{\varepsilon}{2(n-1)} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

for all $n \geq N$. Since ε is arbitrary, we get $A_{\lambda_1 + \lambda_2 + ... + \lambda_n}(x, x, ..., x, y) = 0$. As a result, $x = y$. So, the proof is completed.

Lemma 4: Let A be a modular A-metric on X. If $\{x_n\}_{n\in N} \subset X$ is a convergent sequence in X, then $\{x_n\}_{n\in N}$ is a Cauchy sequence.

Proof: Let $\{x_n\}_{n\in N}$ be a convergent sequence in X. In this case, there exists a $x \in X$ such that $A_\lambda(x_n, x_n, x_n, ..., x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. Then, for given $\varepsilon > 0$, there exist $N_1, N_2 \in N$ such that for all $n \geq N_1$, we have

$$A_{\lambda_i}(x_n, x_n, x_n, ..., x_n, x) < \frac{\varepsilon}{2(n-1)}, \ i = \overline{1, (n-1)}$$

and for all $m \geq N_2$,

$$A_{\lambda_n}(x_m, x_m, x_m, ..., x_m, x) < \frac{\varepsilon}{2}$$

for all $\lambda_i > 0$, $i = \overline{1, n}$. Take $N = \max\{N_1, N_2\}$. Hence, for all $m, n \geq N$, we obtain

$$A_{\lambda_1+\lambda_2+...+\lambda_n}(x_n,x_n,x_n,...,x_n,x_m) \le A_{\lambda_1}(x_n,x_n,...,x_n,x) + A_{\lambda_2}(x_n,x_n,...,x_n,x)$$

$$+A_{\lambda_3}(x_n,x_n,...,x_n,x)$$

$$\vdots$$

$$+A_{\lambda_n}(x_m,x_m,...,x_m,x)$$

$$< (n-1)\frac{\varepsilon}{2(n-1)} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

This implies that $\{x_n\}_{n\in N}$ is a Cauchy sequence.

3. BANACH CONTRACTION THEOREM

The Banach Contraction Theorem which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces was given by Banach in 1922 [2]. In this part, we prove this notable theorem in modular A-metric spaces.

Definition 7: Let A be a modular A-metric on X. If there exists $q \in [0,1)$ such that

$$A_\lambda(Tx,Tx,Tx,...,Tx,Ty) \le qA_\lambda(x,x,x,...,x,y)$$

for all $x,y \in X$ and $\lambda > 0$, then $T : X \to X$ is said to be a contraction mapping.

Theorem 2: Let (X,A) be a complete modular A-metric space and $T : X \to X$ be a contraction mapping. In this case, T has an unique fixed point in X. Moreover, for any $x \in X$, iterative sequence $\{T^k x\}_{k\ge 1}$ converges to an unique the fixed point.

Proof: We write

$$x_1 = Tx_0, \; x_2 = Tx_1 = T^2 x_0$$

for all $x_0 \in X$. In general,

$$x_k = Tx_{k-1} = T^k x_0$$

for all $k \in N$. Then,

$$A_\lambda(x_{k+1},x_{k+1},...,x_{k+1},x) = A_\lambda(Tx_k,Tx_k,...,Tx_k,x)$$

$$\le qA_\lambda(x_k,x_k,...,x_k,x)$$

$$= qA_\lambda(Tx_{k-1},Tx_{k-1},...,Tx_{k-1},x)$$

$$\le q^2 A_\lambda(x_{k-1},x_{k-1},...,x_{k-1},x)$$

$$\vdots$$

$$\le q^k A_\lambda(x_1,x_1,...,x_1,x)$$

for all $\lambda > 0$ and $k \in N$. Therefore, $A_\lambda(x_{k+1},x_{k+1},...,x_{k+1},x) \to 0$ as $k \to \infty$ for all $\lambda > 0$. So, for given $\varepsilon > 0$, there exist $N_1, N_2 \in N$ such that for all $k \ge N_1$, we get

$$A_{\lambda_i}(x_k, x_k, ..., x_k, x) < \frac{\varepsilon}{2(n-1)}, \ i = \overline{1,(n-1)}$$

and for all $m \geq N_2$,

$$A_{\lambda_n}(x_m, x_m, ..., x_m, x) < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. Consequently, for all $k, m \geq N$, we obtain

$$
\begin{aligned}
A_{\lambda_1 + \lambda_2 + ... + \lambda_n}(x_k, x_k, ..., x_k, x_m) &\leq A_{\lambda_1}(x_k, x_k, ..., x_k, x) \\
&\quad + A_{\lambda_2}(x_k, x_k, ..., x_k, x) \\
&\quad \vdots \\
&\quad + A_{\lambda_{(n-1)}}(x_k, x_k, ..., x_k, x) \\
&\quad + A_{\lambda_n}(x_m, x_m, ..., x_m, x) \\
&< (n-1)\frac{\varepsilon}{2(n-1)} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

