# SOME APPLICATIONS OF YOUNG-TYPE INEQUALITIES FOR OPERATORS AND MATRICES

LOREDANA CIURDARIU[1]

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**Abstract.** *The aim of this paper is to present some applications of several new Young-type inequalities given by Alzer, H., Fonseca, C. M. and Kovacec, A., for positive invertible operators on Hilbert spaces. Also will be presented an application of some refinements of Young's inequality given by Dragomir S. S., for positive definite matrices using their eigenvalues.*

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## 1. INTRODUCTION

The famous Young's inequality, state that:

$$a^\nu b^{1-\nu} < \nu\, a + (1 - \nu)b,$$

when a and b are positive numbers, a≠b and ν (0,1), see [20].

This inequality has many applications in various fields and there are a lot of interesting generalizations of this well-known inequality and its reverse, see for example [1, 6-13, 15, 16] and references therein.

As in [1], we consider $A_\nu(a, b) = \nu\, a + (1 - \nu)b$ and $G_\nu(a, b) = a^\nu b^{1-\nu}$.

The following result, given in [10] is a refinement of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah, [15, 16].

**Proposition 1.** For all a, b > 0 we have

$$3\nu\left(A_{\frac{1}{3}}(a, b) - G_{\frac{1}{3}}(a, b)\right) \le A_\nu(a, b) - G_\nu(a, b)$$

if $0 < \nu \le \frac{1}{3}$, and

$$3\nu(1 - \nu)\left(A_{\frac{2}{3}}(a, b) - G_{\frac{2}{3}}(a, b)\right) \le A_\nu(a, b) - G_\nu(a, b)$$

if $\frac{1}{3} \le \nu < 1$.

More recently, in [1] are given new results which extend many generalizations of Young's inequality given before. The below result, from [1] is a generalization of the left-

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[1] Politehnica University of Timisoara, 300006 Timisoara, Romania. E-mail: lc803007@gmail.com.

hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah in [15] and [16], being a very important tool in the demonstration of our next theorems.

**Theorem 1.** Let $\lambda, v$ and $\tau$ be real numbers with $\lambda \geq 1$ and $0 < v < \tau < 1$. Then

$$\left(\frac{v}{\tau}\right)^\lambda < \frac{A_v(a,b)^\lambda - G_v(a,b)^\lambda}{A_\tau(a,b)^\lambda - G_\tau(a,b)^\lambda} < \left(\frac{1-v}{1-\tau}\right)^\lambda$$

for all positive and distinct real numbers a and b.

Moreover, both bounds are sharp.

In [7], related to Young's inequality, often appear the weighted arithmetic mean, geometric mean and harmonic mean defined by $A_v(a,b) = (1-v)\,a + vb$,

$G_v(a,b) = a^{1-v}b^v$ and $H_v(a,b) = A_v^{-1}(a^{-1}, b^{-1}) = [(1-v)\,a^{-1} + vb^{-1}]^{-1}$,

It is necessary to recall, see [1], that for two positive definite matrices A, B, the $\mu$ - weighted arithmetic and geometric mean are defined as

$$A\nabla_\mu B = (1-\mu)A + \mu\,B \text{ and}$$
$$A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\mu A^{\frac{1}{2}}$$

when $\mu \in [0,1]$. If $\mu = \frac{1}{2}$ then we write only $A\nabla B$, $A\sharp B$. We denote the extension of

the $\mu$ -weighted geometric mean by $A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\mu A^{\frac{1}{2}}$, when $\mu \geq 0$.

It is known that for any two square matrices A, B, $A < B$ if $B-A$ is positive semidefinite. Also, $A < B$ if $B-A$ is positive definite [1, 14].

The first matrix version of the Young inequality was proven for invertible matrices A in [17]. Recent improvement of the matrix Young inequality were given for example in [1, 11, 15, 16].

We use the following generalization of Young's inequality given in [6], see inequality (5.8), in order to obtains the matrix analogues in section 2.

