ORIGINAL PAPER

FRACTIONAL INTEGRAL FORMULAS FOR THE MERICHEV-SAIGO MAEDA OPERATORS

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Abstract. *This paper deals with the evaluation of the fractional integrals involving Saigo- Maeda operators of the product of the general class of polynomials and the H-function containing the factor* $x^{\lambda}(x^k+c^k)^{-\rho}$ *in its argument. Some interesting special cases are derived. The results given by Saigo and Raina [6] and Chaurasia and Gupta [2] follow as special cases of the results proved in this paper.*
Keywords: *Fractional calculus, H-function, Appell function* F_3*, Saigo-Maeda operators, generalized hypergeometric function.*

1. INTRODUCTION

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators involving Appell function F_3 are defined by Saigo and Maeda [5] by means of the following equations:

$$(I_{+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}t^{-\alpha'} F_3(\alpha,\alpha',\beta,\beta';\gamma;\ 1-t/x,\ 1-x/t)f(t)dt,\ \Re(\gamma)>0 \qquad (1.1)$$

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha} F_3(\alpha,\alpha',\beta,\beta';\gamma;\ 1-x/t,\ 1-t/x)f(t)dt,\ \Re(\gamma)>0 \qquad (1.2)$$

The general class of polynomials is defined by Srivastava [8, p.1, Eq. (1)] in the following manner:

$$S_w^u[x] = \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s, \qquad w = 0, 1, 2, \ldots \qquad (1.3)$$

where u is an arbitrary positive integer and the coefficients $A_{w,s}$ (w, s $\geq$ 0) are arbitrary constants, real or complex.

The following is the series representation of the H-function as given in [4].

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$$H^{M,N}_{P,Q}\left[z \,\middle|\, \begin{matrix}(a_P,\ A_P)\\(b_Q,\ B_Q)\end{matrix}\right] = \sum_{h=1}^{M}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}\ \chi(\xi)\ x^{\xi}}{\upsilon\,!\,B_h} \tag{1.4}$$

where $\xi = \dfrac{b_h + \upsilon}{B_h}$ $(h=1,\ldots,M)$

$$\chi(\xi) = \frac{\prod_{j=1,\ j\neq h}^{M}\Gamma(b_j - B_j\xi)\ \prod_{j=1}^{N}\Gamma(1-a_j + A_j\xi)}{\prod_{j=M+1}^{Q}\Gamma(1-b_j + B_j\xi)\ \prod_{j=N+1}^{P}\Gamma(a_j - A_j\xi)} \tag{1.5}$$

and which exists for $x \neq 0$ if $\mu^* > 0$ and for $0 < |z| < \beta^{-1}$ if $\mu^* = 0$ where

$$\mu^* = \sum_{j=1}^{Q} B_j - \sum_{j=1}^{P} A_j \quad \textit{and} \quad \beta = \prod_{j=1}^{P}(A_j)^{A_j}\prod_{j=1}^{Q}(B_j)^{-B_j}$$

This result is due to Braaksma [1].A detailed account of the H-function is available from the monograph written by Mathai and Saxena [4]. For our purpose, we recall the generaliged hypergeometric series defined by [4].

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n\ z^n}{(\beta_1)_n\cdots(\beta_q)_n\ n!} = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z), \tag{1.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n = \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in N = \{1,2,3,\ldots\})\end{cases} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\backslash\mathbb{Z}_0) \tag{1.7}$$

and $\mathbb{Z}_0$ denotes the set of nonpositive integers. We now establish

Lemma 1: If $\Re(\gamma) > 0$, $\lambda > 0$, k = 1, 2, 3, …, c is a positive number and ρ is a complex number, then there holds the relation

