ORIGINAL PAPER

A NOVEL TECHNIQUE TO OBTAIN SOLITON SOLUTIONS FOR MODIFIED KORTEWEG-DE VRIES EQUATION OF FRACTIONAL ORDER

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Abstract. *Nonlinear mathematical problems and their solutions attain many attentions. In soliton theory an efficient tool to attain various types of soliton solutions is the improved modified Exp-function technique. Under study article is devoted to find soliton wave solutions of modified KdV equation of fractional-order via a reliable mathematical technique. By use of proposed technique we attain soliton wave solution of various types. The regulation of proposed algorithm is demonstrated by consequent results and computational work. It is concluded that under discussion technique is user friendly with minimum computational work; also we can extend it for physical problems of different nature.*

Keywords: *Improved modified Exp-function technique, travelling wave solutions, fractional calculus, modified Korteweg-de Vries equation.*

1. INTRODUCTION

In the last few years we have observed an extraordinary progress in soliton theory. Solitons have been studied by various mathematician, physicists and engineers for their applications in physical phenomena's. Firstly soliton waves are observed by an engineer John Scott Russell. Wide ranges of phenomena in mathematics and physics are modeled by differential equations. In nonlinear science it is of great importance and interest to explain physical models and attain analytical solutions. In the recent past large series of chemical, biological, physical singularities are feint by nonlinear partial differential equations. At present the prominent and valuable progress are made in the field of physical sciences. The great achievement is the development of various techniques to hunt for solitary wave solutions of differential equations. In nonlinear physical sciences, an essential contribution is of exact solutions because of this we can study physical behaviors and discus more features of the problem which give direction to more applications.

At the disruption between chaos, mathematical physics and probability, factional calculus and differential equations are rapidly increasing branches of research. For accurate clarification of innumerable real-time models of nonlinear occurrence; fractional differential equation (FDEs) of nonlinear structure have accomplished great notice. Because of its recognizable implementation in branch of sciences and engineering it turn out to be a topic of great notice for scientists in conferences. In large fields such as porous structures dynamical processes in self-similar or solute transport, fluid flow, material viscoelastic theory, economics, bio-sciences, control theory of dynamical systems, geology, electromagnetic

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theory, dynamics of earthquakes, statistics, astrophysics, optics, probability, signal processing, chemical physics, and so on implementations of fractional models [5-8] are beneficially exerted. As a consequence, hypothesis of fractional differential equations has shown fast growth [1-8].

In current times, to solve a nonlinear physical problems Wu and He [9] present a well-ordered procedure called Exp-function method. The technique under study has prospective to deal with the complex nonlinearity of the models with the flexibility. It has been used as an effective tool for diversified nonlinear problems arising in mathematical physics. Through the study of publication exhibits that Exp-function method is extremely reliable and has been effective on a huge range of differential equations.

After He et al. Mohyud-Din enlarged the Exp-function method and used this algorithm to find soliton wave solutions of differential equations; Oziz used same technique for Fisher's equation; Yusufoglu for modified BBM equations; Momani for travelling wave solutions of KdV equation of fractional order; Zhu for discrete modified KdV lattice and the Hybrid-Lattice system; Kudryashov for soliton solutions of the generalized evolution equations arising in wave dynamics; Wu et al. for the expansion of compaction-like solitary and periodic solutions; Zhang for high-dimensional nonlinear differential equations. It is to be noticed that after applying under study technique and its modifications i.e. Exp-function method to any ordinary nonlinear differential equation Ebaid proved that $e = f$ and $s = r$ are the only relations of the variables that can be acquired [10-29].

This article is keen to the soliton like solutions of nonlinear fractional order Korteweg-de Vries equation by applying a novel technique. The applications of under study nonlinear equation are very vast in different areas of physical sciences and engineering. Additionally, such type of equation found in different physical phenomenon related to fluid mechanic, astrophysics, solid state physics, chemical kinematics, ion acoustic waves in plasma, control & optimization theory, nonlinear optics etc.

2. PRELIMINARIES AND NOTATION

Some important results of fractional calculus are discussed in under study section. The fractional integral and derivatives defined on [a, b] are given below

Definition 1. A real valued function $h(x), x > 0$ in the space $E_v, v \in R$ is said to be in the space $E_v^n v$ if $h^n \in E_v, n \in N$. There exists a real number $(s > v)$ such that $h(x) = x^s h_1(x),$ where $h_1(x) = E(0,\infty)$.