For any a, b > 0 and $\mu \in [0,1]$ we have:

(1)     $2\frac{v(1-v)}{v^2-v+1}[A(a,b) - L(a,b)] \leq A_v(a,b) - G_v(a,b) \leq 2[A(a,b) - L(a,b)]$,

where $A_v(a,b) = (1-v)\,a + vb$, $G_v(a,b) = a^{1-v}b^v$ , $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$ and

$$L(a,b) = \frac{b-a}{\log b - \log a} \ .$$

We also have to mention the following inequalities used in [7] in the proof of Theorem 4:

For any x > 0 we have,

(2)     $1 - v + vx - x^v \leq \begin{cases} v(x - 1 - \log x) \\ (1-v)(x\log x - x + 1) \end{cases}$

and

(3)     $\frac{x+1}{2} - \frac{x^v + x^{1-v}}{2} \leq \frac{1}{2}\,min\{v, 1-v\}(x-1)\log x$

for any  $\nu \in [0,1]$.

As in [5], it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if  $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space (H; $< . , . >$). The *Gelfand map* establishes a *- isometrically isomorphism $\Phi$ between the set  C(Sp(A))  of all  *continuous functions* defined on the  *spectrum* of A, denoted Sp(A), and the  C$^*$ - algebra C$^*$(A) generated by A and the identity operator   1$_H$ on H as follows: For any $f$, $f \in$ C(Sp(A)) and for any $\alpha, \beta \in \mathbb{C}$ we have

    (i)   $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
    (ii)  $\Phi(fg) = \Phi(f) \Phi(g)$  and  $\Phi(f) = \Phi(f^*)$;
    (iii)  $\| \Phi(f) \| = \|f\| := sup_{t \in Sp(A)}$ |f(t)|;
    (iv)  $\Phi(f_0) = 1_H$   and    $\Phi(f_1) = A$,
where $f_0(t) = 1$ and  $f_1(t) = t$   for   $t \in$ Sp (A).

Using this notation, as in   [5] for example, we define f(A) := $\Phi$ (f) for all   $f \in$ C(Sp(A)) and we call it the *continuous functional calculus*  for a selfadjoint operator A.

It is known that if   A   is a selfadjoint operator and   $f$   is a real valued continuous function on Sp(A), then $f(t) \geq 0$   for any $t \in$ Sp(A)  implies that $f(A) \geq 0$  i.e. $f(A)$  is a *positive operator* on H.

In addition, if and   $f$   and   $g$   are real valued functions on  Sp(A)  then the following property holds: $f(t) \geq g(t)$   for any $t \in Sp(A)$  implies that $f(A) \geq g(A)$  in the operator order of  B(H).

We consider  A, B  two positive operators on a complex Hilbert space (H, $< . , . >$) and the following notations for operators:

$$A \nabla_\nu B = (1 - \nu)A + \nu B, \quad \nu \in [0,1],$$

the weighted operator arithmetic mean and

$$A \nabla_\nu B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\nu A^{\frac{1}{2}} , \quad \nu \in [0,1],$$

the weighted operator geometric mean. We denote the extension of  the weighted operator geometric mean by   $A \#_\nu B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\nu A^{\frac{1}{2}} , \quad \nu \geq 0$, see also [7].

The aim of this paper is to present new inequalities for operators and matrices starting from new generalizations of Young's inequality. In Section 2 are presented a Young-type inequality for positive invertible operators on a complex Hilbert space (H, $< . , . >$)  in Theorem 2 and then, as a consequence, this inequality is given for the particular case $\lambda = n \in \mathbb{N}$  in Proposition 2. In Section 3, Remark 1 is stated an extension of Theorem 3.2 from  [1] and then in Theorem 3 and Theorem 4 are given some matrix inequalities using the spectral decomposition and scalar inequalities formulated in [6, 7].

## 2. THE YOUNG-TYPE INEQUALITIES OPERATORS

The following inequalities operators will use the inequality from Theorem 1 given in previous section.