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[x^{\lambda}(x^k + c^k)^{-\rho}]\right)(x) = x^{\lambda-\alpha-\alpha'+\gamma}\ c^{-k\rho}\ \frac{\Gamma(\lambda+1)\,\Gamma(\lambda-\alpha'+\beta'+1)\,\Gamma(\lambda-\alpha-\alpha'-\beta+\gamma+1)}{\Gamma(\lambda+\beta'+1)\,\Gamma(\lambda-\alpha-\alpha'+\gamma+1)\,\Gamma(\lambda-\alpha'-\beta+\gamma+1)}$$

$$\times\ {}_{3k+1}F_{3k}\left[\begin{matrix}\rho,\Delta(k,\lambda+1),\ \Delta(k,\lambda+\beta'-\alpha'+1),\ \Delta(k,\lambda+\gamma+1-\alpha-\alpha'-\beta);\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad -\dfrac{x^k}{c^k}\\ \Delta(k,\lambda+\beta'+1),\ \Delta(k,\lambda+\gamma+1-\alpha-\alpha'),\ \Delta(k,\lambda+\gamma+1-\alpha'-\beta);\end{matrix}\right] \tag{1.8}$$

where $\Re(\gamma) > 0,\ \Re(\lambda-\alpha') + \min\{-\Re(\alpha'), -\Re(\beta'), \Re(\gamma-\alpha-\beta)\}$ $, \Delta(k,\alpha)$ represent the sequence of parameters

$$\frac{\alpha}{k},\ \frac{\alpha+1}{k},\ \ldots,\ \frac{\alpha+k-1}{k},$$

and ${}_{3k+1}F_{3k}\ (\cdot)$ is the generalized hypergeometric function, defined in [4].

Proof: To prove lemma 1, we first operate the fractional integral operator (1.1) with f (t) = $t^{\lambda}\ (t^k + c^k)^{-\rho}$ and express Appell Function F_3 and $(t^k + c^k)^{-\rho}$ in terms of their equivalent series by means of the formula

$$(t^k + c^k)^{-\rho} = c^{-k\rho} \sum_{q=0}^{\infty} \frac{(\rho)_q}{q!} \left(-\frac{t^k}{c^k}\right)^q \tag{1.9}$$

On interchange the order of integration and summation, which is permissible due to the absolute convergence, is evaluated the inner integral by means of the formula given by Saxena, Ram and Kalla [7].

$$\int_0^x t^{s-1}\ (x-t)^{\gamma-1}\ F_3\ (\alpha, \alpha', \beta, \beta'; \gamma\ ; 1-t/x\ , 1-x/t\ )\ dt$$

$$= \Gamma\ (\gamma)\ x^{\gamma+s-1}\ \Gamma\left[\begin{matrix} s+\alpha',\ s+\beta',\ s+\gamma-\alpha-\beta \\ s+\alpha'+\beta',\ s+\gamma-\alpha,\ s+\gamma-\beta \end{matrix}\right] \tag{1.10}$$

where $\Re(\gamma) > 0$, $\Re(s) >$ max {-$\Re(\alpha')$, - $\Re(\beta')$, $\Re(\alpha+\beta-\gamma)$}, the result (1.8) follows. When $\alpha'=0$, (1.8) reduces to the result given in [2, p. 334, Eq. (1.6)].

Lemma 2: If $\Re(\gamma) > 0$, $\lambda > 0$, k = 1, 2, 3, … , c is a positive number and ρ is a complex number, then we have

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\ [x^{\lambda}\ (x^k + c^k)^{-\rho}\ ]\right)(x) = x^{\lambda-\alpha-\alpha'+\gamma}\ c^{-k\rho}\ \frac{\Gamma(\alpha+\alpha'-\gamma-\lambda)\ \Gamma(\alpha+\beta'-\gamma-\lambda)\ \Gamma(-\beta-\lambda)}{\Gamma(\alpha+\alpha'+\beta'-\gamma-\lambda)\ \Gamma(-\lambda)\ \Gamma(\alpha-\beta-\lambda)}$$

$$\times {}_{3k+1}F_{3k}\left[\begin{matrix} \rho,\ \Delta(k, \lambda-\alpha-\alpha'-\beta'+\gamma+1),\ \Delta(k, \lambda+1),\ \Delta(k, \lambda-\alpha-\beta+1); \\ \\ \Delta(k, \lambda-\alpha-\alpha'+\gamma+1),\ \Delta(k, \lambda-\alpha-\beta'+\gamma+1),\ \Delta(k, \lambda+\beta+1); \end{matrix}\ -\frac{x^k}{c^k}\right] \tag{1.11}$$

where $\Re(\gamma) > 0,\quad \Re(\gamma+\lambda-k\rho)+\min\{-\Re(\alpha'),\ -\Re(\beta'),\ \Re(\gamma-\alpha-\beta)\}+1 > 0$