Definition 2. Riemann-Liouville integral operator of the order $\beta \geq 0$ can be defined as

$$J^{\beta}(x) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} h(t)dt, \beta > 0, x > 0, \\ J^0(x) = h(x). \end{cases} \tag{1}$$

where $h \in E_v, v \geq -1$

Some essential properties of the operator J^{β} are as

$$J^{\beta}J^{\gamma}h(x)=J^{\beta+\gamma}h(x)$$
$$J^{\beta}J^{\gamma}h(x)=J^{\gamma}J^{\beta}h(x) \quad (2)$$
$$J^{\beta}x^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha}$$

For $h\in E_{v}, v\geq -1,\ \beta,\gamma\geq 0$ and $\alpha\geq -1$

There arises some demerits of Riemann-Liouville derivative when we apply it to model real world problems in the form of fractional differential equations. There is a need to overcome this deficiency. For this M. Caputo introduce modified version of fractional differential operator which we used in our article.

Definition 3. Caputo time fractional derivative operator is defined below, for the smallest integer n that exceeds.

$$D_t^{\beta}h(x)=\begin{cases}\dfrac{\partial^{\beta}w(x,t)}{\partial t^{\beta}}=\dfrac{1}{\Gamma(n-\beta)}\displaystyle\int_0^x (x-t)^{n-\beta-1}h(t)dt, -1<<n, n\in N,\\ \dfrac{\partial^{\beta}w(x,t)}{\partial t^{\beta}}, \beta=n.\end{cases} \quad (3)$$

CHAIN RULE

In this segment we used a complex fractional transformation to convert differential equation of fractional order into classical differential equation. We apply the following chain rule

$$\frac{\partial^{\beta}w}{\partial t^{\beta}}=\frac{\partial w}{\partial q}\cdot\frac{\partial^{\beta}q}{\partial t^{\beta}}$$

It is a Jumarie's modification of Riemann-Liouville derivative. There are few results, which are very important and useful

$$D_t^{\beta}w=\sigma_t' \tfrac{dw}{d\xi}D_t^{\beta}\xi \text{ and } D_x^{\beta}w=\sigma_x' \tfrac{dw}{d\xi}D_t^{\beta}\xi .$$

The value of σ_s is determined by assuming a special case given below

$q=t^{\beta}$ and $w=q^{n}$ we have

$$\frac{\partial^{\beta}w}{\partial t^{\beta}}=\frac{\Gamma(1+n\beta)t^{n\beta-\beta}}{\Gamma(1+n\beta-\beta)}=\sigma.\frac{\partial w}{\partial q}=\sigma n t^{n\beta-\beta} \quad (4)$$

Thus we can calculate σ_q as

$$\sigma_q=\frac{\Gamma(1+n\beta)}{\Gamma(1+n\beta-\beta)}$$

Other fractional indexes $(\sigma_x', \sigma_y', \sigma_z')$ can determine in similar way.

3. ANALYSIS OF TECHNIQUE

We suppose the nonlinear FPDE of the form

$$P(v, v_t, v_x, v_{xx}, v_{xxx}, \ldots, D_t^{\gamma} v, D_x^{\gamma} v, D_{xx}^{\gamma} v, \ldots) = 0, \qquad 0 < \gamma \le 1. \tag{5}$$

Where $D_t^{\gamma} v, D_x^{\gamma} v, D_{xx}^{\gamma} v$ are the modified Riemann-Liouville derivative of v with respect to t, x, xx respectively.

Invoking the transformation

$$\xi = px + ry + sz + \Omega\left(\frac{t^{\gamma}}{\Gamma(1+\gamma)}\right) + \xi_0 \tag{6}$$

Here p, r, s, Ω, ξ_0 are all constants with $p, \Omega, \neq 0$

Equation (5) can be converted into an ordinary differential equation of the form

$$Q(v, v', v'', v''', v^{iv}, \ldots) = 0 \tag{7}$$

Where prime signify the derivative of v with respect to ξ.

If possible, integrate equation (7) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero.

According to improved modified Exp-function method, the solution will be

$$v(\xi) = \frac{\sum_{s=1}^{2N} a_s \exp[s\xi]}{\sum_{s=1}^{2N} b_s \exp[s\xi]} + \frac{\sum_{s=1}^{2N} a_{-s} \exp[-s\xi]}{\sum_{s=1}^{2N} b_{-s} \exp[-s\xi]} \tag{8}$$

Where N is positive integers, a_s and b_s are unknown constants.

