ORIGINAL PAPER

THE VARIATIONAL ITERATION METHOD ON K-S EQUATION USING HE'S POLYNOMIALS

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Abstract.*In this paper, we apply a modified version of the variational iteration method (MVIM) for solving K–S equation. The proposed modification is made by introducing He's polynomials in the correction functional of the variational iteration method (VIM). The nonlinear term can be easily handled by the use of He's Polynomials. The use of Lagrange multiplier coupled with He's polynomials are the clear advantages of this technique over the decomposition method.*

Keywords:*Variational iteration method, He's polynomials, K–S equation, nonlinear differential equation*

1. INTRODUCTION

In order to model behavior and effects of many phenomena in different fields of science and engineering by mathematical concepts, nonlinear differential equations are introduced. During the last four decades, there has been a lot of interest in studying dynamical systems that arise from solving the initial value problem for nonlinear partial differential equations. The Kuramoto–Sivashinsky (KS) equation is a model of nonlinear partial differential equation (NLPDE) frequently encountered in the study of continuous media. The Kuramoto-Sivashinsky [henceforth KS] system [1, 2], which arises in the description of stability of flame fronts, reactiondiffusion systems and many other physical settings [3], is one of the simplest nonlinear PDEs that exhibit spatiotemporally chaotic behavior.

The idea of the VIM was first pioneered by He [4]. Then, the VIM is applied by He [5,6] in order to solve autonomous ordinary differential equation as well as delay differential equation. Also, new development and applications of the VIM to nonlinear wave equation, nonlinear fractional differential equations, nonlinear oscillations and nonlinear problems arising in various engineering applications is presented by He and Xh [7]. Variational iteration method is strong and efficient method which can be widely used to handle linear and nonlinear models. The VIM has no specific requirements for nonlinear operators. The method gives the solution in the form of rapidly convergent successive approximations that may give the exact solution if such a solution exists.

Several techniques including decomposition, finite element, Galerkin and cubic spline are employed to solve such equations analytically and numerically. Most of these used schemes are coupled with the inbuilt deficiencies like calculation of the so-called Adomian's polynomials and non compatibility with the physical nature of the problems. In a later work Ghorbani et al. introduced He's polynomials [8, 9] which are compatible with Adomian's polynomials but are easier to calculate and are more user friendly. This very reliable modified

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version [10-12] (MVIM) has been proved useful in coping with the physical nature of the nonlinear problems and hence absorbs all the positive features of the coupled techniques. The basic motivation of this paper is the application of this elegant coupling of He's polynomials and correction functional of VIM for solving K-S equationequations. The numerical results are very encouraging.

Solving for the approximate solitary wave solutions of nonlinear evolution equations has long been a major concern for both mathematicians and physicists. Although various methods for obtaining solitary wave solutions to nonlinear evolution equations have been established, it is not easy to find the exact solutions of some nonlinear evolution equations, particularly nonintegrable nonlinear evolution equations. In this paper,Variational iteration method Using He's Polynomials (MVIM), we study the approximate solitary wave solutions of the Kuramoto-Sivashinsky (KS) equation.

2. MODIFIED VARIATIONAL ITERATION METHOD USING HE'S POLYNOMIALS (MVIM)

To illustrate the basic concept of the MVIM, we consider the following general differential equation:

$$Lu(t)+Nu(t)= g(t), \tag{1}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytic function.

According to the variational iteration method, we can construct the following correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(u_n(\xi) + N\tilde{u}_n(x,t) - g(t)d\xi, \tag{2}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory and $\tilde{u}_n$ is considered as a restricted variation which means $\tilde{u}_n = 0$. Now, we apply He's polynomials

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_n(x,t) + p\int_0^t \lambda(\xi)\left[\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)}\, N(\tilde{u}_n)\right] d\xi - \int_0^T \lambda(\xi)\, g(\xi) d\xi$$

which is the MVIM [10-12] and is formulated by the coupling of VIM and He's polynomials.