Proof: To establish lemma 2, we take $f(t) = t^{\lambda}\ (t^k + c^k)^{-\rho}$ in equation (1.2) and write series expansions for the Appell function and $(t^k + c^k)^{-\rho}$, then interchanging the order of integration and summation, which is permissible due to the absolute convergence and evaluating the inner integral by means of the formula given by Saxena, Ram and Kalla [7].

$$\int_x^{\infty} t^{s-1}\ (t-x)^{\gamma-1}\ F_3\ (\alpha, \alpha', \beta, \beta'; \gamma\ ; 1-x/t, 1-t/x\ )\ dt$$

$$= \Gamma(\gamma)\quad x^{\gamma+s-1}\ \Gamma\left[\begin{matrix} 1+\alpha'-\gamma-s,\ 1+\beta'-\gamma-s,\ 1-\alpha-\beta-s \\ 1-s+\alpha'+\beta'-\gamma,\quad 1-s-\alpha,\quad 1-s-\beta \end{matrix}\right] \tag{1.12}$$

where $\Re(\gamma) > 0$, $\Re(\rho) < 1 + \min\{\Re(\alpha' - \gamma), \Re(\beta' - \gamma), -\Re(\alpha + \beta)\}$ and using the well known relation $(\alpha)_{-n} = \dfrac{(-1)^n}{(1-\alpha)_n}$, the desired result (1.11) is obtained. For $\alpha' = 0$, (1.11) yields the result given in [2].

2. FRACTIONAL INTEGRAL FORMULAS

If $$f(t) = t^{\lambda}(t^k + c^k)^{-\rho} S_w^u [y t^m (t^k + c^k)^{-n}] H_{P,Q}^{M,N}\left[z t^h (t^k + c^k)^{-r}\left|\begin{matrix}(a_P, A_P)\\(b_Q, B_Q)\end{matrix}\right.\right] \tag{2.1}$$

we obtain :

(I) $$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \sum_{s=0}^{[w/u]}\sum_{h=1}^{M}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}(-w)_{us} A_{w,s}\ x(\xi)\ x^{T-\alpha-\alpha'+\gamma} z^{\xi} y^{s}}{s!\ \upsilon!\ B_h\ c^R}$$
$$\times\frac{\Gamma(T+1)\Gamma(T-\alpha'-\beta'+1)\Gamma(T-\alpha-\alpha'-\beta+\gamma+1)}{\Gamma(T+\beta'+1)\Gamma(T-\alpha-\alpha'+\gamma+1)\Gamma(T-\alpha'-\beta+\gamma+1)}$$
$$\times {}_{3k+1}F_{3k}\left[\begin{matrix}\rho+ns+r\xi,\ \Delta(k, T+1),\ \Delta(k, T+\beta'+1-\alpha'),\ \Delta(k, T+\gamma+1-\alpha-\alpha'-\beta);\\ \\ \Delta(k, T+\beta'+1),\ \Delta(k, T+\gamma+1-\alpha-\alpha'),\ \Delta(k, T+\gamma+1-\alpha'-\beta);\end{matrix}\ -\frac{x^k}{c^k}\right] \tag{2.2}$$

where

$T = \lambda + ms + h\xi$ and $R = k\rho + kns + kr\xi$; $T' = \sum_{j=1}^{N} A_j - \sum_{j=N+1}^{P} A_j + \sum_{j=1}^{M} B_j - \sum_{j=M+1}^{Q} B_j > 0$

The result (2.2) is valid for $\Re(\gamma) > 0$, $1 + \Re(\lambda - \alpha' + ms + h\dfrac{b_j}{B_j}) + \min\limits_{1\le j\le M}\{-\Re(\alpha'), -\Re(\beta'), -\Re(\gamma - \alpha - \beta)\} > 0$, $|\arg z| < T'\pi/2$, $T' > 0$, c is a positive number and ρ, m, n, h, r are complex numbers, k = 1, 2, 3, … , u is an arbitrary positive integer and the coefficients $A_{w,s}$ (w, s $\ge$ 0) are arbitrary constants, real or complex.