**Theorem 2.** For any A, B  positive invertible operators on the Hilbert space  H we have

$$\left(\frac{v}{\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(\tau B + (1-\tau)A)A^{-\frac{1}{2}}\right)^\lambda - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

$$< \left(A^{-\frac{1}{2}}(vB + (1-v)A)A^{-\frac{1}{2}}\right)^\lambda - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v\lambda}$$

$$< \left(\frac{1-v}{1-\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(\tau B + (1-\tau)A)A^{-\frac{1}{2}}\right)^\lambda - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

which can be also written,

$$\left(\frac{v}{\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(A\nabla_\tau B)A^{-\frac{1}{2}}\right)^\lambda - A^{-\frac{1}{2}}(A\#_\tau B)A^{-\frac{1}{2}}\right] < \left(A^{-\frac{1}{2}}(A\nabla_v B)A^{-\frac{1}{2}}\right)^\lambda - A^{-\frac{1}{2}}(A\#_v B)A^{-\frac{1}{2}}$$

$$< \left(\frac{1-v}{1-\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(A\nabla_\tau B)A^{-\frac{1}{2}}\right)^\lambda - A^{-\frac{1}{2}}(A\#_\tau B)A^{-\frac{1}{2}}\right]$$

or

$$\left(\frac{v}{\tau}\right)^\lambda [A\#_\lambda(A\nabla_\tau B) - A\#_{\tau\lambda}B] < A\#_\tau(A\nabla_v B) - A\#_{v\lambda}B < \left(\frac{1-v}{1-\tau}\right)^\lambda [A\#_\lambda(A\nabla_\tau B) - A\#_{\tau\lambda}B],$$

for any real numbers $\lambda, v$  and $\tau$  with $\lambda \geq 1$  and   $0 < v < \tau < 1$.

*Proof:* In Theorem 1 if we divide the inequality

$$\left(\frac{v}{\tau}\right)^\lambda < \frac{A_v(a,b)^\lambda - G_v(a,b)^\lambda}{A_\tau(a,b)^\lambda - G_\tau(a,b)^\lambda} < \left(\frac{1-v}{1-\tau}\right)^\lambda$$

by  $b^\lambda$ we get the following:

$$\left(\frac{v}{\tau}\right)^\lambda < \frac{\left(v\frac{a}{b} + (1-v)\right)^\lambda - \left(\frac{a}{b}\right)^{v\lambda}}{\left(\tau\frac{a}{b} + (1-\tau)\right)^\lambda - \left(\frac{a}{b}\right)^{\tau\lambda}} < \left(\frac{1-v}{1-\tau}\right)^\lambda.$$

Now we use the continuous functional calculus as in [7] and we have for an operator C > 0 that

$$\left(\frac{v}{\tau}\right)^\lambda \left[(\tau C + (1-\tau)1_H)^\lambda - C^{\tau\lambda}\right] < (vC + (1-v)1_H)^\lambda - C^{v\lambda}$$

$$< \left(\frac{1-v}{1-\tau}\right)^\lambda \left[(\tau C + (1-\tau)1_H)^\lambda - C^{\tau\lambda}\right].$$

We take $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and we have

$$\left(\frac{v}{\tau}\right)^{\lambda}\left[\left(\tau A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\tau)1_H\right)^{\lambda} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

$$< \left(vA^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-v)1_H\right)^{\lambda} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v\lambda}$$

$$< \left(\frac{1-v}{1-\tau}\right)^{\lambda}\left[\left(\tau A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\tau)1_H\right)^{\lambda} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

or

$$\left(\frac{v}{\tau}\right)^{\lambda}\left[\left(A^{-\frac{1}{2}}(\tau B + (1-\tau)A)A^{-\frac{1}{2}}\right)^{\lambda} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

$$< \left(A^{-\frac{1}{2}}(vB + (1-v)A)A^{-\frac{1}{2}}\right)^{\lambda} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v\lambda}$$

$$< \left(\frac{1-v}{1-\tau}\right)^{\lambda}\left[\left(A^{-\frac{1}{2}}(\tau B + (1-\tau)A)A^{-\frac{1}{2}}\right)^{\lambda} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right].$$

Now if we multiply both sides of previous inequality with $A^{\frac{1}{2}}$ we deduce last inequality of this theorem.