From the definition of Cauchy sequence, above findings indicate that $\{x_k\}_{k \in N}$ is also a Cauchy sequence. Since X is a complete modular A-metric space, there exists a point $x \in X$ such that $x_k \to x$ as $k \to \infty$. By the notion of modular A-metric, the contraction of T and using Lemma 2, we get

$$
\begin{aligned}
A_\lambda(Tx, Tx, ..., Tx, x) &\leq (n-1)A_{\frac{\lambda}{n}}(Tx, Tx, ..., Tx, Tx_k) + A_{\frac{\lambda}{n}}(Tx_k, Tx_k, ..., Tx_k, x) \\
&= (n-1)qA_{\frac{\lambda}{n}}(x, x, ..., x, x_k) + A_{\frac{\lambda}{n}}(x_{k+1}, x_{k+1}, ..., x_{k+1}, x)
\end{aligned}
\tag{3.1}
$$

for all $\lambda > 0$ and $k \in N$. In (3.1), if k tends to ∞, then $A_\lambda(Tx, Tx, ..., Tx, x) = 0$. Thus, $Tx = x$ which means x is a fixed point of T. The last thing we should show to finish the proof is the uniqueness of the fixed point. Assume that z is another fixed point of T. We see that

$$A_\lambda(x, x, ..., x, z) = A_\lambda(Tx, Tx, ..., Tx, Tz) \leq q.A_\lambda(x, x, x, ..., x, z)$$

for all $\lambda > 0$. Since $q \in [0,1)$, we get $(1-q)A_\lambda(x, x, ..., x, z) \leq 0$ for all $\lambda > 0$ which means $A_\lambda(x, x, ..., x, z) = 0$. This implies that $x = z$. Hence, the fixed point of T is unique.

Corollary 1: Let (X, A) be a complete modular A-metric space. Assume that the mapping $T : X \to X$ satisfies

$$A_\lambda(T^k x, T^k x, ..., T^k x, T^k y) \leq qA_\lambda(x, x, ..., x, y)$$

for all $k \in N$ and $x, y \in X$ where $q \in [0,1)$. Then, T has an unique fixed point in X.

Proof: From Theorem 2, T^k has an unique fixed point z. But

$$T^k(Tz) = T(T^k z) = Tz.$$

Thus, Tz is also a fixed point of T^k. Hence $Tz = z$. Then, z is a fixed point of T. Since the fixed point of T is also a fixed point of T^k, then fixed point of T is unique.

Example 2: Take (X, A) as it was taken in Example 1 and assume that $X = R^+$. In addition to these, suppose that (X, A) is a complete modular A-metric space. Define the mapping $T : X \to X$ with

$$T(x) = \frac{x}{4.e^{\frac{x}{4}}}.$$

So, we know that $\dfrac{u}{e^u} \le u$ for all $u \in [0, \infty)$. Therefore, for all $x, y \in X$, we get

$$
\begin{aligned}
A_\lambda(Tx, Tx, ..., Tx, Ty) &= \frac{\lambda}{n}[|Tx - Ty| + |Tx - Ty| + ... + |Tx - Ty|] \\
&= \frac{\lambda}{n}(n-1)|Tx - Ty| \\
&= \frac{\lambda}{n}(n-1)\left|\frac{x}{4.e^{\frac{x}{4}}} - \frac{y}{4.e^{\frac{y}{4}}}\right| \\
&\le \frac{\lambda}{n}(n-1)\left|\frac{x}{4} - \frac{y}{4}\right| \\
&= \frac{\lambda}{n}\frac{1}{4}[|x - y| + |x - y| + ... + |x - y|] \\
&= \frac{1}{4}A_\lambda(x, x, ..., x, y)
\end{aligned}
$$

where $q = \frac{1}{4}$ and for all $\lambda > 0$. Obviously, T is a contraction map in X.

REFERENCES

[1]   Abbas, M., Ali, B., Suleiman, Y. I., *Fixed Point Theory and Applications*, 2015, 64, 2015.
[2]   Banach, S., *Fund Math.*, **3**, 133, 1922.
[3]   Chistyakov, V. V., *Nonlinear Anal.*, **72**, 1, 2010.
[4]   Koshi, S., Shimogaki, T., *J. Fac. Sci. Hokkaido Univ. Ser. I.,* **15**(3-4), 202, 1961.
[5]   Nakano, H., *Tokyo Math. Book Ser.,* **1**, 1950.
[6]   Nakano, H., *Tokyo Math. Book Ser.,* **1**, 1951.
[7]   Orlicz, W., *Bull. Acad. Pol. Sci. Ser. Sci. Math., Astron. Phys.*, **9**, 157, 1961.
[8]   Yamamuro, S., *Trans. Amer. Math. Soc.,* **90**, 291, 1959.