(II) $$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \sum_{s=0}^{[w/u]}\sum_{h=1}^{M}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}(-w)_{us} A_{w,s}\ x(\xi)\ z^{\xi} y^{s}}{s!\ \upsilon!\ B_h\ c^R} x^{T-\alpha-\alpha'+\gamma}$$
$$\times\frac{\Gamma(\alpha+\alpha'-\gamma-T)\Gamma(\alpha+\beta'-\gamma-T)\Gamma(-\beta-T)}{\Gamma(\alpha+\alpha'+\beta'-\gamma-T)\Gamma(-T)\Gamma(\alpha-\beta-T)}$$
$$\times {}_{3k+1}F_{3k}\left[\begin{matrix}\rho+ns+r\xi,\ \Delta(k, T-\alpha-\alpha'-\beta'+\gamma+1),\ \Delta(k, T+1),\ \Delta(k, T-\alpha+\beta+1);\\ \\ \Delta(k, T-\alpha-\alpha'+\gamma+1),\ \Delta(k, T-\alpha-\beta'+\gamma+1),\ \Delta(k, T+\beta+1);\end{matrix}\ -\frac{x^k}{c^k}\right] \tag{2.3}$$

where c is a positive number and ρ, m, n, h, r are complex numbers, k = 1, 2, 3, …; $\Re(\gamma) > 0$, $\Re(\gamma+\lambda-\alpha-k\rho)+\min\{-\Re(\alpha'), -\Re(\beta'), \Re(\gamma-\alpha-\beta)+ms+(h-kr) \max_{1\le j\le N}\left\{\frac{a_j-1}{A_j}\right\}<0$, $|\arg z| < T'\pi/2$, $T' > 0$. u is an arbitrary positive integer and the coefficients $A_{w,s}$ (w, s ≥ 0) are arbitrary constants, real or complex.

The proof of the results (2.2) and (2.3) can be developed on similar lines to that followed for the results (1.6) and (1.9).

3. INTERESTING SPECIAL CASES

(I) If we put α' = 0 in (2.2) and use the identity

$$(I_{0+}^{\alpha+\beta,\ 0,\ -\eta,\ \beta',\ \alpha} f)(x) = (I_{0+}^{\alpha,\ \beta,\ \eta} f)(x),$$

where the right hand sides represents the Saigo operators, we find that

$$\left(I_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \sum_{s=0}^{[w/u]}\sum_{h=1}^{M}\sum_{\upsilon=0}^{\infty} \frac{(-1)^{\upsilon}(-w)_{us} A_{w,s}\ x(\xi)\ z^{\xi}\ y^{s}}{s!\ \upsilon!\ B_h\ c^R} x^{T-\beta}$$
$$\times \frac{\Gamma(T+1)\ \Gamma(T+\eta-\beta+1)}{\Gamma(T-\beta+1)\ \Gamma(T+\alpha+\eta+1)}\ {}_{2k+1}F_{2k}\left[\begin{matrix}\rho+ns+r\xi,\ \Delta(k,T+1),\ \Delta(k,T+\eta+1-\beta);\\ \Delta(k,T+1-\beta),\ \Delta(k,T+\alpha+\eta+1);\end{matrix} -\frac{x^k}{c^k}\right] \tag{3.1}$$

which holds under the same conditions as given with (2.2) for α' = 0. Next if we put α' = 0 in (2.3) and use the identity

$$(I_{-}^{\alpha+\beta,\ 0,\ -\eta,\ \beta',\ \alpha} f)(x) = (I_{-}^{\alpha,\ \beta,\ \eta} f)(x),$$