3. THE K–S EQUATION

Consider the following the K–S equation

$$\frac{\partial u}{\partial t} + v\frac{\partial^4 u}{\partial^4 x} + \frac{\partial^2 u}{\partial^2 x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0, \tag{3}$$

$$u(x,t) = u(x+L,t)\ ,$$

$$u(x,0) = u_0(x,t)\ ,$$

where $L$is a period, has first been derived in 1976 by Kuramoto and Tsuzuki [1] as a modelequation for interfacial instabilities in the context of angular phase turbulence for a system of a reaction–diffusion equation that model the Belouzov–Zabotinskii reaction in three space dimensions,and independently, in 1977, by Sivashinsky [2] to model thermal diffusion instabilities observed in laminar flame fronts in two space dimensions.In the last two decades, many theoretical andnumerical studies were devoted to the K–S equation [13-17].

By setting $v=4$ and $L = 2\pi$

$$\frac{\partial u}{\partial t}+4\frac{\partial^4 u}{\partial^4 x}+\frac{\partial^2 u}{\partial^2 x}+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2=0, \quad 0\le x\le 2\pi \tag{4}$$
$$u(x,t)=u(x+2\pi,t)\,,$$
$$u(x,0)=u_0(x,t)=sin2x\,,$$

The correction functional for Eq. (3.2) is in the following form:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi}+4\frac{\partial^4 u_n(x,\xi)}{\partial^4 x}+\frac{\partial^2 u_n(x,\xi)}{\partial^2 x}+\frac{1}{2}\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\right)d\xi, \tag{5}$$

where u_n is restricted variation$\delta u_n=0$, λ is a Lagrange multiplier andu_0 is an initial approximation or trial function.

With the above correction functional stationary we have:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi}+4\frac{\partial^4 u_n(x,\xi)}{\partial^4 x}+\frac{\partial^2 u_n(x,\xi)}{\partial^2 x}+\frac{1}{2}\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\right)d\xi,$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi}\right)d\xi, \tag{6}$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)\big(1+\lambda(\xi)\big)-\delta\int_0^t \lambda'(\xi)u_n(x,t)\,d\xi \tag{7}$$

By using the following stationary conditions:

$$\delta u_n:\ 1+\lambda(\xi)=0$$
$$\delta u_n:\ \lambda'(\xi)=0$$

This gives the Lagrange multiplier $\lambda(\xi)=-1$ therefore the following iteration formula becomes as:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\xi)}{\partial \xi}+4\frac{\partial^4 u_n(x,\xi)}{\partial^4 x}+\frac{\partial^2 u_n(x,\xi)}{\partial^2 x}+\frac{1}{2}\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\right)d\xi, \tag{8}$$

We can select $u_0(x,y)=sin2x$ from the given condition.

Now we apply the variational iteration method usingHe's polynomials (MVIM):

$$u_0+pu_1+p^2u_2+\cdots = u_0(x,t)-p\int_0^t\left[\frac{\partial u_0}{\partial \xi}+p\frac{\partial u_1}{\partial \xi}+p^2\frac{\partial u_2}{\partial \xi}+\cdots\right]d\xi-p\int_0^t 4\left[\frac{\partial^4 u_0}{\partial^4 x}+p\frac{\partial^4 u_1}{\partial^4 x}+p^2\frac{\partial^4 u_2}{\partial^4 x}+\cdots\right]d\xi-p\int_0^t\left[\frac{\partial^2 u_0}{\partial^2 x}+p\frac{\partial^2 u_1}{\partial^2 x}+p^2\frac{\partial^2 u_2}{\partial^2 x}+\cdots\right]d\xi-\int_0^t\frac{1}{2}\left[\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+\cdots\right]^2 d\xi, \tag{9}$$