**Proposition 2.** For $\lambda = n \in \mathbb{N}$ and A, B positive invertible operators on H we have:

$$\frac{v^n}{\tau^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^k A^{\frac{1}{2}} - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau n} A^{\frac{1}{2}}\right]$$

$$< \sum_{k=0}^{n}\binom{n}{k}(1-v)^{n-k}v^k A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^k A^{\frac{1}{2}} - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{vn} A^{\frac{1}{2}}$$

$$< \frac{(1-v)^n}{(1-\tau)^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^k A^{\frac{1}{2}}\right.$$

$$\left. - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau n} A^{\frac{1}{2}}\right]$$

or

$$\frac{v^n}{\tau^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A\#_k B - A\#_{\tau n}B\right] < \sum_{k=0}^{n}\binom{n}{k}(1-v)^{n-k}v^k A\#_k B - A\#_{vn}B$$

$$< \frac{(1-v)^n}{(1-\tau)^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A\#_k B - A\#_{\tau n}B\right]$$

for any real numbers $v$ and $\tau$ with $0 < v < \tau < 1$.

*Proof:* We use the same method as in Theorem 2.

## 3. A MATRIX ANALOGUE OF A REFINEMENT OF YOUNG'S INEQUALITY

As in [1], we consider $M_n$ the set of $n \times n$ square matrices. We denote by $\lambda_1(H) \leq \lambda_2(H) \ldots \leq \lambda_n(H)$ the eigenvalues of a Hermitian matrix H of order n, in increasing order.

The scalar inequality from Lemma 1 and the Ostrowski's theorem, see [14] allows us to state the following result:

**Theorem 3.** Let $\nu \in (0,1)$ and $A, B$ be positive definite matrices. If $A \leq B$ then we have:

$$A\nabla_\nu B - A\#_\nu B \leq 2\left[A\nabla B + \frac{B-A}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}\right],$$

or

$$A\nabla_\nu B - A\#_\nu B \leq 2\left[A\nabla B - \frac{\frac{\lambda_1(A)}{\lambda_n(B)}-1}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}B\right].$$

*Proof:* We take into account the Hermitian matrix $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ which satisfy the inequality $0 < C \leq I$. As in the proof of Theorem 3.4, see [1], there is a unitary matrix U such that for some $c_i$ we have $U^*CU = diag(c_1, \ldots, c_n) \leq I$ and thus by Ostrowski's theorem we get $\frac{\lambda_1(A)}{\lambda_n(B)} \leq c_i \leq 1$.

If we write the inequality (1) when $0 < b \leq 1$ for these positive real numbers $c_i$, $i = \overline{1,n}$ and replace b by $c_i$ and a by 1 then we have:

$$1 - \nu + \nu c_i - c_i^\nu \leq 2\left[\frac{c_i+1}{2} - \frac{c_i-1}{\log c_i}\right], \qquad i = \overline{1,n},$$

or

$$1 - \nu + \nu c_i - c_i^\nu \leq 2\left[\frac{c_i+1}{2} + \frac{1-c_i}{\log c_i}\right], \qquad i = \overline{1,n},$$

when $c_i \leq 1$. We consider the function $f(t) = \frac{1}{\log(t)}$, $t \in (0,1)$ defined on a compact subinterval of (0,1), function which attains its maximum at its left endpoint and we get:

$$1 - \nu + \nu c_i - c_i^\nu \leq 2\left[\frac{c_i+1}{2} + \frac{1-c_i}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}\right], \qquad i = \overline{1,n},$$

when $c_i \leq 1$, or by calculus, the diagonal matrix inequality,

$$I\nabla_\nu diag(c_1, \ldots, c_n) - I\#_\nu diag(c_1, \ldots, c_n) \leq$$
$$\leq 2\left[I\nabla diag(c_1, \ldots, c_n) + \frac{1}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}\left(I - diag(c_1, \ldots, c_n)\right)\right].$$

Then applying the conjugation $\bullet \to B^{\frac{1}{2}}U \bullet U^*B^{\frac{1}{2}}$ we get the desired inequality.

For the second inequality we proceed like before, but in inequality

$$1 - \nu + \nu c_i - c_i^{\nu} \leq 2\left[\frac{c_i + 1}{2} - \frac{c_i - 1}{\log c_i}\right], \qquad i = \overline{1, n},$$

we replace $-\frac{c_i - 1}{\log c_i}$ by $-\frac{\frac{\lambda_1(A)}{\lambda_n(B)} - 1}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}$, because the function $f(t) = -\frac{t-1}{\log t}$, $t \in (0,1)$ attains its maximum at its left endpoint on a compact subinterval of $(0,1)$.