To determine the value of N, we balance the linear term of highest order of equation (7) with the highest order nonlinear term.

Substituting equation (8) in to the equation (7), equating the coefficients of each power of $\exp(s\xi)$ to zero gives the system of algebraic equations for a_s and b_s then solve the system with the help of Maple 17 to determine these constants.

4. SOLUTION PROCEDURE

Suppose the following fractional differential equation

$$D_t^{\gamma} + 6v^2 v_x + v_{xxx} = 0 \tag{9}$$

Using transformation (6), equation (9) can be converted into an ODE of the form

$$\Omega v' + 6pv^2 v' + p^3 v''' = 0 \tag{10}$$

Integrate once time, we get

$$\Omega v + 2pv^3 + p^3 v'' = 0, \tag{11}$$

Balancing the v'' and v^3 by using homogenous principal, we have

$$N + 2 = 3N$$
$$N = 1$$

Then the trail solution is

$$v(\xi) = \frac{a_1 e^{\xi} + a_2 e^{2\xi}}{b_1 e^{\xi} + b_2 e^{2\xi}} + \frac{a_{-1} e^{-\xi} + a_{-2} e^{-2\xi}}{b_{-1} e^{-\xi} + b_{-2} e^{-2\xi}} \tag{12}$$

Substituting equation (12) in to equation (10), we have

$$\tfrac{1}{A}\left[c_3 \exp(3\xi) + c_2 \exp(2\xi) + c_1 \exp(\xi) + c_0 + c_{-1} \exp(-\xi) + c_{-2} \exp(-2\xi) + c_{-3} \exp(-3\xi)\right] = 0$$
$$A = (b_1 + b_2 \exp(\xi))^3 (b_{-1} + b_{-2} \exp(-\xi))^3$$

where $c_j\,(j = -3, -2, ...2, 3)$ are constants obtained by Maple 17

Equating the coefficients of $\exp(s\xi)$ to be zero, we obtain

$$c_j = 0.\, j = 0, \pm 1, \pm 2, \pm 3 \tag{13}$$

We have following solution sets satisfy the given equation

1st Solution set

$$\left\{\begin{matrix} \Omega = -\dfrac{2p\left(a_{-1}^2 b_1^2 + 2a_{-1}a_1 b_{-1} b_1 + a_1^2 b_{-2}^2 + a_1^2 b_{-1}^2\right)}{b_{-2}^2 b_1^2}, a_2 = 0, a_{-1} = a_{-1}, a_1 = a_1, a_{-2} = \dfrac{a_{-1} b_{-2}}{b_{-1}}, \\ b_{-2} = b_{-2}, b_{-1} = b_{-1}, b_1 = b_1, b_2 = 0 \end{matrix}\right\}$$

We, therefore, obtained the following generalized solitary solution $v(x,t)$ of equation (9)

$$v(x,t) = \frac{\begin{matrix} a_{-1} \exp\left(-xp + \dfrac{2p\left(a_{-1}^2 b_1^2 + 2a_{-1}a_1 b_{-1} b_1 + a_1^2 b_{-1}^2\right)t^{\gamma}}{b_{-1}^2 b_1^2 \Gamma(1+\gamma)}\right) + \\ \dfrac{a_{-1} b_{-2}}{b_{-1}} \exp\left(-2xp + \dfrac{4p\left(a_{-1}^2 b_1^2 + 2a_{-1}a_1 b_{-1} b_1 + a_1^2 b_{-1}^2\right)t^{\gamma}}{b_{-1}^2 b_1^2 \Gamma(1+\gamma)}\right) \end{matrix}}{\begin{matrix} b_1 \exp\left(-xp + \dfrac{4p\left(a_{-1}^2 b_1^2 + 2a_{-1}a_1 b_{-1} b_1 + a_1^2 b_{-1}^2\right)t^{\gamma}}{b_{-1}^2 b_1^2 \Gamma(1+\gamma)}\right) + \\ b_{-2} \exp\left(-2xp + \dfrac{4p\left(a_{-1}^2 b_1^2 + 2a_{-1}a_1 b_{-1} b_1 + a_1^2 b_{-1}^2\right)t^{\gamma}}{b_{-1}^2 b_1^2 \Gamma(1+\gamma)}\right) \end{matrix}} + \frac{a_1}{b_1}$$