we arrive at

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \sum_{s=0}^{[w/u]}\sum_{h=1}^{M}\sum_{\upsilon=0}^{\infty} \frac{(-1)^{\upsilon}(-w)_{us} A_{w,s}\ x(\xi)\ x^{T-\beta} y^{s}}{s!\ \upsilon!\ B_h\ c^R} \frac{\Gamma(\beta-T)\ \Gamma(\eta-T)}{\Gamma(-T)\ \Gamma(\alpha+\beta+\eta-T)}$$
$$\times\ {}_{2k+1}F_{2k}\left[\begin{matrix}\rho+ns+r\xi,\ \Delta(k,T+1),\Delta(k,T-\alpha-\beta-\eta+1);\\ \Delta(k,T-\eta+1),\ \Delta(k,T-\beta+1);\end{matrix} -\frac{x^k}{c^k}\right] \tag{3.2}$$

which holds under the same conditions as given with (2.3) for α'= 0, (3.1) and (3.2) are recently given by Chaurasia and Gupta [2].

(II) If we set h = r = 0, (2.2) and (2.3) yield

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,[x^{\lambda}(x^k+c^k)^{-\rho}]S_w^u[\,y\,x^m(x^k+c^k)^{-n}\,]\right)(x)$$

$$= \frac{x^{\lambda-\alpha-\alpha'+\gamma}}{c^{k\rho}}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}\;A_{w,s}\;y^s}{s\,!}\left(\frac{x^m}{c^{kn}}\right)^s$$

$$\times\frac{\Gamma(\lambda+ms+1)\;\;\Gamma(\lambda+ms-\alpha'+\beta'+1)\;\;\Gamma(\lambda+ms-\alpha-\alpha'-\beta+\gamma+1)}{\Gamma(\lambda+ms+\beta'+1)\;\;\Gamma(\lambda+ms-\alpha-\alpha'+\gamma+1)\;\;\Gamma(\lambda+ms-\alpha'-\beta+\gamma+1)}$$

$$\times\,{}_{3k+1}F_{3k}\left[\begin{matrix}\rho+ns,\,\Delta(k,\,\lambda+ms+1),\,\Delta(k,\lambda+ms+\beta'-\alpha'),\;\Delta(k,\lambda+ms+\gamma+1-\alpha-\alpha'-\beta);\\ \\ \Delta(k,\lambda+ms+\beta'+1),\;\Delta(k,\lambda+ms+\gamma+1-\alpha-\alpha'),\;\Delta(k,\lambda+ms+\gamma+1-\alpha'-\beta);\end{matrix}\;-\frac{x^k}{c^k}\right] \tag{3.3}$$

and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\,[x^{\lambda}(x^k+c^k)^{-\rho}]\,S_w^u[\,y\,x^m(x^k+c^k)^{-n}\,]\right)(x)$$

$$= \frac{x^{\lambda-\alpha-\alpha'+\gamma}}{c^{k\rho}}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}\;A_{w,s}\;y^s}{s\,!}\left(\frac{x^m}{c^{kn}}\right)^s$$

$$\times\frac{\Gamma(\alpha+\alpha'-\gamma-\lambda-ms)\;\;\Gamma(\alpha+\beta'-\gamma-\lambda-ms)\;\;\Gamma(-\beta-\lambda-ms)}{\Gamma(\alpha+\alpha'+\beta'-\gamma-\lambda-ms)\;\;\Gamma(-\lambda-ms)\;\;\Gamma(\alpha-\beta-\lambda-ms)}$$

$$\times\,{}_{3k+1}F_{3k}\left[\begin{matrix}\rho+ns,\,\Delta(k,\;\lambda+ms+1),\,\Delta(k,\,\lambda+ms-\alpha-\alpha'-\beta'+\gamma+1),\;\Delta(k,\,\lambda+ms-\alpha+\beta+1);\\ \\ \Delta(k,\,\lambda+ms-\alpha-\alpha'+\gamma+1),\;\Delta(k,\lambda+ms-\alpha-\beta'+\gamma+1),\;\Delta(k,\lambda+ms+\beta+1);\end{matrix}\;-\frac{x^k}{c^k}\right] \tag{3.4}$$

When $\rho = \alpha' = 0$, Lemma 1 and 2 are reducing to the results given by Saigo and Raina [6]. Finally it may be remarked here that the formulas established in this paper may be useful in obtaining the results for various classical orthogonal polynomials and special functions of one variable.

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