By comparing the co-efficient of like powers of p, weobtain

$$p^{(0)}:\ u_0(x,t)=sin2x,$$

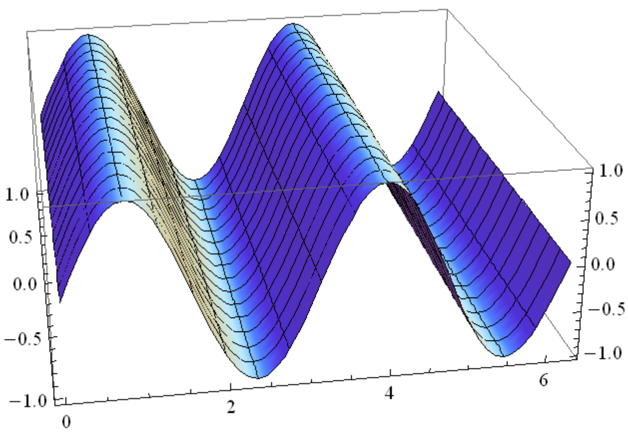

Figure 1. Surface plot of first approximate solution $u_0(x,t)$.

$$p^{(1)}: u_1(x,t)=\mathrm{Sin}[2x]-t(2\mathrm{Cos}[2x]^2+60\mathrm{Sin}[2x])$$

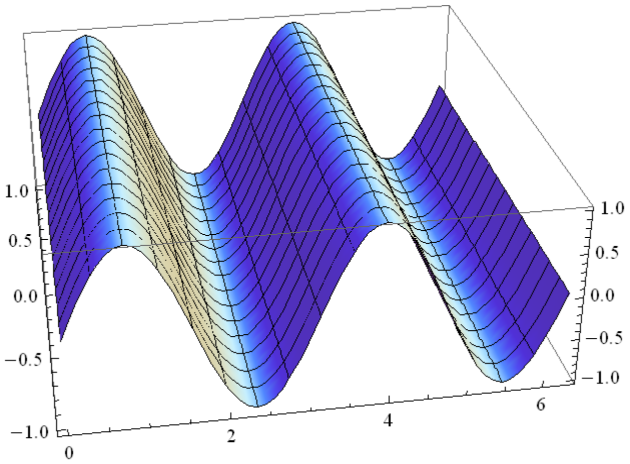

Figure 2. Surface plot of second approximate solution *$u_1(x,t)$*.

$$p^{(2)}: u_2(x,t) = (1 + \mathrm{Cos}[4x] + 60\mathrm{Sin}[2x])\mathrm{t} - (60 + 564\mathrm{Cos}[4x] + 1798\mathrm{Sin}[2x] - 2\mathrm{Sin}[6x])t^2 + (\frac{3604}{3} + 1200\mathrm{Cos}[4x] - \frac{4}{3}\mathrm{Cos}[8x] - 80\mathrm{Sin}[2x] - 80\mathrm{Sin}[6x])t^3$$

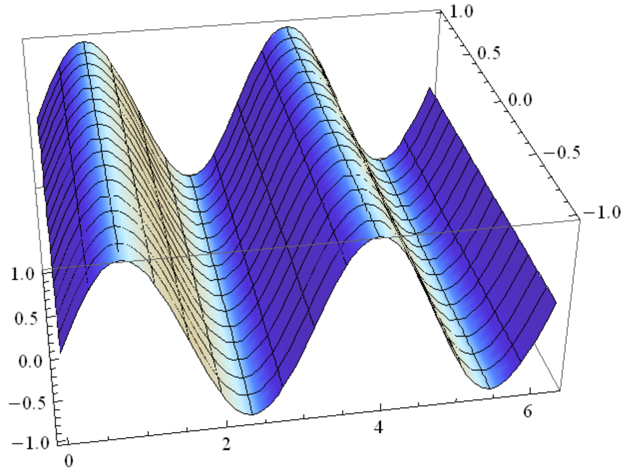

Figure 3. Surface plot of third approximate solution *$u_2(x,t)$*.