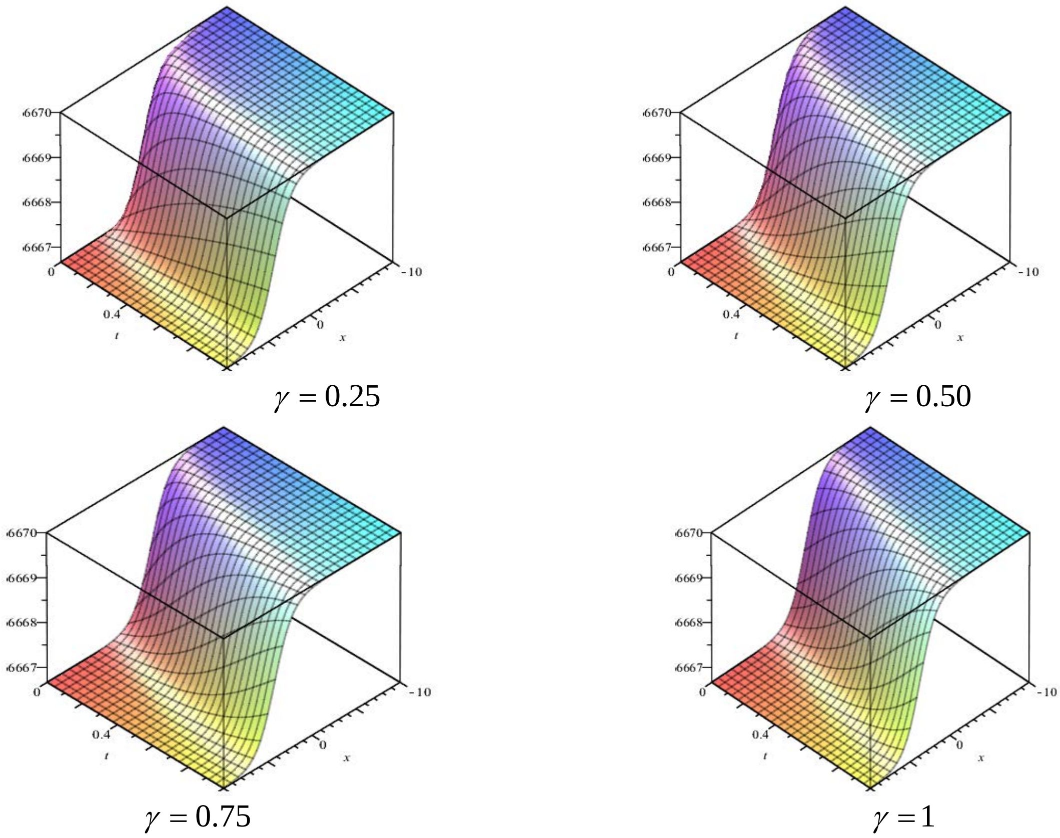

Figure 1. Travelling wave solutions.

2^nd^ Solution set

$$\left\{\begin{aligned}&\Omega = -\frac{2p\left(a_{-2}^2 b_2^2 + 2a_{-2}a_2 b_{-2}b_2 + a_2^2 b_{-2}^2\right)}{b_{-2}^2 b_2^2}, a_{-2} = a_{-2}, a_{-1} = 0, a_2 = a_2, a_1 = \frac{a_{-1}b_1}{b_{-2}},\\ &b_{-2} = b_{-2}, b_{-1} = 0, b_1 = b_1, b_2 = b_2\end{aligned}\right\}$$

We, therefore, obtained the following generalized solitary solution $v(x,t)$ of equation (9)