Then by calculus, the diagonal matrix inequality,

$$I\nabla_{\nu} diag(c_1, \ldots, c_n) - I\#_{\nu} diag(c_1, \ldots, c_n) \leq 2\left[I\nabla diag(c_1, \ldots, c_n) - \frac{\frac{\lambda_1(A)}{\lambda_n(B)} - 1}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}I\right].$$

Then applying the conjugation $\bullet \to B^{\frac{1}{2}}U \bullet U^*B^{\frac{1}{2}}$ we get the desired inequality.

The following result takes place if we put instead of $\lambda = 1$ in inequality (2.1), Theorem 2.1 from [1], $\lambda = n \in \mathbb{N}^*$.

**Remark 1**. Let $\nu$ and $\tau$ be real numbers. If A, B are positive definite matrices, then

$$\frac{\nu^n}{\tau^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k B\#_k A - B\#_{\tau n}A\right] < \sum_{k=0}^{n}\binom{n}{k}(1-\nu)^{n-k}\nu^k B\#_k A - B\#_{\nu n}A$$

$$< \frac{(1-\nu)^n}{(1-\tau)^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k B\#_k A - B\#_{\tau n}A\right]$$

for any real numbers $\nu$ and $\tau$ with $0 < \nu < \tau < 1$.

*Proof:* We use the same method as in [1], starting from inequality

$$\left(\frac{\nu}{\tau}\right)^n\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k a^{n-k}b^k - a^{n(1-\tau)}b^{n\tau}\right]$$

$$< \sum_{k=0}^{n}\binom{n}{k}(1-\nu)^{n-k}\nu^k a^{n-k}b^k - a^{n(1-\nu)}b^{n\nu}$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k a^{n-k}b^k - a^{n(1-\tau)}b^{n\tau}\right],$$

here we put a = 1 and $b = l_i > 0$, see the proof of Lemma 3.1. Now, using the spectral theorem for Hermitian matrices, [14] Theorem 2.5.6, there is a unitary matrix and a real diagonal matrix $\wedge = diag(l_1, \ldots, l_n)$ so that $Q = U^* \wedge U$. Then we have the following matrix inequality for diagonal matrices

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=0}^{n} \binom{n}{k}(1-\tau)^{n-k}\tau^k \wedge^k - \wedge^{n\tau}\right] < \sum_{k=0}^{n}\binom{n}{k}(1-\nu)^{n-k}\nu^k \wedge^k - \wedge^{n\nu}$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k \wedge^k - \wedge^{n\tau}\right],$$

which can be read, as in [1], either as entrywise companion or in the positive semidefinite ordering.

Applying the *-conjugation $\bullet \to U^* \bullet U$ we get

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k Q^k - Q^{n\tau}\right] < \sum_{k=0}^{n}\binom{n}{k}(1-\nu)^{n-k}\nu^k Q^k - Q^{n\nu}$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k Q^k - Q^{n\tau}\right].$$

But $A > 0$ implies $A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}$ are Hermitian positive definite and then by [14], page 494, $Q = A^{-\frac{1}{2}}BA^{\frac{1}{2}}$ is a positive definite *- conjugation of B. Applying here the *-conjugation $\bullet \to A^{\frac{1}{2}} \bullet A^{\frac{1}{2}}$ to last inequality we get the desired inequality.

**Theorem 4.** Let $\nu \in (0,1)$ and A, B be positive definite matrices. If $A \leq B$ then we have:

$$A\nabla_\nu B - A\sharp_\nu B \leq \nu\left[A - B - B\log\frac{\lambda_1(A)}{\lambda_n(B)}\right],$$

or

$$A\nabla_\nu B - A\sharp_\nu B \leq \nu\left[\frac{\lambda_1(A)}{\lambda_n(B)} - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right]B,$$

or

$$A\nabla_\nu B - A\sharp_\nu B \leq (1-\nu)\left[\frac{\lambda_1(A)}{\lambda_n(B)}\log\frac{\lambda_1(A)}{\lambda_n(B)} - \frac{\lambda_1(A)}{\lambda_n(B)} + 1\right]B.$$