$$p^{(3)}: u_3(x,t) = t + t\mathrm{Cos}[4x] + 60t\mathrm{Sin}[2x] - 60t^2 - 60t^2\mathrm{Cos}[4x] + 2t^2\mathrm{Sin}[6x] + 2t^2\mathrm{Sin}[2x] + \frac{3604t^3}{3} - 40t^3\mathrm{Cos}[2x] - 188304t^3\mathrm{Cos}[4x] + 120t^3\mathrm{Cos}[2x]\mathrm{Cos}[4x] - \frac{4}{3}t^3\mathrm{Cos}[8x] + \frac{14392}{3}t^3\mathrm{Cos}[x]\mathrm{Cos}[2x]\mathrm{Sin}[x] - 36040t^3\mathrm{Sin}[2x] + 3352t^3\mathrm{Sin}[6x] - \frac{4}{3}t^3\mathrm{Cos}[2x]\mathrm{Sin}[6x] + 32939t^4\mathrm{Cos}[2x] - 53456t^4\mathrm{Cos}[2x]\mathrm{Cos}[4x] + 302400t^4\mathrm{Cos}[4x] - 1200t^4\mathrm{Sin}[2x] - 5440t^4\mathrm{Cos}[8x] + 5t^4\mathrm{Cos}[2x]\mathrm{Cos}[8x] - 108120t^4\mathrm{Cos}[2x]\mathrm{Sin}[2x] - 102960t^4\mathrm{Sin}[6x] + 1368t^4\mathrm{Cos}[2x]\mathrm{Sin}[6x] - \frac{3990896}{5}t^5\mathrm{Cos}[2x] + \frac{6526912}{5}t^5\mathrm{Cos}[2x]\mathrm{Cos}[4x] - 2576t^5\mathrm{Cos}[2x]\mathrm{Cos}[8x] - \frac{4047512}{15}t^5\mathrm{Cos}[2x]\mathrm{Sin}[6x]$$

$$
\begin{aligned}
&- 2576t^5\text{Cos}[2x]\text{Cos}[8x] - \frac{4047512}{15}t^5\text{Cos}[2x]\text{Sin}[6x] \\
&+ \frac{22617712}{15}t^5\text{Cos}[2x]\text{Sin}[2x] + 8t^5\text{Cos}[2x]\text{Sin}[10x] \\
&+ \frac{6600944}{9}t^6\text{Cos}[2x] - \frac{19653064}{9}t^6\text{Cos}[2x]\text{Cos}[4x] \\
&+ \frac{748816}{9}t^6\text{Cos}[2x]\text{Cos}[8x] - \frac{56}{9}t^6\text{Cos}[2x]\text{Cos}[12x] \\
&+ 1095584t^6\text{Cos}[2x]\text{Sin}[6x] - 1824t^6\text{Cos}[2x]\text{Sin}[10x] \\
&- 96576t^6\text{Cos}[2x]\text{Sin}[2x] + \frac{577280}{21}t^7\text{Cos}[2x] \\
&- \frac{2308480}{21}t^7\text{Cos}[2x]\text{Cos}[4x] - \frac{2881280}{21}t^7\text{Cos}[2x]\text{Cos}[8x] \\
&+ \frac{640}{3}t^7\text{Cos}[2x]\text{Cos}[12x] - \frac{7455104}{9}t^7\text{Cos}[2x]\text{Sin}[2x] \\
&- \frac{51667072}{63}t^7\text{Cos}[2x]\text{Sin}[6x] + \frac{518528}{63}t^7\text{Cos}[2x]\text{Sin}[10x] \\
&- \frac{128}{63}t^7\text{Cos}[2x]\text{Sin}[14x]
\end{aligned}
$$

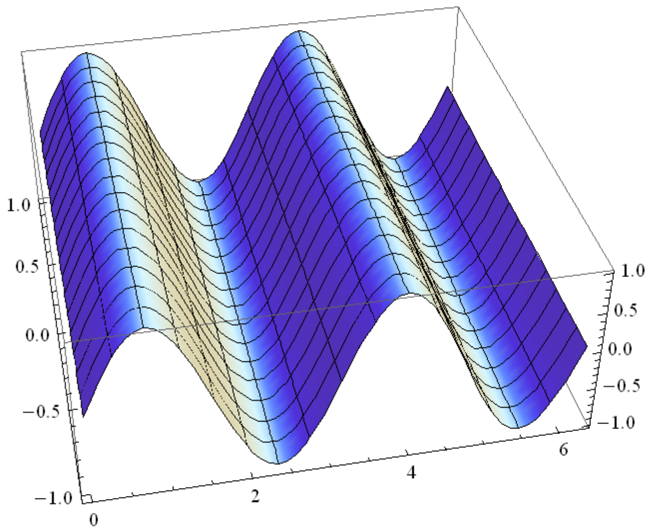

Figure 4. Surface plot of fourth approximate solution $u_3(x,t)$.