$$v(x,t) = \frac{\dfrac{a_{-1}b_1}{b_{-2}}\exp\left(xp - \dfrac{2p\left(a_{-2}^2 b_2^2 + 2a_{-2}a_2 b_{-2}b_2 + a_2^2 b_{-2}^2\right)t^\gamma}{b_{-2}^2 b_2^2\Gamma(1+\gamma)}\right) + a_2\exp\left(2xp - \dfrac{4p\left(a_{-2}^2 b_2^2 + 2a_{-2}a_2 b_{-2}b_2 + a_2^2 b_{-2}^2\right)t^\gamma}{b_{-1}^2 b_1^2\Gamma(1+\gamma)}\right)}{b_1\exp\left(2xp - \dfrac{4p\left(a_{-2}^2 b_2^2 + 2a_{-2}a_2 b_{-2}b_2 + a_2^2 b_{-2}^2\right)t^\gamma}{b_{-2}^2 b_2^2\Gamma(1+\gamma)}\right) + b_2\exp\left(2xp - \dfrac{4p\left(a_{-2}^2 b_2^2 + 2a_{-2}a_2 b_{-2}b_2 + a_2^2 b_{-2}^2\right)t^\gamma}{b_{-2}^2 b_2^2\Gamma(1+\gamma)}\right)} + \frac{a_{-2}}{b_{-2}}$$

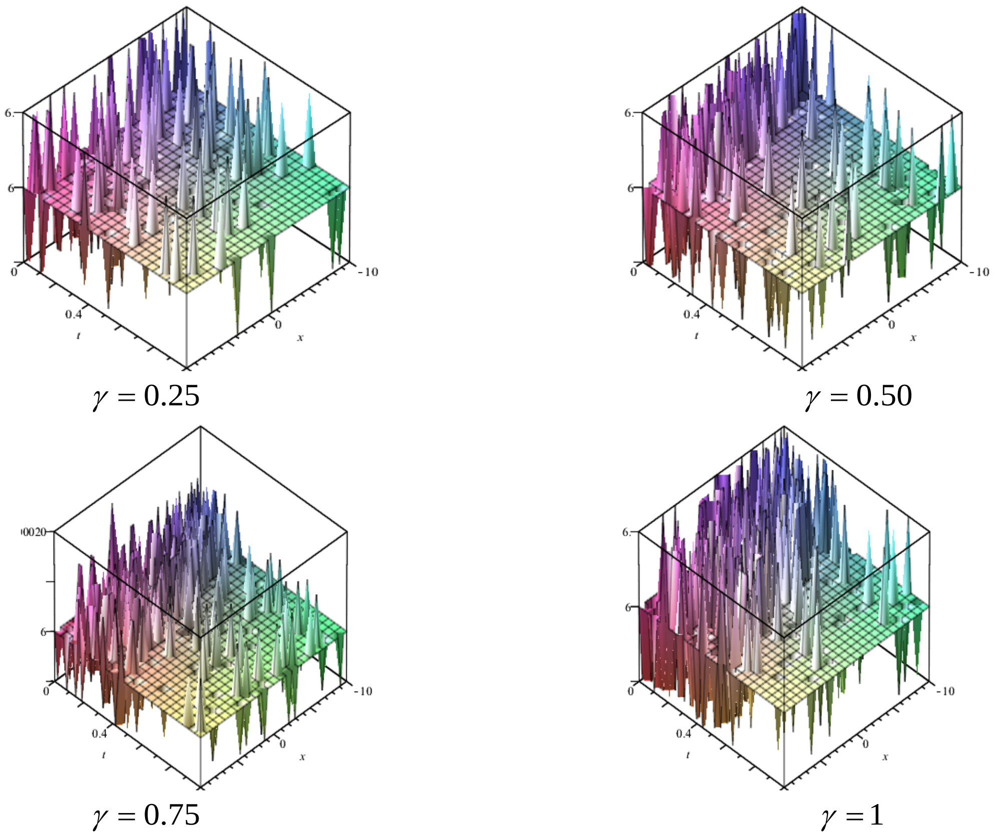

Figure 2. Travelling wave solutions.

3rd Solution set

$$\left\{ p = p, \Omega = \Omega, a_{-2} = -\frac{b_{-1}a_1}{b_2}, a_1 = a_1, a_{-1} = -\frac{b_{-1}a_2}{b_2}, a_2 = a_2, b_{-2} = -\frac{b_{-1}b_1}{b_2}, b_{-1} = b_{-1}, b_1 = b_1, b_2 = b_2 \right\}$$

We, therefore, obtained the following generalized solitary solution of equation (9)

$$v(x,t) = \frac{a_1 \exp\left(xp + \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right) + a_2 \exp\left(2xp + \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right)}{b_1 \exp\left(xp + \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right) + b_2 \exp\left(2xp + \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right)} + \frac{\dfrac{-a_2 b_{-1} \exp\left(-xp - \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right)}{b_2} - \dfrac{a_1 b_{-1} \exp\left(-2xp - \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right)}{b_2}}{b_{-1} \exp\left(-xp - \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right) + \dfrac{b_1 b_{-1} \exp\left(-2xp - \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)} \right)}{b_2}}$$