Moreover, the following inequality takes place:

$$A\nabla_\nu B - \frac{1}{2}(A\sharp_\nu B + A\sharp_{1-\nu}B) \leq \frac{1}{2}min\{\nu, 1-\nu\}\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1\right)\left(\log\frac{\lambda_1(A)}{\lambda_n(B)}\right)B.$$

*Proof:* We take into account the Hermitian matrix $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ which satisfy the inequality $0 < C \leq I$. As in the proof of Theorem 3.4, see [1], there is a unitary matrix U such that for some $c_i$ we have $U^*CU = diag(c_1, \dots, c_n) \leq I$ and thus by Ostrowski's theorem we get $\frac{\lambda_1(A)}{\lambda_n(B)} \leq c_i \leq 1$.

If we write the inequalities (2) and (3) when $0 < x \leq 1$ for these positive real numbers $c_i$, $i = \overline{1,n}$ then we have:

$$1 - v + vc_i - c_i^v \le v(c_i - 1 - \log c_i), \qquad i = \overline{1,n},$$

or

$$1 - v + vc_i - c_i^v \le (1 - v)[c_i \log c_i - c_i + 1], \qquad i = \overline{1,n},$$

Moreover, we also have:

$$\frac{c_i + 1}{2} - \frac{c_i^v + c_i^{1-v}}{2} \le \frac{1}{2} min\{v, 1 - v\}(c_i - 1) \log c_i, \qquad i = \overline{1,n}.$$

We consider the functions $f(x) = -\log x$, $g(x) = x - 1 - \log x$, $h(x) = x \log x - x + 1$ and $t(x) = (x - 1)\log x$, $x \in (0,1)$ defined on a compact subinterval of $(0,1)$, function which attains its maximum at its left endpoint and we get:

$$1 - v + vc_i - c_i^v \le v\left(c_i - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right), \qquad i = \overline{1,n},$$

$$1 - v + vc_i - c_i^v \le v\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right), \qquad i = \overline{1,n},$$

$$1 - v + vc_i - c_i^v \le (1 - v)\left(\frac{\lambda_1(A)}{\lambda_n(B)}\log\frac{\lambda_1(A)}{\lambda_n(B)} - \frac{\lambda_1(A)}{\lambda_n(B)} + 1\right), \qquad i = \overline{1,n},$$

and

$$\frac{c_i + 1}{2} - \frac{c_i^v + c_i^{1-v}}{2} \le \frac{1}{2} min\{v, 1 - v\}\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1\right)\log\frac{\lambda_1(A)}{\lambda_n(B)}, \qquad i = \overline{1,n}.$$

when $c_i \le 1$.

By calculus, the diagonal matrix inequalities become,

$$I\nabla_v diag(c_1, \dots, c_n) - I\#_v diag(c_1, \dots, c_n) \le v\left[diag(c_1, \dots, c_n) - I - \log\frac{\lambda_1(A)}{\lambda_n(B)}I\right],$$

$$I\nabla_v diag(c_1, \dots, c_n) - I\#_v diag(c_1, \dots, c_n) \le v\left[diag(c_1, \dots, c_n) - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right]I,$$

$$I\nabla_v diag(c_1, \dots, c_n) - I\#_v diag(c_1, \dots, c_n) \le (1 - v)\left(\frac{\lambda_1(A)}{\lambda_n(B)}\log\frac{\lambda_1(A)}{\lambda_n(B)} - \frac{\lambda_1(A)}{\lambda_n(B)} + 1\right)I$$

and

$$\frac{diag(c_1, \dots, c_n) + I}{2} - \frac{[diag(c_1, \dots, c_n)]^v + [diag(c_1, \dots, c_n)]^{1-v}}{2} \le$$
$$\le \frac{1}{2} min\{v, 1 - v\}\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1\right)\left(\log\frac{\lambda_1(A)}{\lambda_n(B)}\right)I, \qquad i = \overline{1,n}.$$

respectively.

Then applying the conjugation $\bullet \to B^{\frac{1}{2}}U\bullet U^*B^{\frac{1}{2}}$ we get the desired inequalities.

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