$$
\begin{aligned}
u(x,t) =\ & t + t\text{Cos}[4x] + 60t\text{Sin}[2x] - 60t^2 - 60t^2\text{Cos}[4x] + 2t^2\text{Sin}[6x] + 2t^2\text{Sin}[2x] \\
&+ \frac{3604t^3}{3} - 40t^3\text{Cos}[2x] - 188304t^3\text{Cos}[4x] + 120t^3\text{Cos}[2x]\text{Cos}[4x] \\
&- \frac{4}{3}t^3\text{Cos}[8x] + \frac{14392}{3}t^3\text{Cos}[x]\text{Cos}[2x]\text{Sin}[x] - 36040t^3\text{Sin}[2x] \\
&+ 3352t^3\text{Sin}[6x] - \frac{4}{3}t^3\text{Cos}[2x]\text{Sin}[6x] + 32939t^4\text{Cos}[2x] \\
&- 53456t^4\text{Cos}[2x]\text{Cos}[4x] + 302400t^4\text{Cos}[4x] - 1200t^4\text{Sin}[2x] \\
&- 5440t^4\text{Cos}[8x] + 5t^4\text{Cos}[2x]\text{Cos}[8x] - 108120t^4\text{Cos}[2x]\text{Sin}[2x] \\
&- 102960t^4\text{Sin}[6x] + 1368t^4\text{Cos}[2x]\text{Sin}[6x] - \frac{3990896}{5}t^5\text{Cos}[2x] \\
&+ \frac{6526912}{5}t^5\text{Cos}[2x]\text{Cos}[4x] - 2576t^5\text{Cos}[2x]\text{Cos}[8x]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4047512}{15}t^5\text{Cos}[2x]\text{Sin}[6x]+\frac{22617712}{15}t^5\text{Cos}[2x]\text{Sin}[2x]\\
&+8t^5\text{Cos}[2x]\text{Sin}[10x]+\frac{6600944}{9}t^6\text{Cos}[2x]-\frac{19653064}{9}t^6\text{Cos}[2x]\text{Cos}[4x]\\
&+\frac{748816}{9}t^6\text{Cos}[2x]\text{Cos}[8x]-\frac{56}{9}t^6\text{Cos}[2x]\text{Cos}[12x]\\
&+1095584t^6\text{Cos}[2x]\text{Sin}[6x]-1824t^6\text{Cos}[2x]\text{Sin}[10x]\\
&-96576t^6\text{Cos}[2x]\text{Sin}[2x]+\frac{577280}{21}t^7\text{Cos}[2x]-\frac{2308480}{21}t^7\text{Cos}[2x]\text{Cos}[4x]\\
&-\frac{2881280}{21}t^7\text{Cos}[2x]\text{Cos}[8x]+\frac{640}{3}t^7\text{Cos}[2x]\text{Cos}[12x]\\
&-\frac{7455104}{9}t^7\text{Cos}[2x]\text{Sin}[2x]-\frac{51667072}{63}t^7\text{Cos}[2x]\text{Sin}[6x]\\
&+\frac{518528}{63}t^7\text{Cos}[2x]\text{Sin}[10x]-\frac{128}{63}t^7\text{Cos}[2x]\text{Sin}[14x]+\cdots
\end{aligned}
$$

4. CONCLUSION

In this paper, we successfully applied the variational iterationmethod using He's polynomials (MVIM) for findingthe approximate solution ofKuramoto–Sivashinsky (KS) equation. The obtained approximate numerical solutions maintain a good accuracy compared with the exact solution.The use of Lagrange multiplier coupled withHe's polynomials are the clear advantages of this techniqueover the decomposition method.

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