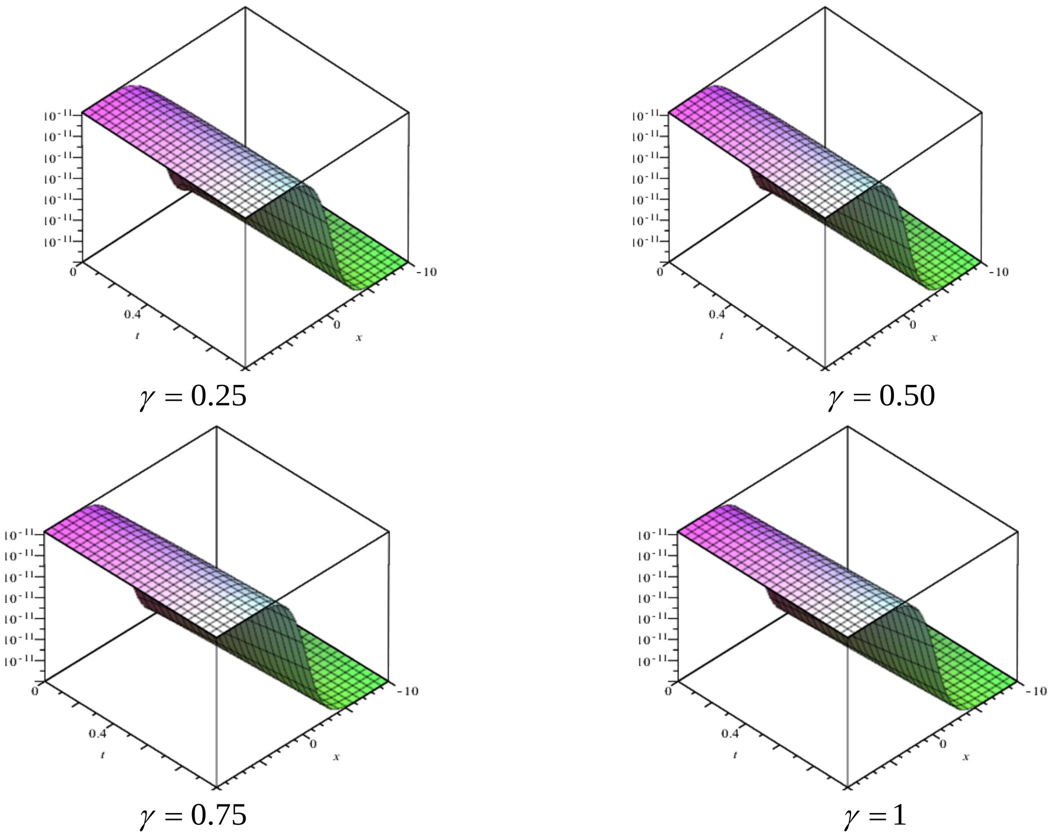

Figure 3. Travelling wave solutions.

4th Solution set

$$\left\{ p = p, \Omega = \Omega, a_{-2} = a_{-2}, a_{-1} = a_{-1}, a_1 = -\frac{a_{-2} b_1}{b_{-2}}, a_2 = -\frac{a_{-1} b_1}{b_{-2}}, b_{-2} = b_{-2}, b_{-1} = 0, b_1 = b_1, b_2 = 0 \right\}$$

We, therefore, obtained the following generalized solitary solution $v(x,t)$ of equation (9)

$$v(x,t) = \frac{a_{-1} \exp\left(-xp - \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)}\right) + a_{-2} \exp\left(-2xp - \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)}\right)}{b_{-2} \exp\left(-2xp - \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)}\right)} - \frac{\frac{a_{-2} b_1 \exp\left(xp - \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)}\right)}{b_{-2}} - \frac{a_{-1} b_1 \exp\left(2xp + \frac{2\Omega t^{\gamma}}{\Gamma(1+\gamma)}\right)}{b_{-2}}}{b_1 \exp\left(-xp - \frac{\Omega t^{\gamma}}{\Gamma(1+\gamma)}\right)}$$

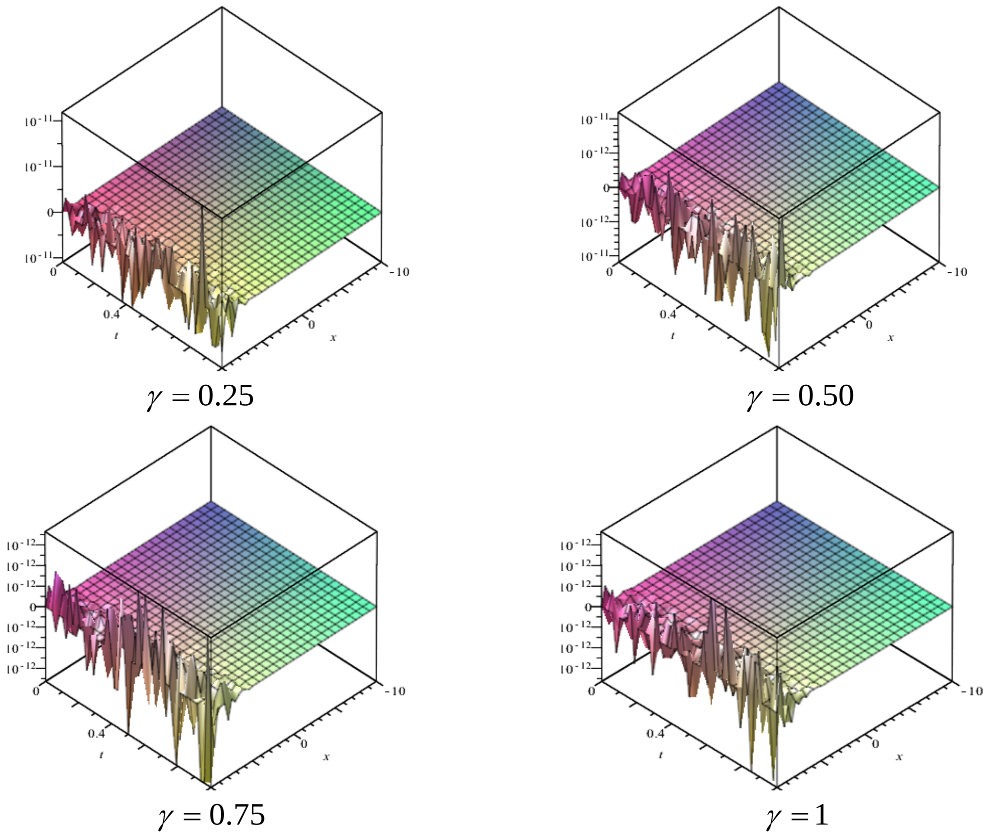

Figure 4. Travelling wave solutions.

5. RESULTS AND DISCUSSION

From the above figures we note that soliton is a wave which preserve its shape after it collides with another wave of the same kind. By solving nonlinear modified KdV equation of fractional order, we attain desired solitary wave solutions for different value of γ such as 0.25, 0.5, 0.75 and 1. The solitary wave moves toward right if the velocity is positive or left directions if the velocity is negative and the amplitudes and velocities are controlled by various parameters. Solitary waves show more complicated behaviors which are controlled by various parameters. Figures signifies graphical representation for different values of parameters. In both cases, for various values of parameters, we attain identical solitary wave solutions which obviously comprehend that final solution does not effectively based upon these parameters. So we can choose arbitrary values of such parameters. Since the solutions depend on arbitrary parameters, we choose different parameters as input to our simulations.

6. CONCLUSION

This article is devoted to attain, test and analyze the novel soliton wave solutions and physical properties of nonlinear partial differential equation. For this, fractional order nonlinear modified KdV equation is considered and we apply improved modified Exp function method. We attain desired soliton solutions of various types for different values of parameters. It is guaranteed the accuracy of the attain results by backward substitution into the original equation with Maple 17. The scheming procedure of this method is simplest, straight and productive. We observed that the under study technique is more reliable and have minimum computational task, so widely applicable. In precise we can say, this method is quite competent and much operative for evaluating exact solution of NLEEs. The validity of given algorithm is totally hold up with the help of the computational work, the graphical representations and successive results. Results obtained by this method are very encouraging and reliable for solving any other type of NLEEs. The graphical representations clearly indicate the solitary solutions.

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